The impact of negotiable cost-paying on basic models of network formation *

VERY PRELIMINARY DRAFT

By Norma Olaizola† and Federico Valenciano‡

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Abstract

This paper studies the impact of “liberalizing” the cost-paying of links on some basic models of network formation.

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†BRiDGE group (http://www.bridgebilbao.es), Departamento de Fundamentos del Análisis Económico I, Universidad del País Vasco, Avenida Lehendakari Aguirre 83, 48015 Bilbao, Spain; norma.olaizola@ehu.es.

‡BRiDGE group (http://www.bridgebilbao.es), Departamento de Economía Aplicada IV, Universidad del País Vasco, Avenida Lehendakari Aguirre 83, 48015 Bilbao, Spain; federico.valenciano@ehu.es.
1 Introduction

A model introduced by Olaizola and Valenciano (2015b) merges two basic models of strategic network formation, Jackson and Wolinsky (1996) connections model and Bala and Goyal (2000) two-way flow model, and integrates them as extreme cases. The basic idea consists of assuming that two types of links can be formed: strong links and weak ones. *Strong links* work better and must be supported by the two players involved, their cost is $2c$ and each player must pay $c > 0$ for it. The flow through them suffers some decay: that is to say, only $\delta \in (0, 1)$ out of a unit of information at one node reaches the other. *Weak links* work worse, i.e. the level of fluidity is $\alpha \in (0, \delta)$, and they are supported by only one player who pays for its cost which is $c$. This link-formation model “bridges the gap” between the two benchmark models in the following sense: if $\alpha = 0$ this is equivalent to Jackson and Wolinsky (1996) model, and when $\alpha = \delta$ this is equivalent to Bala and Goyal (2000) model.

This bridge-model can be further specified in two different ways. One is assuming a strictly noncooperative environment where coordination is not possible. This does not preclude the formation of strong links if it is assumed that a doubly supported link becomes necessarily strong. In this setting *Nash equilibrium*, strict or not, is the adequate stability notion as in Bala and Goyal (2000) setting. Another possible scenario is allowing for pairwise coordination for the formation of strong links. In this case *pairwise stability* is the right stability notion as in Jackson and Wolinsky (1996). In Olaizola and Valenciano (2015b) the efficient networks for the bridge-model, i.e. those that maximize the aggregate payoff, which are the same in both scenarios, are characterized and their stability is studied from both points of view.

In this paper we consider several variations of this model. We first relax the assumption on the way of sharing the cost of strong links. When pairwise coordination is possible, the assumption that the cost of a strong link must be shared equally by the two players forming it lacks a clear motivation. If any two players can coordinate to form a link, why cannot they negotiate the way of sharing its cost? This point of view is adopted and its consequences established, first for the variation of the model just outlined, then for some other natural variations of Olaizola and Valenciano (2015b).

2 The impact of liberalizing cost-paying

Let us first briefly review the model introduced by Olaizola and Valenciano (2015b) with no negotiable costs of strong links outlined above. Let $\delta$ $(0 < \delta < 1)$ be the fraction of the value of information at one node that reaches the other node through a *strong* link, and let $\alpha$ $(0 < \alpha \leq \delta < 1)$ be the fraction of the value of information at one node that reaches the other through a *weak* link. For a network $g$ and a pair of nodes $i \neq j$, let $P_{ij}(g)$ denote the set of paths in $g$ from $i$ to $j$. For each $p \in P_{ij}(g)$, let $\ell(p)$ denote the length of $p$ and $\omega(p)$ the number of weak links in $p$. Then $i$’s valuation...
of the unit of information originating from $j$ that arrives via $p$ is

$$I_i(p) = \delta^{(p) - \omega(p)} \alpha^{\omega(p)}.$$

If information at $j$ reaches $i$ via the best possible route from $j$ to $i$, then $i$’s valuation of information originating from $j$ is

$$I_{ij}(g) = \max_{p \in P_{ij}(g)} I_i(p),$$

and $i$’s overall information is

$$I_i(g) = \sum_{j \in N(i(g))} I_{ij}(g).$$

Thus player $i$’s payoff in $g$ is:

$$\Pi_i(g) = I_i(g) - c \mu^d_i(g) = \sum_{j \in N(i(g))} \max_{p \in P_{ij}(g)} \delta^{(p) - \omega(p)} \alpha^{\omega(p)} - c \mu^d_i(g),$$

(1)

where $\mu^d_i(g)$ is the number of links (weak or strong) in which $i$ is involved.

As to efficiency, there is the following result:

**Proposition 1** (Proposition 3, Olaizola and Valenciano (2015b)) If the payoff function is given by (1) with $0 \leq \alpha \leq \delta < 1$, then the unique efficient profile is:

(i) The strong-complete graph if $c < \min\{\delta - \delta^2, 2(\delta - \alpha)\}$.

(ii) The weak-complete graph if $2(\delta - \alpha) < c < 2(\alpha - \alpha^2)$

and $c(n - 4) < 2n\alpha - 4\delta - 2(n - 2)\delta^2$.

(iii) All-encompassing stars of strong links if

$$\delta - \delta^2 < c < \min\{2(\delta - \alpha) + (n - 2)(\delta^2 - \alpha^2), \delta + (n - 2)\delta^2/2\},$$

(2)

and

$$c(n - 4) > 2n\alpha - 4\delta - 2(n - 2)\delta^2.$$  

(3)

(iv) All-encompassing stars of weak links if

$$\max\{2(\delta - \alpha) + (n - 2)(\delta^2 - \alpha^2), 2(\alpha - \alpha^2)\} < c < 2\alpha + (n - 2)\alpha^2.$$  

(4)

(v) The empty network if

$$c > \max\{2\alpha + (n - 2)\alpha^2, \delta + (n - 2)\delta^2/2\}.$$  

As to the stability of these efficient networks we have:
Proposition 2 (Proposition 6, Olaizola and Valenciano (2015b)) If the payoff function is given by (6) with $0 \leq \alpha \leq \delta < 1$, we have:

(i) A pairwise stable network has at most one non-trivial weak component (which is strong if $\alpha = 0$), and has at most one non-trivial strong component.

(ii) If $0 < c < \min\{\delta - \delta^2, \delta - \alpha\}$, then the strong-complete graph is the unique pairwise stable network.

(iii) If $\delta - \alpha < c < \alpha - \alpha^2$ and $\delta < 2\alpha/(1 + \alpha)$, then weak-complete graphs are the unique pairwise stable networks.

(iv) If $\delta - \delta^2 < c < \delta - \alpha$, then all-encompassing stars of strong links are pairwise stable.

(v) If $\delta - \alpha^2 < c < \alpha + (n - 2)\alpha^2$, then all-encompassing periphery-sponsored stars of weak links are pairwise stable.

(vi) If $\max\{(\delta - \alpha)(1 + (n - 2)\alpha), \delta - \alpha^2\} < c < \alpha$, then all-encompassing stars of weak links (periphery-sponsored, center-sponsored or mixed-sponsored) are pairwise stable.

(vii) If $c > \delta - \alpha$, then in a pairwise stable network a peripheral player cannot be connected by a strong link. If $c > \alpha$, then in a pairwise stable network a peripheral player cannot be sponsored by a weak link.

For none of the efficient structures the region where they are efficient and the region where they are pairwise stable coincide. Those with strong links (strong-complete and all-encompassing stars of strong links) are pairwise stable only within a subset of the region where they are efficient. For instance, all-encompassing stars of strong links are pairwise stable only if

$$
\delta - \delta^2 < c < \delta - \alpha,
$$

while they are efficient in the much wider region where (2) and (3). As to efficient structures formed by weak links, the regions where they are efficient and that where they are pairwise stable are different. These results are obtained under the assumption that the cost of a strong link, $2c$, is to be equally shared by the two players supporting it and the cost of a weak link is paid by the player who initiates it. The point of this paper is to study the effect of relaxing these assumptions by allowing players to negotiate the way of sharing the cost of links.

2.1 Scenario I

We consider first the following variation of Olaizola and Valenciano (2015b) model:

Scenario I: Coordination is possible for the formation of strong links (of cost $2c$, and fluidity $\delta$) and the cost shares of each strong link can be negotiated by the two players who form it. Weak links (of cost $c$, and fluidity $\alpha < \delta$) are those supported by only one player, who pays its cost, and can be unilaterally formed or as a result of a withdrawal of support of a strong link by a player paying no more than $c$ for it.

Note that assuming negotiable costs of strong links does not affect the results relative to efficiency, dependent on aggregate cost of the links that form a network and unaffected by the way of sharing them, i.e. Proposition 1 applies to this scenario. But it
may affect stability. In fact, the extension of Jackson and Wolinsky’s notion of pairwise stability provided in Olaizola and Valenciano (2015b) must be further revised in this setting. To begin with, a complete description of a network $g$ in this setting requires specifying the part of the cost paid by each of the two players supporting every strong link, i.e. now (1) must be rewritten introducing the matrix of shares of costs

$$\Pi_i(g, c) = I_i(g) - c_i(g, c) = \sum_{j \in N(i|g)} \max_{p \in \mathcal{P}_{ij}(g)} \delta^{(p)} - \omega^{(p)} \alpha^{(p)} - \sum_{ij \in g} c_{ij}. \quad (6)$$

where $c_{ij}$ is the investment of player $i$ in the link, weak or strong, connecting $i$ and $j$. For a matrix of costs $c = (c_{ij})_{i,j \in N}$ to make sense and be consistent with the model we assume: $c_{ii} = 0$; $c_{ij} = c$ and $c_{ji} = 0$, if $g_{ij} = 1$ and $g_{ji} = 0$. As for strong links we have the following necessary conditions of feasibility:

$$c_{ij}, c_{ji} \geq 0 \quad \text{and} \quad c_{ij} + c_{ji} = 2c. \quad (7)$$

We say that $c$ is a matrix of costs consistent for $g$ if these conditions hold. But this is not enough to guarantee that network $g$ is stable for the allocation of costs expressed by $c$. Moreover, in order to adapt Jackson and Wolinsky’s (1996) pairwise stability notion to this scenario, the adaptation proposed in Olaizola and Valenciano (2015b) must be further revised. To begin with, the actions allowed for players, given a network $g$ and a matrix of costs $c$ consistent for $g$, w.r.t. which a notion of stability is to be formulated must be specified. We assume the following options $(g, c) \to (g', c')$:

- **Investing in a new link** $ij \notin g$: then $i$ and $j$ become connected by a weak link if $ji \notin g$, and by a strong link if $ji \in g$, in both cases $i$ must pay $c$, i.e. $g'_{ij} = 1$ and $c'_{ij} = c$.

- **Deleting a weak link** $ij \in g$ s.t. $ji \notin g$: then $i$ and $j$ become disconnected, i.e. $g'_{ij} = 0$ and $c'_{ij} = 0$.

- **Withdrawing support of a strong link** $ij \in g$: if player $i$ withdraws support from the link, there are two cases:
  - if $c_{ij} > c$, $i$ and $j$ become disconnected: $g'_{ij} = g'_{ji} = 0$ and $c'_{ij} = c'_{ji} = 0$.
  - if $c_{ij} \leq c$, $i$ and $j$ become connected by a weak link supported by $j$ at cost $c$, i.e. $g'_{ij} := 0$ and $c'_{ij} = c'_{ji} = c$.

- **Switching support**: by combining a deletion/withdrawing with an investment in a new one.

- **Creating a new strong link** any two players not connected by a strong link in $g$, (i.e. disconnected or connected by a weak one) create a strong link: $g'_{ij} = g'_{ji} = 1$ and agree how to share its cost $(c'_{ij}, c'_{ji})$ s.t. $c'_{ij}, c'_{ji} \geq 0$ and $c'_{ij} + c'_{ji} = 2c$.

Thus the first four are unilateral actions, while the last one requires pairwise coordination and includes two actions: forming the link and agreeing the way of sharing its cost. The following definition formalizes the idea of stability w.r.t. these actions.

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1Alternatively it can be assumed that in both cases $i$ and $j$ become disconnected. But this does not seem consistent with the possibility of forming a strong link by making double an existing weak one.
Definition 1 A network $g$ admits a cost share equilibrium allocation (CSE-allocation) $c$, if there exists a matrix of costs consistent for $g$ s.t. no player has an incentive to any unilateral move nor pair of players to create a new strong link.

Depending on the network and the values of the parameters, a cost share equilibrium allocation may not exist, and in general, they are not unique when they exist. But when a network admits a cost share equilibrium allocation, it represents an outcome stable w.r.t. admissible unilateral and bilateral actions which adapts pairwise stability to the current scenario.

The goal now is to establish conditions under which each of the five efficient networks, characterized by Proposition 1, admit a cost share equilibrium allocation and to compare them with the conditions for pairwise stability. We discuss one by one the five efficient structures.

1. The strong-complete network.

Let $g$ be a strong-complete network. Just observe that if the cost of each link in $g$ is equally shared by the two players forming it, this is a CSE-allocation in the whole region where the strong-complete network is pairwise stable (Proposition 2-(ii)). In other words, whenever a strong-complete network is pairwise stable, $c = (c_{ij})_{i,j \in N}$ with $c_{ij} = c_{ji} = c$, for all $i \neq j$ is a CSE-allocation for it. Now assume $c = (c_{ij})_{i,j \in N}$ with $c_{ij} \neq c_{ji}$. Assume w.l.o.g. $c_{ij} > c_{ji}$. Then, if $\delta - c_{ij} \geq \delta^2$, $i$ has no incentive to withdraw support to its strong link with $j$, and $j$ has no incentive to withdraw support to its strong link with $i$ if $\delta - c_{ji} \geq \max\{\delta^2, \alpha\}$. That is

$$c_{ij} \leq \delta - \delta^2 \text{ and } c_{ji} \leq \min\{\delta - \delta^2, \delta - \alpha\}.$$ 

Both conditions along with $c_{ij} + c_{ji} = 2c$ are compatible if

$$c \leq \min\{\delta - \delta^2, \delta - (\delta^2 + \alpha) / 2\}.$$ 

It can easily be checked that these conditions are weaker than those under which the strong-complete network is pairwise stable (Proposition 2-(ii)). Interestingly enough, the possibility of asymmetry in the way of sharing the cost widens the region where this totally symmetric structure can be stabilized. Figure 1 represents the region for $n = 20$ and $\alpha = 0.2$ (Fig.1a) and $\alpha = 0.6$ (Fig.1b), bounded by continuous think lines, while the region where it is pairwise stable is the smaller region where part of its boundary is represented by a dash line.

The extreme case $\alpha = 0$ corresponds to Jackson and Wolinsky (1996) model. In this case the above condition becomes $c \leq \delta - \delta^2$ which is the region where the complete network in their model is efficient and pairwise stable (Propositions 1 and 2, Jackson and Wolinsky, 1996), and consequently negotiable costs does not widen the stability.
region of complete networks, stable in their model.

2. The weak-complete networks.

Let \( g \) be a weak-complete network. In Scenario I each weak link in \( g \) is supported and paid for by only one of the two players. In this particular context, the only action not included in the repertoire considered for pairwise stability and possible in Scenario I is forming a strong link and freely agreeing how to share its cost. Therefore within the region where this structure is pairwise stable (see Proposition 2-((iii))), no action in Scenario I may improve a player’s payoff, but perhaps forming a strong link. Let us see how this option actually further restrict this region. In Olaizola and Valenciano (2015b) model the only way in which this could be done is by “doubling” a weak link, for which there is no incentive if \( c \geq \delta - \alpha \). But now players forming a strong link can agree on how to share its cost. Assume weak link \( ij \) is supported by \( i \), and they envisage forming a strong link and paying \( c_{ij} \) and \( c_{ji} \) for it. Both players would have incentives if \( \alpha - c < \delta - c_{ij} \) and \( \alpha < \delta - c_{ji} \), which is possible if \( c < 2 (\delta - \alpha) \). In other words, no player has incentive to make a strong link if \( c \geq 2 (\delta - \alpha) \). Therefore, the weaker condition \( c \geq \delta - \alpha \) for pairwise stability (Proposition 2-((iii))) must be replaced by this stronger one for the existence of CSE-allocation. Thus the region where weak-complete networks are stable shrinks when costs are negotiable. Figure 2 represents the region for \( n = 20 \) and \( \alpha = 0.2 \) (Fig.2a) and \( \alpha = 0.6 \) (Fig.2b), bounded by continuous think lines, while the region where it is pairwise stable is the greater

Figure 1 a: Strong-complete

\[ \alpha = 0.2 \]

Figure 1 b: Strong-complete

\[ \alpha = 0.6 \]
3. The all-encompassing star of strong links.

Let \( g \) be an all-encompassing star of strong links. Let \( i_o \) be its center and \( j \) any peripheral node. Let \( c_{i_o j} \) and \( c_{j i_o} \) be the shares of the cost to be paid by each of them. For such a center-spokes pairwise allocation of cost to be feasible it is needed that condition (7) holds, which becomes

\[
c_{i_o j}, c_{j i_o} \geq 0 \quad \text{and} \quad c_{i_o j} + c_{j i_o} = 2c.
\]

It seems natural to expect that in a reasonable agreement the center would not pay more than a peripheral player for the link connecting them. Thus we assume\(^2\) \( c_{i_o j} < c_{j i_o} \). For this allocation to be a stable agreement it must hold \( \delta - c_{i_o j} \geq \alpha \), otherwise the center would not be interested in supporting a strong link with a peripheral player at that cost. Then it must hold

\[
c_{i_o j} \leq \delta - \alpha. \quad (8)
\]

Similarly, for a peripheral player to have an incentive to pay \( c_{j i_o} \) for a strong link with the center, it must hold

\[
\delta(1 + (n - 2)\delta) - c_{j i_o} \geq \alpha(1 + (n - 2)\delta) - c
\]

i.e.

\[
c_{j i_o} \leq (\delta - \alpha)(1 + (n - 2)\delta) + c. \quad (9)
\]

Otherwise player \( j \)’s payoff would improve by switching, i.e. by withdrawing support to the strong link with \( i_o \) and investing \( c \) in a weak one with \( i_o \). On the other hand, no two peripheral players are interested in creating a strong link if \( \delta - c \leq \delta^2 \), i.e.

\[
c \geq \delta - \delta^2, \quad (10)
\]

\(^2\)There also exist CES-allocations with \( c_{i_o j} > c_{j i_o} \), but within a smaller region.
because in that case at least one of the two players would not improve his/hers payoff.

Thus the four conditions (7), (8), (9) and (10) are necessary for there to be a margin for negotiating the shares of the cost, in other words, for \( c \) to be a cost share equilibrium allocation for \( g \). There is room for these conditions to hold if (summing up (8) and (9))

\[
c_{i,o} + c_{j,o} = 2c \leq (\delta - \alpha)(2 + (n - 2)\delta) + c,
\]
i.e. if \( c \leq (\delta - \alpha)(2 + (n - 2)\delta) \). Finally, the aggregate payoff must be non-negative, i.e. \( c \leq \delta + (n - 2)\delta^2/2 \). In sum we have

\[
c \leq \min\{\delta + (n - 2)\delta^2/2, (\delta - \alpha)(2 + (n - 2)\delta)\}. \tag{11}
\]

Note that pairwise stability conditions for stars of strong links (5) require \( c \leq \delta - \alpha \), while cost share equilibrium allocations exist for a considerably wider set of values of the parameters assuming cost-payment negotiable. It follows also from (11) that the number of players contributes to widening the range of values of the other parameters where cost share equilibrium allocations exist, while the greater \( \alpha \) the more stringent this condition becomes. Figure 3 represents this region for \( n = 20 \) and \( \alpha = 0.2 \) (Fig.3a) and \( \alpha = 0.6 \) (Fig.3b), bounded by continuous think lines, while the region where it is pairwise stable is the considerable smaller region where part of its boundary is represented by a dash line..

The extreme case \( \alpha = 0 \) corresponds to Jackson and Wolinsky (1996) model. In this case (11) becomes:

\[
c \leq \delta + (n - 2)\delta^2/2,
\]
which along with $\delta - \delta^2 \leq c$ determines the region where all-encompassing stars are efficient networks in their model (Proposition 1, Jackson and Wolinsky, 1996). This means that in Jackson and Wolinsky (1996) model all-encompassing stars can be stabiliized by cost share equilibrium allocations within the whole region where they are efficient.

4. All-encompassing stars of weak links.

Assume $g$ is an all-encompassing star of weak links. Under the conditions for which such structure is pairwise stable, that is, Proposition 2-(v) or (vi) if it is periphery-sponsored, and (vi) otherwise, no unilateral action can improve the payoff of a player, nor a new strong link between two peripheral players. There only remains to study the possibility of creating a new strong link between the center and a peripheral player. Assume peripheral $j$ supports the weak link with the center $i_o$. Player $j$ has an incentive to form a strong link with $i_o$ and pay $c_{ji_o}$ for it if $\alpha + (n - 2) \alpha^2 - c < \delta + (n - 2) \alpha \delta - c_{ji_o}$, and $i_o$ has an incentive to pay for it $c_{i_oj}$ if $\alpha < \delta - c_{i_oj}$. There is no room for both conditions if

$$c \geq (\delta - \alpha) (2 + (n - 2) \alpha).$$

Again negotiable costs reduces the region of stability (See Figure 4).

5. The empty network.

In Scenario I the empty network remains stable in the same region where it is pairwise stable, that is, for $c \geq \delta$. In sum, the preceding discussion can be summarized in the following result.

**Proposition 3** If the payoff function is given by (6), in Scenario I:
(i) A cost share equilibrium allocation exists for the strong-complete network whenever the following condition holds

\[ c \leq \min\{\delta - \delta^2, \delta - (\delta^2 + \alpha) / 2\}. \]

(ii) A cost share equilibrium allocation exists for weak-complete networks whenever the following conditions hold

\[ 2(\delta - \alpha) \leq c \leq \alpha - \alpha^2. \]

(iii) A cost share equilibrium allocation exists for all-encompassing stars of strong links whenever the following conditions hold

\[ \delta - \delta^2 \leq c \leq \min\{\delta + (n - 2)\delta^2/2, (\delta - \alpha)(2 + (n - 2)\delta)\}. \]

(iv) The allocation of costs of a periphery-sponsored all-encompassing star of weak links is a cost share equilibrium allocation whenever the following conditions hold

\[ \max\{\delta - \alpha^2, (\delta - \alpha)(2 + (n - 2)\alpha)\} \leq c \leq \alpha + (n - 2)\alpha^2. \]

(v) The allocation of costs of any all-encompassing star of weak links is a cost share equilibrium allocation whenever the following conditions hold

\[ \max\{\delta - \alpha^2, (\delta - \alpha)(2 + (n - 2)\alpha)\} \leq c \leq \alpha. \]

(vi) The trivial allocation of costs \((c_{ij} = 0 \text{ for all } i, j)\) of the empty network is a cost share equilibrium allocation whenever the following condition holds

\[ c \geq \delta. \]

In particular, we have the following conclusion for Jackson and Wolinsky model in Scenario I:

**Corollary 1** In particular, when \(\alpha = 0\), i.e. in Jackson and Wolinsky (1996) model:

(i) A cost share equilibrium allocation exists for the strong-complete network whenever \(c \leq \delta - \delta^2\), i.e. whenever it is efficient.

(ii) A cost share equilibrium allocation exists for all-encompassing stars of strong links whenever \(\delta - \delta^2 \leq c \leq \delta + (n - 2)\delta^2/2\), i.e. whenever it is efficient.

In conclusion, the possibility of negotiating the way of sharing the costs of strong links favors their formation and stability to the detriment of weak ones, in the sense that it widens the range of values where the efficient structures (complete and all-encompassing stars) formed by strong links are stable, while it reduces that where such efficient structures formed by weak links are stable.
3 Work in progress

In Scenario I, cost-sharing of links is negotiable only for strong links. This seems natural in a context where weak links are those singly supported and are unilaterally formed. But consider the following variation of the model:

Scenario II: Coordination is necessary for the formation of strong links (of cost $2c$, and fluidity $\delta$) and possible for weak links (of cost $c$, and fluidity $\alpha < \delta$). The cost of strong links and weak links can be freely shared by the players forming them. Singly supported weak links may exist and be formed unilaterally, in which case the player who forms it pays for its cost.

This and other variations are investigated.

References


