Rationing problems with ex-ante conditions

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Abstract
An extension of the standard rationing model is introduced. Agents are not only identified by their respective claims on some amount of a scarce resource, but also by some exogenous ex-ante conditions (characteristics), different from claims (e.g., endowments, entitlements, wealth, obligations, assets). Inequalities in the ex-ante conditions induce compensations between agents which influence the final distribution. Within this framework, we provide a generalization of the constrained equal awards rule. We characterize this generalized rule by means of consistency, path-independence and compensated exemption. Finally, we use the corresponding dual properties to characterize a generalization of the constrained equal losses rule.

Keywords: rationing, equal awards rule, equal losses rule, claims problem, ex-ante conditions

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1. Introduction
A standard rationing problem is an allocation problem in which each individual in a group of agents has a claim on a quantity of some (perfectly divisible) resource (e.g., money) and the available amount of this resource is insufficient to satisfy all claims. Assignment of taxes, bankruptcy situations and the distribution of emergency supplies are examples of rationing problems. Standard rationing problems have been widely studied in the literature.\textsuperscript{1} Since ancient times, several solutions to this simple problem have been proposed (see Aumann and Maschler, 1985; O’Neill, 1982), based mainly on equalizing gains or losses from claims, or by using a proportional yardstick.

Standard rationing analysis considers claims to be the only relevant information affecting the final distribution. Recently, several authors have studied complex rationing
situations in which not only claims, but also individual rights or other entitlements, affect the final distribution. Hougaard et al. (2012, 2013a,b) and Pulido et al. (2002, 2008) introduce references or baselines based on past experience or exogenous entitlements in order to refine the claims of agents. Indeed, Hougaard et al. (2013a) consider baselines as consolidated rights represented by positive numbers. The authors propose that agents are first assigned their baselines truncated by the claims before allocating the resulting deficit, or surplus, using a standard rationing rule in which the claims are the truncated baselines (in the case of a deficit) or the gap between each claim and its respective truncated baseline (in the case of a surplus).

In the above models, the references or baselines can be interpreted as objective evaluations of the real needs of agents that usually differ from their claims. They can also be understood as a tentative allocation becoming upper or lower bounds for the final distribution depending on whether they are feasible or not.

In this paper, we aim to prioritize (fully or partially\(^2\)) some agents over others based on their exogenous characteristics, not necessarily related to the allocation problem. Let us illustrate this point with some examples.

Imagine there are \( n \) agents and each agent \( i \) has an initial stock or endowment of resource \( \delta_i \geq 0 \). Furthermore, let us suppose that there is scarcity and that the available amount \( r > 0 \) of resource to be currently distributed does not cover the claims of agents. It seems unfair to treat equally agents with different endowments, even in the case of having equal claims. In this paper, we propose to prioritize an agent with a small endowment with respect to another agent with a larger endowment by compensating as much as possible the gap between endowments. Consider, for instance, a distribution of irrigation water among a group of farmers in a drought period. Moreover, imagine that each farmer has a reservoir to collect rainwater, but the current level (endowment of water) of the reservoirs are not all equal. Even in the case that the crop extension owned by each farmer is equal, the distribution of water should be affected by inequalities between the water reserves of farmers.

Another situation where ex-ante inequalities between agents arise is in the distribution of grants or subsidies by a public institution. The access to grants requires to fulfill some conditions related to the socio-economic situation of agents (e.g., salary, assets, debts, debts, debts, debts, debts, debts).

\(^2\)A rule fully prioritizes an agent \( i \) over an agent \( j \) if this rule assigns agent \( i \) all his claim whenever agent \( j \) is assigned a strictly positive amount (see Moulin, 2000). That is, if \( c_i \) and \( c_j \) are the corresponding claims of agents \( i \) and \( j \) and \( x_i \) and \( x_j \) their corresponding allocations assigned by the rule, then it must hold that if \( x_j > 0 \), then \( x_i = c_i \). On the other hand, a rule partially prioritizes in awards an agent \( i \) over an agent \( j \) if this rule assigns more awards to agent \( i \) than to agent \( j \) whenever agent \( j \) is assigned a strictly positive amount and agent \( i \) does not receive all his claim. That is, if \( x_j > 0 \) and \( x_i < c_i \), then \( x_j < x_i \). Finally, a rule partially prioritizes in losses an agent \( i \) over an agent \( j \) if this rule assigns less losses to agent \( i \) than to agent \( j \) whenever agent \( i \) is assigned a strictly positive loss and agent \( j \) does not lose all his claim. That is, if \( c_i - x_i > 0 \) and \( c_j - x_j < c_j \), then \( c_i - x_i < c_j - x_j \).
family members) that affect the final distribution of funds. In our model, we propose to summarize these aspects in a monetary value $\delta_i \in \mathbb{R}$ related to the net wealth of agents. Notice that for some agents this value might be negative (for instance, if debts are larger than assets) while positive for others. It seems reasonable that inequalities in the net wealth affect the distribution of funds, giving (full or partial) priority to agents with worse socio-economic environment. An allocation problem that fits in this situation might be the distribution of scholarships. In this case, other aspects that might refine the value $\delta_i$ are the family income or the previous success in academic results.

Notice that in the above examples there are exogenous characteristics of agents, different from claims (e.g., endowments, rights, wealth, entitlements, obligations, needs, health), that might suggest payoff compensations in favour of some agents and to the detriment of others. The ex-ante condition of each agent $i$ is gathered in the parameter $\delta_i$. The next numerical example illustrates how we aim these ex-ante conditions affect the final distribution.

**Example 1.** Consider a three-person rationing problem where the claims are $c_1 = 2.5$, $c_2 = 3$ and $c_3 = 2.5$ and the ex-ante conditions are $\delta_1 = 0$, $\delta_2 = 1.5$ and $\delta_3 = 4.5$, respectively. Imagine that we want to distribute an amount $r = 3$ following an egalitarian awards principle but taking these ex-ante conditions into account. Agent 1 is the first agent to be assigned awards since he has the worst ex-ante condition (in the two aforementioned applications, either this agent has the lowest water reserve, or he is the poorest agent, respectively). Thus, agent 1 receives $\delta_2 - \delta_1 = 1.5$ units of resource in order to compensate the inequality in ex-ante conditions with respect to the

![Figure 1: Equalizing awards with ex-ante conditions.](image-url)
agent with the second worst ex-ante condition. At this point there are still 1.5 units left to be distributed. Finally, agents 1 and 2 share equally this amount (0.75 units each) and agent 3 does not receive anything. This holds since neither agent 1, nor agent 2 have been fully compensated with respect to agent 3. Indeed, notice that the gap between the ex-ante conditions of agents 1 and 3 is larger than the amount received by agent 1, i.e. \( \delta_3 - \delta_1 = 4.5 > 1.5 + 0.75 = 2.25 \). An analogous argument applies if we compare the gap between the ex-ante conditions of agents 2 and 3 with the amount received by agent 2, i.e. \( \delta_3 - \delta_2 = 3 > 0.75 \). A graphic representation of this procedure can be found in Figure 1, which is inspired by the hydraulic representation of rationing rules given by Kaminski (2000).

In the above example, notice that agents 1 and 2 have deserved full priority with respect to agent 3, while agent 1 has deserved partial priority over agent 2.

Let us remark that the values of the ex-ante conditions \( \delta_i \) are not allocated. Indeed, what is relevant is not the numerical value of the ex-ante condition of an agent, but its relative value with respect to the ex-ante conditions of the rest of agents. Specifically, as the above example shows, bilateral compensations are induced by the inequalities in the ex-ante conditions of any pair of agents. Asymmetric allocations were previously analysed in Moulin (2000). This author assigns weights to agents and distributes awards or losses (up to the value of the claims) proportionally with respect to the weights. He also combines these weighted solutions with full priority rules. In our approach, the asymmetries are included in the definition of the problem and the rules that we apply preserve the idea of equal distribution.

To end this section, let us mention some additional interesting applications. First, consider food supply to refugees, where partial and full priority, depending on the characteristics of the refugees (age or health), might be introduced by means of ex-ante condition parameters. Another application is the introduction of partial priority in bankruptcy situations. Kaminski (2006) assigns to different categories of claimants lexicographic full priorities. The model we introduce allows to combine partial and full priority between categories of claimants or between claimants in the same category. Indeed, there is an extensive literature on bankruptcy laws discussing the insertion of partial priority in bankruptcy codes (e.g., Bebchuk and Fried, 1996; Bergström et al., 2004; Warren, 1997). Moreover, our model might be applied to allocate resources in a context in which a same group of agents faces a sequence of rationing problems at different periods of time. The distribution in the current period is influenced by the amount received in previous periods, that can be considered as an ex-ante condition for the current rationing problem. Finally, inequalities in the ex-ante conditions might be also useful to analyse taxation problems when differences in net wealth of agents are relevant in the final allocation.

The remainder of the paper is organized as follows. In Section 2, we introduce the main notations, we define a rationing problem with ex-ante conditions and we extend
the constrained equal awards rule and the constrained equal losses rule to this new framework. In Section 3, we characterize axiomatically both rules by extending, and in some cases adapting, the axioms used to characterize the rules in standard rationing problems (Herrero and Villar, 2001). In Section 4, we conclude.

2. Rationing problems with ex-ante conditions and rules

In this section, previously to analyse rationing problems with ex-ante conditions, let us first introduce some notations and recall the definition of a standard rationing problem.

We denote by $N$ the set of natural numbers that we identify with the universe of potential agents, and by $\mathcal{N}$ the family of all finite subsets of $N$. Given $S \in \mathcal{N}$, we denote by $s$ the cardinality of $S$.

Given a finite subset of agents $N = \{1,2,\ldots,n\} \in \mathcal{N}$, a standard rationing problem is to distribute $r \geq 0$ among these $n$ agents with claims $c = (c_1,c_2,\ldots,c_n) \in \mathbb{R}_N^+$. It is assumed that $r \leq \sum_{i \in N} c_i$ since otherwise no rationing problem exists. We denote a standard rationing problem by the pair $(r,c) \in \mathbb{R}_+ \times \mathbb{R}_N^+$.

A feasible allocation for $(r,c)$ is represented by a vector $x = (x_1,x_2,\ldots,x_n) \in \mathbb{R}_N^+$ such that $0 \leq x_i \leq c_i$ and $\sum_{i \in N} x_i = r$, where $x_i$ represents the payoff associated to agent $i \in N$. A rationing rule associates a unique allocation to each standard rationing problem. Two well-known rationing rules are the constrained equal awards (CEA) rule and the constrained equal losses (CEL) rule.

**Definition 1. (CEA).** For any standard rationing problem $(r,c) \in \mathbb{R}_+ \times \mathbb{R}_N^+$ the CEA rule is defined as

$$CEA_i(r,c) = \min \{c_i, \lambda\}, \text{ for all } i \in N,$$

where $\lambda \in \mathbb{R}_+$ satisfies $\sum_{i \in N} \min \{c_i, \lambda\} = r$.

**Definition 2. (CEL).** For any standard rationing problem $(r,c) \in \mathbb{R}_+ \times \mathbb{R}_N^+$ the CEL rule is defined as

$$CEL_i(r,c) = \max \{0, c_i - \lambda\}, \text{ for all } i \in N,$$

where $\lambda \in \mathbb{R}_+$ satisfies $\sum_{i \in N} \max \{0, c_i - \lambda\} = r$.

Next, we introduce rationing problems with ex-ante conditions.

**Definition 3.** Let $N \in \mathcal{N}$ be a finite subset of agents. A rationing problem with ex-ante conditions for $N$ is a triple $(r,c,\delta)$, where $r \in \mathbb{R}_+$ is the amount of resource, $c \in \mathbb{R}_N^+$ is the vector of claims such that $r \leq \sum_{i \in N} c_i$, and $\delta \in \mathbb{R}_N$ is the vector of ex-ante conditions.

The aim of a rationing problem with ex-ante conditions is to distribute an amount of a scarce resource fairly but taking into account the inequalities in the ex-ante conditions.
We denote by $R^N$ the set of all rationing problems with ex-ante conditions and agent set $N$, and by $R = \bigcup_{N \in \mathcal{N}} R^N$ the family of all rationing problems with ex-ante conditions.

The definition of an allocation rule for these problems does not differ essentially from the standard definition.

**Definition 4.** A generalized rationing rule is a function $F$ that associates to each rationing problem with ex-ante conditions $(r, c, \delta) \in R^N$, where $N \in \mathcal{N}$, a unique allocation $x = F(r, c, \delta) = (F_1(r, c, \delta), F_2(r, c, \delta), \ldots, F_n(r, c, \delta)) \in \mathbb{R}_+^N$ such that

- $\sum_{i \in N} x_i = r$ (efficiency) and
- $0 \leq x_i \leq c_i$, for all $i \in N$.

Next, we extend the CEA rule to this new framework.

**Definition 5.** (Generalized equal awards rule, GEA). For any $(r, c, \delta) \in R^N$, where $N \in \mathcal{N}$, the GEA rule is defined as

$$GEA_i(r, c, \delta) = \min \{c_i, (\lambda - \delta_i)_+\} \text{, for all } i \in N,$$

where $\lambda \in \mathbb{R}$ satisfies $\sum_{i \in N} GEA_i(r, c, \delta) = r$.

Notice that the GEA rule is well defined. Indeed, by applying Bolzano’s Theorem to the continuous function $\varphi(\lambda) = \sum_{i \in N} \varphi_i(\lambda) = \sum_{i \in N} \min \{c_i, (\lambda - \delta_i)_+\}$, the existence of a value $\lambda$, such that $\varphi(\lambda) = r$, is guaranteed since

$$\varphi\left(\min_{i \in N} \{\delta_i\}\right) = 0 \leq r \leq \varphi\left(\max_{i \in N} \{c_i + \delta_i\}\right) = \sum_{i \in N} c_i.$$

Moreover, let us suppose that there exist $\lambda, \lambda' \in \mathbb{R}$, with $\lambda < \lambda'$, such that $\varphi(\lambda) = \varphi(\lambda') = r$. As the reader may verify, $\varphi_k(\lambda)$ is a non-decreasing function for all $k \in N$. Hence, we have that $\varphi_k(\lambda) \leq \varphi_k(\lambda')$ for all $k \in N$. Therefore, we obtain $r = \sum_{k \in N} \varphi_k(\lambda) \leq \sum_{k \in N} \varphi_k(\lambda') = r$ and thus $\varphi_k(\lambda) = \varphi_k(\lambda')$ for all $k \in N$. We conclude that the solution is unique and so it is well defined for all problems.

Notice that, in contrast to the standard rationing problems when $r < \sum_{i \in N} c_i$, the value of $\lambda$ in the formula of the GEA rule might not be unique. For instance, in the two-person problem $(r, c, \delta) = (2, (2, 2), (0, 3))$ the unique solution is $GEA(2, (2, 2), (0, 3)) = (2, 0)$ but $\lambda \in [2, 3]$.

Obviously, the GEA rule generalizes the CEA rule. That is, the allocation assigned by the GEA rule when applied to a problem without inequalities in ex-ante conditions coincides with the allocation of the CEA rule applied to the corresponding standard rationing problem (without ex-ante conditions).

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3From now on, we use the following notation: for all $a \in \mathbb{R}$, $(a)_+ = \max\{0, a\}$. 

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Proof. Let \( \lambda \) of the parameter \( \text{CEA} \) all his claim: that is, if agents \( i, j \) between the payoff of agents. Therefore, if there is a difference between the payoff of two agents, \( i \neq j \), it is because of the agent with the smallest payoff has received all his claim: that is, if \( \text{CEA}_i(r, c) < \text{CEA}_j(r, c) \), then \( \text{CEA}_i(r, c) = c_i \). This principle can be extended to rationing problems with ex-ante conditions by minimizing the differences between the payoff plus the corresponding ex-ante condition of agents. This feature of the GEA rule is used later and it is crucial to prove the property of consistency.

Remark 1. If \( \delta = (\alpha, \alpha, \ldots, \alpha) \in \mathbb{R}^N \), then \( \text{GEA}(r, c, \delta) = \text{CEA}(r, c) \).

Let us illustrate the application of the rule with an example.

Example 2. The rationing problem with ex-ante conditions proposed in Example 1 can be formally defined as

\[
(r, c, \delta) = (3, (2.5, 3, 2.5), (0, 1.5, 4.5)).
\]

The allocation assigned by the GEA rule is \( \text{GEA}(r, c, \delta) = (2.25, 0.75, 0) \) where the value of the parameter \( \lambda \) in the formula is 2.25, as the reader may check.

Proposition 1. Let \( (r, c, \delta) \in \mathbb{R}^N \), \( N \in \mathcal{N} \), and let \( x^* \in \mathbb{R}^N \) be such that \( x^*_i \leq c_i \), for all \( i \in N \), and \( \sum_{i \in N} x^*_i = r \). The following statements are equivalent:

1. \( x^* = \text{GEA}(r, c, \delta) \).
2. For all \( i, j \in N \) with \( i \neq j \), if \( x^*_i + \delta_i < x^*_j + \delta_j \), then either \( x^*_j = 0 \), or \( x^*_i = c_i \).

Proof. 1 \( \Rightarrow \) 2) Let us suppose that \( x^* = \text{GEA}(r, c, \delta) \) and there exist \( i, j \in N \), such that \( x^*_i + \delta_i < x^*_j + \delta_j \) but \( x^*_j > 0 \) and \( x^*_i < c_i \). Hence, \( \lambda - \delta_j > 0 \), \( x^*_i = (\lambda - \delta_i)_+ \), and so

\[
x^*_i + \delta_i = (\lambda - \delta_i)_+ + \delta_i \geq \lambda \geq \min\{c_j + \delta_j, \lambda\} = \min\{c_j, (\lambda - \delta_j)_+\} + \delta_j = x^*_j + \delta_j.
\]

Hence, we reach a contradiction with the hypothesis \( x^*_i + \delta_i < x^*_j + \delta_j \) and we conclude that either \( x^*_i = c_i \), or \( x^*_j = 0 \).

2 \( \Rightarrow \) 1) Let us suppose that for all \( i, j \in N \) with \( x^*_i + \delta_i < x^*_j + \delta_j \), it holds that either \( x^*_j = 0 \), or \( x^*_i = c_i \), but \( x^* \neq \text{GEA}(r, c, \delta) \). Then, by efficiency, there exist \( i, j \in N \) such that

\[
0 \leq x^*_i < \text{GEA}_i(r, c, \delta) \leq c_i \text{ and } c_j \geq x^*_j > \text{GEA}_j(r, c, \delta) \geq 0.
\]

This means that \( x^*_i < c_i \), \( \lambda - \delta_i > 0 \) and \( (\lambda - \delta_j)_+ < c_j \). However,

\[
x^*_j + \delta_j > \text{GEA}_j(r, c, \delta) + \delta_j = (\lambda - \delta_j)_+ + \delta_j \geq \lambda \geq \min\{c_i + \delta_i, \lambda\} = \min\{c_i, (\lambda - \delta_i)_+\} + \delta_i = \text{GEA}_i(r, c, \delta) + \delta_i > x^*_i + \delta_i.
\]

By assumption, it should hold that either \( x^*_j = 0 \), or \( x^*_i = c_i \), but this contradicts (1). Hence we conclude that \( x^* = \text{GEA}(r, c, \delta) \).
Now, we extend the idea of equalizing losses to rationing problems with ex-ante conditions. An agent’s loss is the difference between his claim and his assigned payoff. If an agent has a better ex-ante condition than another, then he may suffer a higher loss compared to this other agent. We define the generalized equal losses rule as follows:

**Definition 6.** (Generalized equal losses rule, GEL). For any \((r,c,δ) \in \mathbb{R}^N\), where \(N \in \mathbb{N}\), the GEL rule is defined as

\[
GEL_i(r,c,δ) = \max\{0, c_i - (\lambda + δ_i)\}, \text{ for all } i \in N,
\]

where \(\lambda \in \mathbb{R}\) satisfies \(\sum_{i \in N} GEL_i(r,c,δ) = r\).

The GEL rule assigns losses in an egalitarian way, but taking into account that no agent can receive a negative payoff and that the differences among ex-ante conditions might induce bilateral compensations of losses between agents.

The reader can check that the GEL rule is well defined by using similar arguments to those for the case of the GEA rule.

**Remark 2.** If \(δ = (α, α, \ldots, α) \in \mathbb{R}^N\), then \(GEL(r,c,δ) = CEL(r,c)\).

Let us illustrate the application of the GEL rule with an example.

**Example 3.** Consider the rationing problem with ex-ante conditions given in Example 1, \((r,c,δ) = (3, (2.5, 3, 2.5), (0, 1.5, 4.5))\). Notice that the total loss is \(c_1 + c_2 + c_3 - r = 5\). Agent 3 is the first agent to be assigned losses since he has the best ex-ante condition. In the first step, this agent suffers the maximum loss, all his claim, since the amount that he claims is not enough to compensate the difference between his own ex-ante condition and the second best ex-ante condition, i.e. \(c_3 = 2.5 < δ_3 - δ_2 = 3\). At this point there are still 2.5 units of losses left to be allocated. In the next step, 1.5 units of losses are assigned to agent 2 in order to fully compensate the difference between ex-ante conditions, i.e. \(δ_2 - δ_1 = 1.5\). Finally, the remaining unit is equally divided between both agents. Therefore, the losses allocation is \((0.5, 2, 2.5)\) and so the assigned payoff vector is \(GEL(r,c,δ) = (c_1 - 0.5, c_2 - 2, c_3 - 2.5) = (2, 1, 0)\).

In the standard rationing framework, the CEA and the CEL are dual rules. This means that one rule distributes the total gain \(r\), in the primal problem \((r,c)\), in the same way as the other rule distributes the total loss \(ℓ = \sum_{i \in N} c_i - r\), in the dual problem \((ℓ,c)\). The idea of duality can be adapted for rationing problems with ex-ante conditions but taking into account that the vector \(δ\), which represents the ex-ante conditions, become \(-δ\) when passing from the primal problem \((r,c,δ)\) to the dual problem \((ℓ, c, -δ)\). Next, we formally define the concept of dual rules.
Definition 7. \( F^* \) is the dual rule of \( F \) if, for all \( N \in \mathcal{N} \) and all \((r,c,\delta) \in \mathcal{R}^N\),
\[
F^*(r,c,\delta) = c - F(\ell,c,-\delta),
\]
where \( \ell = \sum_{i \in N} c_i - r \).

The duality of the GEA rule and the GEL rule is maintained as it is the duality of the CEA rule and the CEL rule in the standard framework. The proof is straightforward and it is left to the reader.

Proposition 2. The GEA and the GEL are dual rules of each other.

3. Axiomatic characterizations

The CEA and the CEL rules (for standard rationing problems) have been characterized in several studies: among others, see Dagan (1996), Herrero and Villar (2001, 2002), Schummer and Thomson (1997) and Yeh (2004, 2006, 2008).

Herrero and Villar (2001) characterizes the CEA rule by means of three axioms: consistency, path-independence and exemption. In this section, we characterize the GEA rule inspired by these axioms. Specifically, we adapt the properties of consistency and path-independence, and we introduce a new property, compensated exemption.

Path-independence states that if we apply a rule to a problem but the available amount of resource diminishes suddenly, the new allocation obtained by applying once again the same rule (to the new amount and with the original claims) is equal to the one obtained when using the previous allocation as claims.\(^4\)

Definition 8. A generalized rationing rule \( F \) satisfies path-independence if for all \( N \in \mathcal{N} \) and all \((r,c,\delta) \in \mathcal{R}^N\) with \( \sum_{i \in N} c_i \geq r \) \( r' \geq r \) it holds
\[
F(r,c,\delta) = F(r,F(r',c,\delta),\delta).
\]

Because of claim boundedness (see Definition 4), if a rule satisfies path-independence, then it is monotonic with respect to \( r \). That is, for all \( N \in \mathcal{N} \), all \( c \in \mathbb{R}_N^+ \) and all \( r,r' : \{r \leq r' \leq \sum_{i \in N} c_i \} \Rightarrow \{F(r,c,\delta) \leq F(r',c,\delta)\} \). This property is known as resource monotonicity.

Consistency is a property that requires that when we re-evaluate the resource allocation within a subgroup of agents using the same rule, the allocation should not change. To define this property we use the following notation. Given a vector \( x \in \mathbb{R}_N^+ \) and a subset \( S \subseteq N \), we denote by \( x|_S \in \mathbb{R}_S^+ \) the vector \( x \) restricted to the members of \( S \).

\(^4\)Notice that the ex-ante conditions, contrary to the claims, remain unaltered. This is because the ex-ante conditions arise from exogenous characteristics of agents and these characteristics are not related to any tentative allocation.
Definition 9. A generalized rationing rule $F$ is consistent if for all $(r, c, \delta) \in R^N$, all $N \in \mathcal{N}$ and all $T \subseteq N$, $T \neq \emptyset$, it holds

$$F(r, c, \delta)|_T = F\left(r - \sum_{i \in N \setminus T} F_i(r, c, c_i|_T, \delta_i|_T)\right).$$

Before defining compensated exemption, let us remark that in the standard rationing framework, exemption is a property that ensures that an agent with a small enough claim will not suffer from rationing. Specifically, for the two-person case $N = \{i, j\}$, a solution $(x_i, x_j) = F(r, (c_i, c_j))$ satisfies exemption if $x_k = c_k$ whenever $c_k \leq \frac{r}{2}$ for some $k \in N$.

In our framework, exemption applies just after the inequalities in ex-ante conditions have been compensated (as much as possible). Indeed, as we have discussed in Example 1, compensations are induced by pairwise comparisons of the ex-ante conditions of each pair of agents. These two remarks lead us to require exemption for any two-person problem after carrying out eventual compensations between agents. We formally define this property as follows:

Definition 10. A generalized rationing rule $F$ satisfies compensated exemption if for any two-person rationing problem with ex-ante conditions $(r, c, \delta) \in R^N$, with $N = \{i, j\}$, it holds that

if $\min\{r, c_i\} \leq \frac{r - (\delta_i - \delta_j)}{2}$ then $F_i(r, c, \delta) = \min\{r, c_i\}$.

To better understand this property let us first suppose that $c_i \leq r$. In case $\delta_i - \delta_j > 0$, the agent $j$ must be first compensated. If the claim of agent $i$ is smaller than the average of the remaining amount of resource (after compensating agent $j$), $c_i \leq \frac{r - (\delta_i - \delta_j)}{2}$, then the claim of agent $i$ can be considered small enough and agent $i$ should not be rationed.

On the other hand, in case $\delta_j - \delta_i > 0$ it is agent $i$ that must be compensated. Hence, notice that the property can be rearranged as follows:

if $c_i - (\delta_j - \delta_i) \leq \frac{r - (\delta_j - \delta_i)}{2}$ then $x_i = c_i$.

In this expression, $c_i - (\delta_j - \delta_i)$ represents the non-satisfied part of the claim of agent $i$ after being compensated. If this adjusted claim is smaller than the average of the remaining amount of resource $\frac{r - (\delta_j - \delta_i)}{2}$, then the non fulfilled part of the claim can be considered small enough and must be additionally satisfied.

Notice that if there are no inequalities in the ex-ante conditions, i.e. $\delta_i = \delta_j$, then compensated exemption recovers the classical exemption property for the two-person case.

Finally, let us remark that in case $c_i > r$ we just truncate the claim by $r$.

The next proposition states that the GEA rule satisfies all these properties. The proof is provided in the Appendix.
Proposition 3. The GEA rule satisfies path-independence, consistency and compensated exemption.

Next we show that if a rule satisfies path-independence and compensated exemption, then it also satisfies a kind of equal treatment of equals property for the two-person case.

Proposition 4. If a generalized rationing rule $F$ satisfies compensated exemption and path-independence, then for any two-person rationing problem with ex-ante conditions $(r, c, \delta) \in \mathcal{R}^N$, with $N = \{1, 2\}$, it holds that\footnote{We use the following notation: for all $a \in \mathbb{R}$, $|a| = \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{if } a < 0. \end{cases}$}

\[
\text{if } c_1 + \delta_1 = c_2 + \delta_2 \text{ and } r > |\delta_1 - \delta_2| \text{ then } F_1(r, c, \delta) + \delta_1 = F_2(r, c, \delta) + \delta_2.
\]

Proof. Let $x^* = F(r, c, \delta)$. Suppose on the contrary, that $c_1 + \delta_1 = c_2 + \delta_2$ and $r > |\delta_1 - \delta_2|$, but, w.l.o.g.,

\[
x_1^* + \delta_1 < x_2^* + \delta_2.
\]

From (2), it comes that $x_1^* + \delta_1 < \frac{x_1^* + x_2^* + \delta_1}{2} + \frac{x_1^* + \delta_2}{2}$ and thus

\[
x_1^* = F_1(r, c, \delta) < \frac{r + \delta_2 - \delta_1}{2}.
\]

Now we claim that there exists $r' > r$ such that $F_1(r', c, \delta) = \frac{r + \delta_2 - \delta_1}{2}$. Notice that $\frac{r + \delta_2 - \delta_1}{2} > 0$ since $x_1^* \geq 0$. Moreover, $\frac{r + \delta_2 - \delta_1}{2} \leq c_1$ since $c_1 + \delta_1 = c_2 + \delta_2$. Since $F$ satisfies path-independence it also satisfies resource monotonicity (see page 9). Hence, $F$ is a continuous and increasing function in $r$. Therefore, by continuity, since $F_1(0, c, \delta) = 0$, $F_1(c_1 + c_2, c, \delta) = c_1$ and $F$ is an increasing function in $r$, there exists $r' \in [0, c_1 + c_2]$ such that $F_1(r', c, \delta) = \frac{r + \delta_2 - \delta_1}{2}$. Now, by (3), we have $F_1(r, c, \delta) < F_1(r', c, \delta)$. Hence, by resource monotonicity, we conclude $r' > r$ and the proof of the claim is done.

Let us denote $x' = F(r', c, \delta)$. Notice that $\min\{r, x_1^*\} \leq x_1' = \frac{r - (\delta_2 - \delta_1)}{2}$ which implies, by compensated exemption applied to the problem $(r, x', \delta)$, that $F_1(r, x', \delta) = \min\{r, x_1'\} = \min\{r, \frac{r + \delta_2 - \delta_1}{2}\} = \frac{r + \delta_2 - \delta_1}{2}$, where the last equality follows from $r > |\delta_2 - \delta_1|$. Finally, by path-independence, we obtain

\[
F(r, c, \delta) = F(r, F(r', c, \delta), \delta) = F(r, x', \delta) = \left(\frac{r + \delta_2 - \delta_1}{2}, \frac{r + \delta_1 - \delta_2}{2}\right)
\]

and we conclude that $F_1(r, c, \delta) + \delta_1 = \frac{r + \delta_1 + \delta_2}{2} = F_2(r, c, \delta) + \delta_2$. \hfill \qed

Notice that if $\delta_1 = \delta_2$, then the statement of Proposition 4 reads as follows: if $c_1 = c_2$, then $F_1(r, c, \delta) = F_2(r, c, \delta)$, which is the classic equal treatment of equals for two-person standard rationing problems.

The next theorem characterizes the GEA rule by means of the properties we have presented.
Theorem 1. The GEA is the unique rule that satisfies path-independence, compensated exemption and consistency.

Proof. By Proposition 3, we know that the GEA rule satisfies path-independence, consistency and compensated exemption. Next, we show uniqueness. Let $F$ be a rule satisfying these properties. If $|N| = 1$, it is straightforward. Consider now the two-person case $N = \{1, 2\}$ and $(r, c, \delta) \in R^{1,2}$. Let us suppose that, w.l.o.g., $\delta_1 \leq \delta_2$ and denote $x^* = (x^*_1, x^*_2) = F(r, c, \delta)$. We consider three cases:

**Case 1:** $r \leq \delta_2 - \delta_1$. Then,

$$\min\{r, c_1\} \leq r = \frac{r}{2} + \frac{r}{2} \leq \frac{r - (\delta_1 - \delta_2)}{2}.$$  

Hence, by compensated exemption, we have that $x^*_1 = \min\{r, c_1\}$ and $x^*_2 = (r - c_1)_+$, and the solution $F$ is uniquely determined.

**Case 2:** $r > \delta_2 - \delta_1 \geq c_1$. Then,

$$\min\{r, c_1\} = c_1 \leq \delta_2 - \delta_1 = \frac{\delta_2 - \delta_1}{2} + \frac{\delta_2 - \delta_1}{2} < \frac{r - (\delta_1 - \delta_2)}{2}.$$  

Hence, by compensated exemption, we have that $x^*_1 = \min\{r, c_1\} = c_1$ and $x^*_2 = r - c_1$, and the solution $F$ is also uniquely determined.

**Case 3:** $r > \delta_2 - \delta_1$ and $c_1 > \delta_2 - \delta_1$. We consider two subcases:

**Subcase 3.a:** $c_1 + \delta_1 = c_2 + \delta_2$. Since $r > \delta_2 - \delta_1$, by Proposition 4 we have that $x^*_1 + \delta_1 = x^*_2 + \delta_2$; taking into account that $x^*_1 + x^*_2 = r$, the solution $F$ is uniquely determined.

**Subcase 3.b:** $c_1 + \delta_1 \neq c_2 + \delta_2$. First, if $\min\{r, c_1\} \leq \frac{r - (\delta_1 - \delta_2)}{2}$, then by compensated exemption $x^*_1 = \min\{r, c_1\}$ and $x^*_2 = (r - c_1)_+$, and the solution $F$ is uniquely determined. Similarly, if $\min\{r, c_2\} \leq \frac{r - (\delta_1 - \delta_2)}{2}$, then by compensated exemption $x^*_2 = \min\{r, c_2\}$ and $x^*_1 = (r - c_2)_+$, and the solution $F$ is uniquely determined. Otherwise,

$$\min\{r, c_i\} + \delta_i > \frac{r + \delta_1 + \delta_2}{2}, \text{ for all } i \in \{1, 2\}. \quad (4)$$

By the hypothesis of Subcase 3.b

$$c_i + \delta_i \leq c_j + \delta_j, \text{ where } i, j \in \{1, 2\} \text{ with } i \neq j. \quad (5)$$

Now we claim that for $r' = 2c_i + \delta_i - \delta_j$, we have that $x' = F(r', c, \delta)$ is such that $x'_i = c_i$ and $x'_j = c_i + \delta_i - \delta_j$. To verify this, first notice that, by (5), $r' < c_i + c_j$. Moreover, we show that $c_i + \delta_i - \delta_j \geq 0$. Suppose on the contrary that $c_i < \delta_i - \delta_j$. If $i = 1$ and $j = 2$, we obtain a contradiction with the hypothesis of Case 3; if $i = 2$ and $j = 1$ then $c_2 < \delta_1 - \delta_2 \leq 0$, getting again a contradiction. Notice that the second inequality follows from the assumption $\delta_1 \leq \delta_2$. 

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Now, since \( c_i + \delta_i - \delta_j \geq 0 \), we have

\[
\min\{r', c_i\} + \delta_i = \min\{2c_i + \delta_i - \delta_j, c_i\} + \delta_i = c_i + \delta_i = \frac{r' + \delta_i + \delta_j}{2},
\]

and so \( \min\{r', c_i\} = \frac{r' - (\delta_i - \delta_j)}{2} = c_i \). Hence, by compensated exemption, we have that \( x'_i = c_i \) and, by efficiency, \( x'_i = r' - x'_i = c_i + \delta_i - \delta_j \), and the proof of the claim is done.

On the other hand, \( r' = 2c_i + \delta_i - \delta_j \geq 2 \min\{r, c_i\} + \delta_i - \delta_j > r \), where the last inequality follows from (4). Therefore, by path-independence, we obtain

\[
F(r, c, \delta) = F(r, F(r', c, \delta), \delta) = F(r, x', \delta).
\]

Finally, since \( x'_i + \delta_j = c_i + \delta_i = x'_i + \delta_i \) and \( r > \delta_2 - \delta_1 \), where the inequality comes from the hypothesis of Case 3, applying Proposition 4 to the problem \((r, x', \delta)\) we obtain

\[
F_i(r, c, \delta) + \delta_i = F_i(r, x', \delta) + \delta_i = F_j(r, x', \delta) + \delta_j = F_j(r, c, \delta) + \delta_j,
\]

where the first and the last equalities come from path-independence. Hence, by efficiency, the solution \( F \) is uniquely determined. Therefore, we conclude that, for the two-person case, the \( GEA \) rule is the unique rule that satisfies path-independence and compensated exemption.

Let \( |N| \geq 3 \) and suppose that \( F \) and \( F' \) satisfy the three properties, but \( F \neq F' \). Hence, there exists \((r, c, \delta) \in \mathcal{R}^N\) such that \( x = F(r, c, \delta) \neq F'(r, c, \delta) = x' \). This means that there exist \( i, j \in N \) such that \( x_i > x'_i \), \( x_j < x'_j \) and, w.l.o.g., \( x_i + x_j \leq x'_i + x'_j \). However, since \( F \) and \( F' \) are consistent,

\[
(x_i, x_j) = F(r - \sum_{k \in N \setminus \{i, j\}} x_k, (c_i, c_j), (\delta_i, \delta_j)) \quad \text{and} \quad (x'_i, x'_j) = F'(r - \sum_{k \in N \setminus \{i, j\}} x'_k, (c_i, c_j), (\delta_i, \delta_j)).
\]

Since \( F = F' \) for the two-person case and path-independence implies resource monotonicity, we have that

\[
(x'_i, x'_j) = F'(x'_i + x'_j, (c_i, c_j)(\delta_i, \delta_j)) = F(x'_i + x'_j, (c_i, c_j)(\delta_i, \delta_j)) \geq F(x_i + x_j, (c_i, c_j)(\delta_i, \delta_j)) = (x_i, x_j),
\]

in contradiction with \( x_i > x'_i \). Hence, we conclude that \( F = F' = GEA \). \( \square \)

The properties in Theorem 1 are independent as the reader can verify in Examples 4, 5 and 6 in the Appendix.

In what follows, we characterize the \( GEL \) rule using the fact that the \( GEA \) and the \( GEL \) are dual rules of each other (see Proposition 2). The properties that characterize the \( GEL \) rule are the corresponding dual properties that characterize the \( GEA \) rule.
Definition 11. $P^*$ is the dual property of $P$ if for every rule $F$ it holds that $F$ satisfies $P$ if and only if its dual rule $F^*$ satisfies $P^*$.

The dual property of compensated exemption is compensated exclusion (see Proposition 5 in the Appendix).

Definition 12. A generalized rationing rule $F$ satisfies compensated exclusion if for any two-person rationing problem with ex-ante conditions $(r, c, \delta) \in \mathbb{R}^N$, with $N = \{i, j\}$, it holds that
\[
\text{if } \min\{\ell, c_i\} \leq \frac{\ell - (\delta_j - \delta_i)}{2} \text{ then } F_i(r, c, \delta) = (r - c_j)_+.
\]

Parallel to standard rationing problems (without ex-ante conditions), the dual property of path-independence is composition (see Proposition 6 in the Appendix).

Definition 13. A generalized rationing rule $F$ satisfies composition if for all $N \in \mathcal{N}$, all $(r, c, \delta) \in \mathbb{R}^N$ and all $r_1, r_2 \in \mathbb{R}_+$ such that $r_1 + r_2 = r$, it holds
\[
F(r, c, \delta) = F(r_1, c, \delta) + F(r_2, c - F(r_1, c, \delta), \delta).
\]

Before providing the characterization result for the GEL rule, note that Herrero and Villar connect the properties that characterize a rule for a standard rationing problem with the properties that characterize the corresponding dual rule.

“Theorem 0 (Herrero and Villar, 2001). If a rule $F$ is characterized by a set of independent properties $P = \{P_1, P_2, \ldots, P_k\}$ and if for any $P_i$ there exists a dual property $P^*_i$, then the dual rule $F^*$ is characterized by the corresponding set of dual properties $P^* = \{P^*_1, P^*_2, \ldots, P^*_k\}$. Moreover, the properties in $P^*$ are also independent”.

This result can be extended directly to our generalized framework. This allows us to characterize the GEL rule by the corresponding dual properties that characterize the GEA rule.

Theorem 2. The GEL is the unique rule that satisfies composition, compensated exclusion and consistency.

Proof. We know that the GEA and the GEL are dual rules of each other (Proposition 2). Moreover, compensated exclusion is the dual property of compensated exemption (Proposition 5), composition is the dual property of path-independence (Proposition 6) and consistency is dual of itself. Therefore, the result follows from Theorem 0 of Herrero and Villar (2001).

\footnote{Recall that $\ell$ denotes the total loss. In this case $\ell = c_i + c_j - r$.}
4. Conclusions

We have presented an extension of the standard rationing model. The aim of this extension is to take into account ex-ante inequalities between agents involved in the rationing process and to try to compensate for these inequalities. Two of the principal rationing rules (equal awards and equal losses) have been generalized and characterized within this new framework.

As we have previously mentioned in the Introduction, Hougaard et al. (2013a) propose an extension of the standard rationing model. They consider a vector of baselines \( b = (b_i)_{i=1,...,n} \), where \( b_i \) is interpreted as a tentative allocation for agent \( i \). Moreover, they denote by \( t_i(c, b) = \min\{c_i, b_i\} \) the corresponding truncated baseline. These authors use the CEA rule in the baselines model as follows:

\[
\tilde{CEA}(r, c, b) = \begin{cases} 
  t(c, b) + CEA\left(r - \sum_{i \in \mathcal{N}} t_i(c, b), c - t(c, b)\right) & \text{if } \sum_{i \in \mathcal{N}} t_i(c, b) \leq r \\
  t(c, b) - CEA\left(\sum_{i \in \mathcal{N}} t_i(c, b) - r, t(c, b)\right) & \text{if } \sum_{i \in \mathcal{N}} t_i(c, b) > r 
\end{cases}
\]

That is, the allocation is made in a two-step process: first, truncated baselines are assigned and, after that, the surplus or the deficit with respect to the available amount of resource is shared equally.

We would like to point out that this allocation rule can be reinterpreted as the GEA rule for rationing problems with ex-ante conditions by taking \( \delta^* = -t(c, b) \), i.e. \( \tilde{CEA}(r, c, b) = GEA(r, c, \delta^*) \). Notice that the truncated baselines are embedded in our model as debts to agents and thus they are represented by a negative value.

On the other way around, differences between ex-ante conditions, which induce compensations, could be viewed as a preliminary allocation. In this sense, we try to fully compensate (if possible) each agent \( i \) by the amount \( (\max_{j \in \mathcal{N}} \delta_j) - \delta_i \), that is, the difference with respect to the agent with the best ex-ante condition. Specifically, if we take \( b^*_i = (\max_{j \in \mathcal{N}} \delta_j) - \delta_i \) for all \( i \in \mathcal{N} \) and \( r \geq \sum_{i \in \mathcal{N}} t_i(c, b^*) \), then it holds that \( GEA(r, c, \delta) = \tilde{CEA}(r, c, b^*) \). However, if \( r < \sum_{i \in \mathcal{N}} t_i(c, b^*) \), this interpretation does not fit into the baselines model and \( GEA(r, c, \delta) \neq \tilde{CEA}(r, c, b^*) \).

Let us also remark that, in contrast to the baselines model, ex-ante conditions are not upper bounds (in case of deficit) or lower bounds (in case of surplus). This is reflected in the formula of the GEA rule where a single expression is valid for all cases. Finally, if we consider the equal losses principle both models are not clearly connected.

Two final remarks might inspire future research. First, it would be interesting to also adapt some characterizations of the CEA and the CEL rules provided in the literature (see page 9) to our framework. Second, there are two important rationing rules that have not yet been analysed in our new framework: the Talmudic rule and the proportional rule. Both solutions are self-dual (dual of itself) rules. Self-duality establishes a symmetry principle in the behaviour of the rule when distributing awards and losses.
A natural proposal to generalize the Talmudic rule would be

\[ GT(r, c, \delta) = GEA \left( \min \left\{ r, \frac{\sum_{i \in N} c_i}{2} \right\}, \frac{c}{2}, \delta \right) + GEL \left( \max \left\{ 0, r - \frac{\sum_{i \in N} c_i}{2} \right\}, \frac{c}{2}, \delta \right). \]

It can be easily verified that this rule is self-dual and that if there are not inequalities in ex-ante conditions, then it corresponds to the classical Talmudic rule for standard rationing problems.

However, it is not so clear that the extension of the proportional solution to the ex-ante conditions context would likewise maintain the self-duality property. A contribution in this sense would be valuable since in most bankruptcy codes the proportional rule is mixed with lexicographic priority consideration.

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Appendix

Proof of Proposition 3 First, we prove path-independence. If \( r = r' \), the result is straightforward. If \( r < r' \), we claim that

\[ GEA(r, c, \delta) = GEA(r, GEA(r', c, \delta), \delta). \]

Let \( i \in N \). By definition we have

\[ GEA_i(r, c, \delta) = \min \{ c_i, (\lambda - \delta_i) + \} \quad \text{with} \quad \sum_{k \in N} GEA_k(r, c, \delta) = r, \]

\[ GEA_i(r', c, \delta) = \min \{ c_i, (\lambda' - \delta_i) + \} \quad \text{with} \quad \sum_{k \in N} GEA_k(r', c, \delta) = r' \quad \text{and} \]

\[ GEA_i(r, GEA(r', c, \delta), \delta) = \min \{ \min \{ c_i, (\lambda' - \delta_i) + \}, (\lambda'' - \delta_i) + \} \]

\[ \text{with} \quad \sum_{k \in N} GEA_k(r, GEA(r', c, \delta), \delta) = r. \]

First, we show

\[ \lambda < \lambda'. \quad (6) \]

Suppose on the contrary, that \( \lambda \geq \lambda' \). Then, for all \( i \in N \),

\[ GEA_i(r, c, \delta) = \min \{ c_i, (\lambda - \delta_i) + \} \geq \min \{ c_i, (\lambda' - \delta_i) + \} = GEA_i(r', c, \delta), \]
and summing all the above inequalities, we obtain
\[
r = \sum_{i \in N} GEA_i(r, c, \delta) \geq \sum_{i \in N} GEA_i(r', c, \delta) = r',
\]
which contradicts \( r < r' \).

Let us suppose now that \( GEA(r, c, \delta) \neq GEA(r, GEA(r', c, \delta), \delta) \). Then, there exists \( i^* \in N \) such that \( GEA_{i^*}(r, c, \delta) < GEA_{i^*}(r, GEA(r', c, \delta), \delta) \). Then, we have
\[
GEA_{i^*}(r, c, \delta) = \min\{c_{i^*}, (\lambda - \delta_{i^*})_+\} < \min\{\min\{c_{i^*}, (\lambda' - \delta_{i^*})_+\}, (\lambda'' - \delta_{i^*})_+\}
\]
which leads to \( \min\{c_{i^*}, (\lambda - \delta_{i^*})_+\} = (\lambda - \delta_{i^*})_+ \). Taking this into account, and substituting in (7), we have
\[
(\lambda - \delta_{i^*})_+ < \min\{\min\{c_{i^*}, (\lambda' - \delta_{i^*})_+\}, (\lambda'' - \delta_{i^*})_+\} \leq (\lambda'' - \delta_{i^*})_+.
\]
Hence, \( \lambda - \delta_{i^*} \leq (\lambda - \delta_{i^*})_+ < (\lambda'' - \delta_{i^*})_+ = \lambda'' - \delta_{i^*} \) which implies
\[
\lambda < \lambda''.
\]
Combining (6) and (8) we obtain, for all \( j \in N \setminus \{i^*\} \),
\[
GEA_j(r, c, \delta) = \min\{c_j, (\lambda - \delta_j)_+\} \leq \min\{c_j, \min\{(\lambda' - \delta_j)_+, (\lambda'' - \delta_j)_+\}\}
\]
which proves that the \( GEA \) rule satisfies path-independence.

Next, we prove consistency. Let \( (r, c, \delta) \in R^N \) and \( T \subseteq N \), with \( T \neq \emptyset \). Let us denote \( x^* = GEA(r, c, \delta) \). By Proposition 1 it holds that, for all \( i, j \in T \) with \( i \neq j \),
if $x_i^* + \delta_i < x_j^* + \delta_j$, then either $x_j^* = 0$, or $x_i^* = c_i$. Since $x_i^*$ is feasible in the reduced problem $(r - \sum_{i \in N \setminus T} x_i^*, c_{T}, \delta_T)$ and again by Proposition 1, we conclude that $x_i^* = GEA(r - \sum_{i \in N \setminus T} x_i^*, c_{T}, \delta_T)$ which proves consistency.

Finally, we prove compensated exemption. If $r = 0$, the result is straightforward. Let $(r, c, \delta) \in \mathcal{R}^{(i,j)}$, $r > 0$, be a two-person rationing problem with ex-ante conditions and let $x^* = GEA(r, c, \delta)$. Suppose on the contrary, that w.l.o.g., $\min \{r, c_i\} \leq \frac{r-(\delta_i-\delta_j)}{2}$ but $x_i^* < \min \{r, c_i\}$. Hence, by efficiency, $x_i^* = r - x_i^* > 0$.

We consider two cases:

**Case 1:** $r < c_i$. In this case $r \leq \frac{r-(\delta_i-\delta_j)}{2}$, or, equivalently,

$$r + \delta_i \leq \delta_j \text{ and thus } \delta_j \geq \delta_i. \quad (10)$$

Moreover, since $x^* = GEA(r, c, \delta)$ and $x_i^* < c_i$, we have $x_i^* = \min \{c_i, (\lambda - \delta_i)^+\} = (\lambda - \delta_i)^+ = \lambda - \delta_i$, since, otherwise, from (10) $0 > \lambda - \delta_i \geq \lambda - \delta_j$, and then $x_j^* = 0$, which implies a contradiction.

On the other hand, since $x^* = GEA(r, c, \delta)$ and $x_j^* > 0$, we get

$$0 < x_j^* = \min \{c_j, (\lambda - \delta_j)^+\} = \min \{c_j, \lambda - \delta_j\} \leq \lambda - \delta_j.$$

However, if $\lambda - \delta_j > 0$ we would have that, by (10), $\lambda > \delta_j \geq r + \delta_i$ and thus $r < \lambda - \delta_i = x_i^*$ which is a contradiction.

**Case 2:** $r > c_i$. In this case, by hypothesis, we get

$$c_i \leq \frac{r-(\delta_i-\delta_j)}{2}. \quad (11)$$

Since we are assuming that $x_i^* < c_i < r$, we have $x_i^* = \min \{c_i, (\lambda - \delta_i)^+\} = (\lambda - \delta_i)^+$. If $\lambda - \delta_i \geq 0$, then $r = x_i^* + x_j^* = \lambda - \delta_i + x_j^* \leq \lambda - \delta_i + \lambda - \delta_j$, where the last inequality follows from $0 < x_j^* = \min \{c_j, (\lambda - \delta_j)^+\} = \min \{c_j, \lambda - \delta_j\}$. Using this inequality in (11), we get $c_i \leq \lambda - \delta_i$, which implies that $x_i^* = c_i$, in contradiction with our hypothesis. On the other hand, if $\lambda - \delta_i < 0$, then $x_i^* = 0$ and $r = x_j^* \leq \lambda - \delta_j$. Hence $r + \delta_j \leq \lambda$ and so, by substitution in (11), we get $c_i \leq \frac{\lambda - \delta_j}{2} < 0$, which is a contradiction.

We conclude that the GEA rule satisfies compensated exemption. \hfill \square

**Example 4.** Let $F$ be a generalized rationing rule defined as follows, for all $(r, c, \delta) \in \mathcal{R}^N$, $N \in \mathcal{N}$, we have

$$F(r, c, \delta) = GEA(r, c, \delta).$$

This rule satisfies consistency and path-independence but does not satisfy compensated exemption.

**Example 5.** Let $(r, c, \delta) \in \mathcal{R}^N$, $N \in \mathcal{N}$, and let us denote by $\hat{c}_i = \min \{r, c_i\}$ the truncated claim of agent $i \in N$. Up to reordering agents, there exist natural numbers
\( k_1, k_2, \ldots, k_m \) such that \( k_1 + k_2 + \ldots + k_m = n \) and

\[
\hat{c}_1 + \delta_1 = \hat{c}_2 + \delta_2 = \ldots = \hat{c}_k + \delta_k \\
< \hat{c}_{k_1+1} + \delta_{k_1+1} = \hat{c}_{k_1+2} + \delta_{k_1+2} = \ldots = \hat{c}_{k_1+k_2} + \delta_{k_1+k_2} \\
< \hat{c}_{k_1+k_2+1} + \delta_{k_1+k_2+1} = \ldots = \hat{c}_{k_1+k_2+k_3} + \delta_{k_1+k_2+k_3} \\
\vdots \\
< \hat{c}_{k_1+\ldots+k_{m-1}+1} + \delta_{k_1+\ldots+k_{m-1}+1} = \ldots = \hat{c}_{k_1+\ldots+k_m} + \delta_{k_1+\ldots+k_m}.
\]

Notice that we have divided agents in \( m \) groups according to the value \( \hat{c}_j + \delta_j \), where this value is constant within groups and strictly increasing across groups. Let us denote each group by \( N_j = \{ i \in N : 1 \leq i \leq k_1 \} \) and \( N_t = \{ i \in N : k_1 + \ldots + k_{t-1} + 1 \leq i \leq k_1 + \ldots + k_t \} \), for all \( t \in \{2, \ldots, m\} \). Then, we can define recursively an allocation rule by assigning payoffs to the members of each group as follows.

**Step 1 (group \( N_1 \)):**

If \( \sum_{i \in N_1} c_i \geq r \) then \( x_i = GEA_i(r, c_{|N_1}, \delta_{|N_1}) \), for all \( i \in N_1 \), and \( x_i = 0 \), otherwise. Stop.

If not, \( \sum_{i \in N_1} c_i < r \), we assign \( x_i = c_i \), for all \( i \in N_1 \) and we proceed to the next step.

**Step t (2 \( \leq t \leq m \), groups \( N_2 \) to \( N_m \)):**

If \( \sum_{i \in N_t} c_i \geq r - \sum_{j=1}^{i-1} c_j \) then \( x_i = GEA_i \left( r - \sum_{j=1}^{i-1} c_j, c_{|N_t}, \delta_{|N_t} \right) \), for all \( i \in N_t \) and \( x_i = 0 \), for all \( i \in N_t \) with \( k = t + 1, t + 2, \ldots, m \). Stop.

If not, \( \sum_{i \in N_t} c_i < r - \sum_{j=1}^{i-1} c_j \), we assign \( x_i = c_i \), for all \( i \in N_t \) and we proceed to the next step.

This rule satisfies consistency and compensated exemption but does not satisfy path-independence.

**Example 6.** Let \( N \in \mathbb{N} \) with \( |N| \geq 3 \). Define\(^7\) \( N_1 = \{ i, j \} \subseteq N \) such that \( i < k \) and \( j < k \) for all \( k \in N \setminus \{ i, j \} \) and \( N_2 = N \setminus N_1 \). Let \( C_{N_1} = c_i + c_j \), \( C_{N_2} = \sum_{k \in N_2} c_k \), \( \Delta_{N_1} = \delta_i + \delta_j \), and \( \Delta_{N_2} = \sum_{k \in N_2} \delta_k \). Next, let us denote by \( z = (z_1, z_2) \) the allocation obtained by applying the GEA rule to the two-subgroup problem; that is

\[
z = (z_1, z_2) = GEA(r, (C_{N_1}, C_{N_2}), (\Delta_{N_1}, \Delta_{N_2})).
\]

Then, define \( F \) as follows: if \( |N| \leq 2 \), \( F(r, c, \delta) = GEA(r, c, \delta) \); if \( |N| \geq 3 \)

\[
F_k(r, c, \delta) = \begin{cases} 
GEA_k(z_1, (c_i, c_j), (\delta_i, \delta_j)) & \text{if } k \in N_1, \\
GEA_k(z_2, (c_k)_{k \in N_2}, (\delta_k)_{k \in N_2}) & \text{if } k \in N_2.
\end{cases}
\]

\(^7\) That is, \( N_1 \) is formed by the two agents associated to the smallest natural numbers in \( N \).
Rule $F$ satisfies compensated exemption and path-independence but it is not consistent.

**Proposition 5.** Compensated exemption and compensated exclusion are dual properties.

**Proof.** Let $(r,c,\delta) \in \mathcal{R}^{(1,2)}$ be a two-person rationing problem with ex-ante conditions and let us suppose that $F$ and $F^*$ are dual rules, that is, $F^*(r,c,\delta) = c - F(\ell,c,-\delta)$.

Hence, we claim that if $F$ satisfies compensated exemption, then $F^*$ satisfies compensated exclusion. To verify this, suppose, w.l.o.g., that, for the problem $(r,c,\delta)$, we have

$$\min\{\ell,c_1\} \leq \ell - (\delta_2 - \delta_1).$$

(12)

Notice that (12) is the same condition as that used in the definition of compensated exemption when we apply rule $F$ to the problem $(\ell,c,-\delta)$. Hence, since $F$ satisfies compensated exemption and by (12), we have

$$F^*_1(r,c,\delta) = c_1 - F_1(\ell,c,-\delta) = c_1 - \min\{c_1,\ell\} = \max\{0,c_1 - \ell\}
= \max\{0,c_1 - (c_1 + c_2 - r)\} = (r - c_2)_+,$$

which proves that $F^*$ satisfies compensated exclusion.

Similarly, we claim that if $F$ satisfies compensated exclusion, then $F^*$ satisfies compensated exemption. Let us suppose, w.l.o.g., that for the problem $(r,c,\delta)$, we have

$$\min\{r,c_1\} \leq \frac{r - (\delta_1 - \delta_2)}{2}.$$  

(13)

Notice that (13) is the same condition as that used in the definition of compensated exclusion when we apply rule $F$ to the problem $(\ell,c,-\delta)$. Hence, since $F$ satisfies compensated exclusion, we have that

$$F^*_1(r,c,\delta) = c_1 - F_1(\ell,c,-\delta) = c_1 - (\ell - c_2)_+$$

$$= c_1 - \max\{0,\ell - c_2\} = \min\{c_1,c_1 + c_2 - \ell\} = \min\{c_1,r\},$$

which proves that $F^*$ satisfies compensated exemption.

$\square$

**Proposition 6.** Path-independence and composition are dual properties.

**Proof.** Let us suppose that $F$ and $F^*$ are dual rules, that is, $F^*(r,c,\delta) = c - F(\ell,c,-\delta)$.

We claim that if $F$ satisfies composition, then $F^*$ satisfies path-independence. To verify this, let $r \geq r_1 \geq 0$ and define $r_2 = r - r_1$ and $\ell_1 = \sum_{i \in N} c_i - r_1$. Hence,

$$\ell = \sum_{i \in N} c_i - r = \ell_1 - r_2,$$  

and so $\ell_1 \geq \ell$.

(14)
On the one hand, we have

\[ F^*(r_1, c, \delta) = c - F(\ell_1, c, -\delta) = c - (F(\ell, c, -\delta) + F(r_2, c - F(\ell, c, -\delta), -\delta)) \]

\[ = F^*(r, c, \delta) - F(r_2, c - F(\ell, c, -\delta), -\delta), \]  
where the first and the last equalities follow from the definition of dual rule, and the remaining equality follows from the composition property of \( F \) and (14).

By definition of dual rule, we have

\[ F^*(r_1, r^*, (r,c,\delta), \delta) = F^*(r_1, r, c, \delta) - F(r - r_1, r^*, r, c, \delta, -\delta) \]

\[ = F^*(r, c, \delta) - F(r_2, c - F(\ell, c, -\delta), -\delta). \]  

Thus, taken into account (15) and (16), we conclude that \( F^* \) satisfies path-independence.

Similarly, we claim that if \( F \) satisfies path-independence, then \( F^* \) satisfies composition. To verify this, let \( r_1 + r_2 = r \), where \( r_1, r_2 \in \mathbb{R}_+ \) and \( \ell_1 = \sum_{i \in N} c_i - r_1 \). Notice that \( \ell_1 \geq \ell \). By path-independence and by definition of dual rule, we have

\[ F(\ell, c, -\delta) = F(\ell, F(\ell_1, c, -\delta), -\delta) = F(\ell_1, c, -\delta) - F^*(r_2, F(\ell, c, -\delta), -\delta). \]  

Then, by definition of dual rule and by (17), we have

\[ F^*(r, c, \delta) = c - F(\ell, c, -\delta) = c - (F(\ell_1, c, -\delta) - F^*(r_2, F(\ell_1, c, -\delta), -\delta)) \]

\[ = F^*(r_1, c, \delta) + F^*(r_2, F(\ell_1, c, -\delta), -\delta) \]

\[ = F^*(r_1, c, \delta) + F^*(r_2, c - F^*(r_1, c, \delta), -\delta). \]

Therefore, \( F^* \) satisfies composition.

References