Frame-Dependent Utility Theory

By Mark Schneider\textsuperscript{1,2}

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A large literature on non-expected utility models has developed preference functionals which are non-linear in probabilities to explain attitudes toward risk. In this paper, we introduce a frame-dependent utility model which resolves many of the paradoxes that motivated non-expected utility models while retaining expected utility analysis for any given decision. In particular, we embed the von Neumann-Morgenstern model of risk preference in a model which also accounts for the decision maker's risk perception. A correspondence between risk perception and risk preference then provides a unified explanation for the classical anomalies.

Keywords: Allais Paradox; Framing Effects; Expected Utility

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1. **Introduction**

A large literature has demonstrated that preferences often depend on context-specific features of a decision problem. Well-established models have formalized the idea that preferences are reference-dependent (e.g., Kahneman and Tversky 1979; Koszegi and Rabin 2006, 2007), and state-dependent (e.g., Karni 1993). While preferences are also known to be frame-dependent (e.g. Tversky and Kahneman, 1981), an explicit model of frame-dependent utility has not been provided. In this paper, we provide one approach to address this gap. In particular, we embed the von Neumann-Morgenstern model of risk preference in a model which also accounts for the decision maker's risk perception and the framing of alternatives. A correspondence between risk perception and risk preference then provides a unified explanation for many of the classical decision anomalies.

The frame dependent utility (FDU) model developed here is motivated by emerging evidence from neuroscience suggesting there are separate systems in the brain which are sensitive to risk and reward. Building on this notion, we consider a model involving a risk-sensitive (R) system which is risk-averse and a payoff-sensitive (P) system which is risk-seeking. The model makes the switch between the R and P systems endogenous: the R system increases in activation if the perceived risk is greater than the perceived reward, and the P system increases in activation if the perceived reward is greater. The resulting choice arises from the interaction of risk-seeking impulses from the P system and risk-averse signals from the R system.

The paper is organized as follows: Section 2 provides some background. Section 3 develops the model and Section 4 applies the model to decision making paradoxes. Section 5 applies the model to reinterpret the data from Tversky and Kahneman (1992). Section 6 discusses related literature. Section 7 concludes.
2. **Background**

Recent neuroscience studies suggest there are separate systems in the brain which are sensitive to risk and reward. In particular, studies implicate the nucleus accumbens in encoding rewards (Knutson et al. 2001, Ambroggi et al. 2008) and the insula in encoding fear, anxiety, and risk (Paulus and Stein 2006; Stein et al. 2007, Preuschoff et al. 2008). Emerging theory and evidence suggests the following stylized facts:

(i) The insula is involved in the encoding of risk\(^3\) (Knutson and Bossaerts 2007, Preuschoff et al., 2008; Bossaerts 2010).

(ii) Activation in the insula precedes risk-averse choices\(^4\) (Kuhnen and Knutson 2005, Rolls et al., 2008, Rudorf et al., 2012).


(iv) The nucleus accumbens is involved in the encoding of reward (Knutson et al. 2001, Ambroiggi et al. 2008).


While we highlight the role of specific neural systems, our approach is general and we distinguish between a risk-sensitive and a payoff-sensitive system without necessarily equating them with specific brain regions. The distinction between an R and P system is new, but is

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\(^3\) In the studies noted in (i) and (iii), risk is measured by the variance of possible returns.

\(^4\) The evidence implicating the insula in risk-aversion is not unequivocal, however, as other studies (e.g., Paulus, et al. 2003) observed the insula to correlate with risk-seeking choices.
related to plausible intuitions about dual processes in decision making, and may be interpreted as a dual process model with conflicts between a cautious self and an impulsive self. In particular, one can conceive of a cautious self that is particularly sensitive to risk and an impulsive self that focuses on rewards. Behavior depends on whether the decision maker ‘listens’ to the recommendation of the cautious or the impulsive self. In the model developed in the following sections, the relative dominance of the two selves is determined endogenously through changes in the framing of alternatives and the decision maker’s risk perception. In the words of noted decision theorist Ronald Howard\textsuperscript{5}, the frame-dependent model developed here formalizes a decision maker’s conflict between “greed and fear”.

The structure of the frame-dependent utility (FDU) model is outlined in Figure 1. First, the information and choice alternatives are framed in some way. We decompose choices into two levels – a process level and a choice level. At the process level, the utility function of either the R or P system is selected to evaluate the alternatives. At the choice level, the decision maker reveals the preferences of the selected utility function.

![Figure 1. Overview of Frame-Dependent Utility Theory](image)

We formalize the relationships in Figure 1 by embedding the von Neumann-Morgenstern model of risk preference in a model which also accounts for the decision maker’s risk

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\textsuperscript{5} Professor Howard attended my presentation of the frame-dependent model at the awards session of the Decision Analysis Society at the 2014 INFORMS Annual Meeting in San Francisco and voiced this comment in person.
perception. Our key assumption is then a correspondence between risk preference and risk perception - that people are risk-seeking when the perceived reward exceeds the perceived risk and are risk-averse when the perceived risk exceeds the perceived reward. This is the main substantive assumption driving the results in this chapter. In this respect, while we present the model in the context of an R and P system to endow the model with an axiomatic and a neuroscientific foundation, the essence of the model remains unchanged regardless of whether the reader views the existence of these systems as a reality or a metaphor.

3. Preliminaries

Let \( \mathcal{M} = [a, b] \subset \mathbb{R}, a < b \), be an interval of possible outcomes where \( 0 \in \mathcal{M} \), and a lottery, \( f \), is a probability distribution on \( \mathcal{M} \) (i.e., \( f : \mathcal{M} \to [0,1] \), and \( \sum_{x \in \mathcal{M}} f(x) = 1 \)). Denote the set of lotteries by \( \mathcal{F} \). The support of \( f \) is given by \( \text{supp}(f) = \{x \in \mathcal{M} : f(x) > 0\} \). We consider only lotteries with finite support. In this chapter, we study choices between two lotteries \( f, g \in \mathcal{F} \) where \( f \) has support \( \{x_1, x_2, ..., x_n\} \) with corresponding probabilities \( \{f(x_1), f(x_2), ..., f(x_n)\} \) and \( g \) has support \( \{y_1, y_2, ..., y_n\} \) with corresponding probabilities \( \{g(y_1), g(y_2), ..., g(y_n)\} \).

Leland and Schneider (2015) define a presentation or frame, \( F \), of size \( k \) as a matrix:

\[
\begin{bmatrix}
\mathbf{x} & \mathbf{p} \\
\mathbf{y} & \mathbf{q}
\end{bmatrix}
\]

where \( \mathbf{x}, \mathbf{y}, \mathbf{p}, \) and \( \mathbf{q} \) are vectors such that \( \dim(\mathbf{x}) = \dim(\mathbf{y}) = \dim(\mathbf{p}) = \dim(\mathbf{q}) = k \), \( \sum_{i=1}^{k} p_i = \sum_{i=1}^{k} q_i = 1 \), \( \sum_{i:x_i=x} p_i = f(x) \) and \( \sum_{i:y_i=y} q_i = g(y) \). We denote the \( i^{th} \) entry of \( \mathbf{x}, \mathbf{y}, \mathbf{p}, \) and \( \mathbf{q} \) by \( x_i, y_i, p_i, \) and \( q_i \), respectively, where the \( i^{th} \) probability of \( \mathbf{p} \) corresponds to the \( i^{th} \) outcome of \( \mathbf{x} \) and the \( i^{th} \) probability of \( \mathbf{q} \) corresponds to the \( i^{th} \) outcome of \( \mathbf{y} \). Throughout we use
italicized font to denote outcomes in the support of a lottery (e.g., $x \in \text{supp}(f)$), and bold font for outcomes in a frame (e.g., $x_i \in \text{x}$). A more intuitive display of a frame is given in Figure 2.

**Figure 2: A generic choice frame**

<table>
<thead>
<tr>
<th>$f$</th>
<th>$g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(x_1, y_1)$</td>
<td>$(y_1)$</td>
</tr>
<tr>
<td>$(x_2, y_2)$</td>
<td>$(y_2)$</td>
</tr>
<tr>
<td>$(x_i, y_i)$</td>
<td>$(y_i)$</td>
</tr>
<tr>
<td>$(x_n, y_n)$</td>
<td>$(y_n)$</td>
</tr>
<tr>
<td>$x_1$</td>
<td>$y_1$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$y_2$</td>
</tr>
<tr>
<td>$x_i$</td>
<td>$y_i$</td>
</tr>
<tr>
<td>$x_n$</td>
<td>$y_n$</td>
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<tr>
<td>$p_1$</td>
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<td>$p_2$</td>
<td>$q_2$</td>
</tr>
<tr>
<td>$p_i$</td>
<td>$q_i$</td>
</tr>
<tr>
<td>$p_n$</td>
<td>$q_n$</td>
</tr>
</tbody>
</table>

We consider a decision maker who compares the $i^{th}$ outcome of $x$ with the $i^{th}$ outcome of $y$, and compares the $i^{th}$ probability of $p$ with the $i^{th}$ probability of $q$. Without loss of generality we illustrate the model in a setting where the decision maker chooses between a risky lottery $f$ and a safer lottery $g$. A sufficient, but not necessary condition for $f$ to be riskier than $g$ is for $g$ to second-order stochastically dominate $f$. While it may not always be clear whether one lottery is riskier than another, this distinction is unambiguous in the examples we discuss. In cases where this distinction may not be readily apparent, one may employ a standard metric of riskiness such as variance or standard deviation. Alternatively, one may use a formal index designed to rank the riskiness of a lottery, such as that developed by Aumann and Serrano (2008).

To highlight the comparisons being made in the discussions to follow, we will typically present a choice between alternatives in the form of Figure 2. For the classic examples from the literature, we highlight the modal choice of experimental subjects in bold font.

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Let $F(z)$ and $G(z)$ denote the cumulative distribution functions for two arbitrary lotteries $f$ and $g$, respectively. Then $g$ second-order stochastically dominates $f$ if $\int_{-\infty}^{z} [F(z) - G(z)]dz \geq 0$ for all $x \in M$ with strict inequality at some $x$. 

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6
3.1 Completing the Frame

One seemingly rigid aspect of our setup is that the row-vectors representing the lotteries must have the same dimension for a given frame. The natural question that arises is how to compare two lotteries with different support sizes? To address this, we let $x_i$ be a null zero outcome denoted $x_i = \emptyset$, if $p_i = 0$. One natural approach is to ‘extend’ the row vector with the smaller number of outcomes by adding null zero outcomes to ‘complete’ the frame. We adopt the convention that null zero outcomes are treated as zero outcomes in our calculations, but we retain the distinct notation since these zero values are qualitatively different from a payoff of $0$ with positive probability: A payoff of $0$ is an actual outcome that might obtain for the decision maker, whereas zeros used to complete the frame simply enable the decision maker to compare the extra outcomes in the lottery with a larger support size to ‘nothing’. Since null zero outcomes used to complete a frame represent ‘nothing’ rather than an actual payoff of $0$, we impose that they are included after all outcomes in the support of a lottery are presented in a frame. Our approach to completing the frame implies a bias toward complexity for gains since it penalizes simpler representations of lotteries (those with fewer outcomes with positive probability in the frame) if the zero values are compared against positive outcomes. For losses, this approach implies an aversion to complexity since it will penalize lotteries with more outcomes in the frame if the zero values are compared against losses. To illustrate, consider the frame in Figure 3:

\[\begin{array}{cccccccc}
(x_1, y_1) & (p_1, q_1) & (x_2, y_2) & (p_2, q_2) & (x_i, y_i) & (p_i, q_i) & (x_n, y_n) & (p_n, q_n) \\
\hline
f & 400 & 0.25 & 300 & 0.25 & 200 & 0.25 & 100 & 0.25 \\
g & 500 & 0.25 & 200 & 0.75 & \emptyset & 0 & \emptyset & 0 \\
\end{array}\]
In Figure 3, we predict that lottery $f$ will often be perceived more favorably than lottery $g$, since one’s perception is attracted to the comparisons between 200 and $\emptyset$ and between 100 and $\emptyset$. In contrast, the payoff of 500 under $g$ may viewed as only marginally better than the payoff of 400 under $f$. But if the lotteries are re-framed as in Figure 4, we predict a shift in preference toward $g$ since the null zero outcomes are no longer present and the perceived differences now favor $g$ relative to $f$.

**Figure 4: Illustration of a frame without null zero outcomes**

\[
\begin{array}{cccccccc}
(x_1, y_1) & (p_1, q_1) & (x_2, y_2) & (p_2, q_2) & (x_i, y_i) & (p_i, q_i) & (x_n, y_n) & (p_n, q_n) \\
f & 400 & 0.25 & 300 & 0.25 & 200 & 0.25 & 100 & 0.25 \\
g & 500 & 0.25 & 200 & 0.25 & 200 & 0.25 & 200 & 0.25 \\
\end{array}
\]

We will see additional examples of how ‘completing the frame’ by adding null zero outcomes to the lottery with smaller support size has empirical content in Section 4.5.

### 3.2 Types of Frames

In our analysis, we will focus on two types of choice presentations. The first type which we refer to as a *parallel frame* is a presentation which sets $p_i = q_i$ for all $i = 1, 2, ... n$. For any arbitrary lotteries $f, g \in \mathcal{F}$, the second type of presentation which we refer to as the *minimalist* or *efficient frame* is a presentation in which (i) and (ii) hold:

(i): There are no non-null values $x_i, x_j \in x$, and $y_i, y_j \in y$ such that $x_i = x_j$ or $y_i = y_j$ for $i \neq j$.

(ii): If $|\text{supp}(f)| = n$ and $|\text{supp}(g)| = m$, with $m \geq n$, the frame is of size $m$ and all $x_i \in x$ such that $x_i \notin \text{supp}(f)$ are null zero outcomes.

In other words, the minimalist frame has no redundancy in that the same outcome does not appear in the same row-vector more than once (except for null zero outcomes needed to
complete the frame), and the size of the frame is equal to the larger support size of the two lotteries. A generic minimalist frame between arbitrary $f, g \in \mathcal{F}$ is illustrated in Figure 5.

**Figure 5: A generic minimalist frame**

<table>
<thead>
<tr>
<th>$f$</th>
<th>$x_1$</th>
<th>$p_1$</th>
<th>...</th>
<th>$x_n$</th>
<th>$p_n$</th>
<th>$\emptyset$</th>
<th>0</th>
<th>...</th>
<th>$\emptyset$</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g$</td>
<td>$y_1$</td>
<td>$q_1$</td>
<td>...</td>
<td>$y_n$</td>
<td>$q_n$</td>
<td>$y_{n+1}$</td>
<td>$q_{n+1}$</td>
<td>...</td>
<td>$y_m$</td>
<td>$q_m$</td>
</tr>
</tbody>
</table>

An additional property a frame might possess is (support) monotonicity: $x_1 \geq x_2 \geq \cdots \geq x_n$ for all non-null $x_i \in x$ and $y_1 \geq y_2 \geq \cdots \geq y_m$ for all non-null $y_i \in y$. We refer to such a presentation as a *decreasing monotonic frame*, and if the inequalities are reversed, as an *increasing monotonic frame*. We next show that despite the myriad different ways in which lotteries can be framed, FDU makes unique predictions whenever lotteries are presented in monotone minimalist frames.

**Proposition 1:** For a choice between $f$ and $g$:

(i) There exists a unique frame that is minimalist and decreasing (support) monotonic.

(ii) There exists a unique frame that is minimalist and increasing (support) monotonic.

**Proof:** Note that we can always construct a minimalist and decreasing support monotonic frame for any two lotteries in two steps: First, monotonically order all outcomes in the support of each lottery in decreasing order. Second, complete the frame by adding null zero outcomes after the outcomes of the lottery with smaller support size until the dimension of each vector in the frame equals the larger support size of the two lotteries.

To see that the frame is unique, note that if a decreasing monotonic frame is also minimalist then the frame is strictly monotonically decreasing (over the support of each lottery) and payoffs with positive probability are presented such that $x_1 > x_2 > \cdots > x_n$ and $y_1 > y_2 > \cdots > y_m$. Without loss of generality, let $m \geq n$. Then the strict monotonic ordering implies that
the $i^{th}$ best outcome of $f$ is in the same column as the $i^{th}$ best outcome of $g$ for $1 \leq i \leq n$, and a null zero outcome is in the same column as the $i^{th}$ best outcome of $g$ for $n + 1 \leq i \leq m$. Part (ii) follows analogously.

In the model developed in this paper, a decision maker who compares the $i^{th}$ outcome of $x$ with the $i^{th}$ outcome of $y$, and compares the $i^{th}$ probability of $p$ with the $i^{th}$ probability of $q$, will not be sensitive to whether a monotonic frame is increasing or decreasing, in which case her behavior is the same for any frame that is minimalist and monotonic.

While the way people mentally frame lotteries is an empirical question, it seems plausible that people naturally think in a minimalist frame since it is the simplest problem representation, and that they think in a parallel frame only when one of the lotteries is degenerate. The case when one lottery is degenerate is special since that is the only case where it is not ambiguous which between-lottery comparisons are being made: Each outcome in the non-degenerate lottery is compared to the unique outcome in the degenerate lottery.

### 3.3 Assumptions and Representation

We model a decision maker with two preference orderings – one ordering reflecting his sensitivity toward risk (R) and the other reflecting his sensitivity to rewards or payoffs (P). It is as if the decision maker is torn between two alternatives, and can justify each of them by one of these orderings. To model this internal conflict, we take the existence of an R and P system as primitive. Let $\succeq_s$ denote the preference relation for system $s$, $s \in \{P,R\}$ with strict preference and indifference given by $>_s$ and $\sim_s$, respectively. Likewise, denote a ‘revealed preference’ relation for the decision maker by $\succeq$, where $\succeq$ determines the actual choice made by the decision maker given any two lotteries $f$ and $g$. 


Our setup may also be viewed as a dual-self model in which the individual is composed of two selves – an impulsive (reward seeking) self and a cautious self (risk-averse), where $\succeq_s$ is the preference relation for each self $s \in \{P,R\}$. The final preference of the decision maker is an aggregation of these two preference relations where the aggregation rule selects a “dictator” in each decision, based on the perceptions of risk and reward. We first impose the von Neumann–Morgenstern Axioms separately for each system (or each “self”). Assumptions 1-3 thus assume that each system has consistent preferences.

**Assumption 1 (Ordering).** $\succeq_s$ is a complete and transitive binary relation on $\mathcal{F}$.

**Assumption 2 (Continuity).** For any $f, g, h \in \mathcal{F}$, if $f \succeq_s g \succeq_s h$, then there exists $\alpha \in [0,1]$ such that $\alpha f + (1 - \alpha)h \sim_s g$.

**Assumption 3 (Independence).** For all $f, g, h \in \mathcal{F}$:

$$f \succeq_s g \iff \alpha f + (1 - \alpha)h \succeq_s \alpha g + (1 - \alpha)h \text{ for all } \alpha \in [0,1].$$

The observations in Section 2 suggest activation increases in the Payoff-sensitive system in response to greater potential rewards, while activation increases in the Risk-sensitive system in response to greater risk. We thus assume the decision maker decides based on the P-system whenever the perceived upside exceeds the perceived downside risk, and that she decides based on the R-system when the reverse holds. Our next assumption thus connects the two systems of preferences.

**Assumption 4 (Preference Aggregation Rule).** There exists an index of perceived upside potential of $f$ relative to $g$, $\pi(f, g)$, and an index of perceived downside risk of $f$ relative to $g$, $\rho(f, g)$, such that for all $f, g \in \mathcal{F}$:
(i) \( \pi(f, g) > \rho(f, g) \) implies \( f \succeq g \Leftrightarrow f \succeq_P g \).

(ii) \( \pi(f, g) \leq \rho(f, g) \) implies \( f \succeq g \Leftrightarrow f \succeq_R g \).

Assumption 4 states that the payoff sensitive system is dominant whenever the perceived upside potential of the riskier lottery, \( f \), exceeds the perceived downside risk, and that the risk-sensitive system is dominant if the perceived downside risk is greater. In some sense, Assumption 4 may seem tautological, but it has an important implication in representing preferences as in Proposition 2. In addition to serving as a simplifying approximation, the discrete processing implicitly assumed in Assumption 4 may also have empirical support. For instance, in their survey paper on neuroeconomics, Camerer et al. (2005) discuss a ‘winner-take-all’ feature of neural processing: “When two distinct groups of neurons convey different information about the external world, the resulting perceptual judgment often adopts the information of one neuronal group and entirely suppresses the information carried by the other” (p. 25). In a similar sense, Assumption 4 posits that one system will govern the decision at any time, rather than weighing the inputs from both systems. As Camerer et al. note, “the brain doesn’t invariably integrate (i.e., average) over the signals carried by individual neurons” (p. 25).

**Proposition 2:** For all \( f, g \in \mathcal{F} \), Assumptions 1-4 hold if and only if there exists a tuple \( \langle u_p, u_R, \pi, \rho \rangle \) such that \( f \succeq_P g \Leftrightarrow U_p(f) \geq U_p(g) \), \( f \succeq_R g \Leftrightarrow U_R(f) \geq U_R(g) \), and

\[
\begin{align*}
    f \succeq g & \iff V(f) \geq V(g) \\
    \text{where } V(f) &= \begin{cases} 
    U_p(f), & \pi(f, g) > \rho(f, g) \\
    U_R(f), & \pi(f, g) \leq \rho(f, g)
    \end{cases}
\end{align*}
\]

(1)

Furthermore, \( U_p(f) = \sum_x f(x)u_p(x) \), \( U_R(f) = \sum_x f(x)u_R(x) \), and \( u_p(x) \) and \( u_R(x) \) are each unique up to a positive linear transformation.
Proof: By von Neumann and Morgenstern (1947), Assumptions 1–3 imply \( f \succeq_s g \iff U_s(f) \succeq U_s(g) \), for \( s \in \{P, R\} \). Assumption 4, in the presence of Assumptions 1–3, implies \( f \succeq g \iff U_R(f) \succeq U_R(g) \) when \( \pi(f, g) > \rho(f, g) \) and \( f \succeq g \iff U_R(f) \succeq U_R(g) \) when \( \pi(f, g) \leq \rho(f, g) \).

The EU functional form is chosen for simplicity and to demonstrate that the paradoxes can be explained even if each system has consistent preferences. Finally, we assume the R system is risk-averse and the P system is risk-seeking:

**Assumption 5 (Risk preferences).** \( u_R(x) \) is concave and \( u_P(x) \) is convex.

For any choice between lotteries \( f \) and \( g \), the model satisfies first order stochastic dominance for any increasing utility functions \( u_R(x) \) and \( u_P(x) \) since the same utility function is used to evaluate both lotteries in a given frame.

### 3.4 Risk Perception

In this section, we explicitly model a decision maker’s risk perception. For a generic choice presentation, \( F \), as in Figure 2, define sets \( U \) and \( D \) as follows:

\[
U := \{i: p_i > q_i, x_i > 0\} \cup \{i: p_i < q_i, x_i, y_i < 0\} \cup \{i: x_i > y_i\}
\]

\[
D := \{i: p_i < q_i, y_i > 0\} \cup \{i: p_i > q_i, x_i, y_i < 0\} \cup \{i: x_i < y_i\}
\]

\( U \) is the set of all columns in the choice presentation which favor the riskier lottery \( f \) relative to \( g \). It consists of all pairs \( (x_i, y_i) \), for which \( f \) yields a higher payoff than \( g \), and all pairs \( (p_i, q_i) \), for which \( f \) has a higher probability of gain than \( g \) or a lower probability of loss. \( D \) is the set of all columns which favor \( g \) and is defined analogously to \( U \). The partition of \( F \) into \( U \) and \( D \) is
reminiscent of the elation-disappointment decomposition of a lottery in Gul (1991), although our decomposition allows for comparisons between lotteries.

To illustrate the frame partition into sets $U$ and $D$, consider the frame for a choice between a binary lottery and a sure thing in Figure 6. In the figure, the column vectors in $U$ are in bold, but the column vectors in $D$ are not. Under a frame partition:

$$U = \{(x_1, y_1), (p_2, q_2)\} = \{(100, 50), (0.75, 0.5)\}.$$  
$$D = \{(x_2, y_2), (p_1, q_1)\} = \{(25, 50), (0.25, 0.5)\}.$$

**Figure 6: Frame Partition**

<table>
<thead>
<tr>
<th></th>
<th>$x_1, y_1$</th>
<th>$p_1, q_1$</th>
<th>$x_2, y_2$</th>
<th>$p_2, q_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f$</td>
<td><strong>100</strong></td>
<td>0.25</td>
<td>25</td>
<td><strong>0.75</strong></td>
</tr>
<tr>
<td>$g$</td>
<td>50</td>
<td>0.50</td>
<td>50</td>
<td>0.50</td>
</tr>
</tbody>
</table>

Figure 6 partitions the frame into the column vectors favoring the riskier lottery $f$ and the column vectors favoring $g$. Given a frame partition, we can model the agent’s perception of upside and downside risk as in Definitions 1 and 2:

**Definition 1 (Payoff Perception):** Let $x_i, x_j \in x$, and $y_i, y_j \in y$ denote payoffs such that $i, j \in U (i, j \in D)$. The perceived payoff upside (perceived payoff downside) $\mu(x_i, y_i)$ of the $i^{th}$ outcome column vector in a frame is a function satisfying properties $1 – 4$:

1. **Ordering.** If $[x_j, y_j]$ is a strict subset of $[x_i, y_i]$, then $\mu(x_i, y_i) < \mu(x_j, y_j)$.

2. **Diminishing Absolute Sensitivity:** $\mu$ exhibits diminishing absolute sensitivity if for any $x_i, y_i > 0$ and any $\epsilon > 0$, $\mu(x_i + \epsilon, y_i + \epsilon) < \mu(x_i, y_i)$

3. **Increasing Proportional Sensitivity:** $\mu$ exhibits increasing proportional sensitivity if, for any $x_i, y_i > 0$ and any $\alpha > 1$, $\mu(\alpha x_i, \alpha y_i) > \mu(x_i, y_i)$

4. **Reflection:** If $x_i, y_i, x_j, y_j > 0$, then $\mu(x_i, y_i) < \mu(x_j, y_j) \iff \mu(-x_i, -y_i) < \mu(-x_j, -y_j)$.
Properties 1, 2, and 4 were introduced by Bordalo et al. (2012) as properties of salient payoffs. In the appendix we show how properties 1 - 3 can be derived by imposing simple restrictions on the perceptions of differences. Note that if $[x_i, y_i]$ is a strict subset of $[x_i, y_j]$, then $y_i$ and $x_i$ have both a larger absolute difference and a larger ratio than $y_j$ and $x_j$. Ordering implies that the perceived difference between $y_i$ and $x_i$ is greater than that between $y_j$ and $x_j$. For instance, ordering implies that the perceived difference between 60 and 20 is greater than that between 40 and 30. Diminishing absolute sensitivity implies that for a fixed absolute difference, the perceptual system is more sensitive to larger ratios. This implies, for instance, that the perceived difference between 100 and 1 is greater than that between 200 and 101. Increasing proportional sensitivity implies, for a fixed ratio, the perceptual system is more sensitive to larger absolute differences. This implies, for instance, that the perceived difference between 2000 and 1000 is greater than that between 200 and 100. Reflection ensures that perception is most sensitive to deviations from 0. We characterize a decision maker’s probability perception analogously.

**Definition 2 (Probability Perception):** Let $p_i, p_j \in p$ and $q_i, q_j \in q$ denote probabilities such that $i, j \in U$ $(i, j \in D)$. The perceived probability upside (perceived probability downside) $\phi(p_i, q_i)$ of the $i^{th}$ probability column vector in a frame is a function satisfying properties $\square$ 1 – 3.

1. **Ordering.** If $[p_j, q_j]$ is a strict subset of $[p_i, q_i]$, then $\phi(p_j, q_j) < \phi(p_i, q_i)$.

2. **Diminishing Absolute Sensitivity:** $\mu$ exhibits diminishing absolute sensitivity if for any $p_i, q_i > 0$ and any $\epsilon > 0$, $\phi(p_i + \epsilon, q_i + \epsilon) < \phi(p_i, q_i)$

3. **Increasing Proportional Sensitivity:** $\mu$ exhibits increasing proportional sensitivity if, for any $p_i, q_i > 0$ and any $\alpha > 1$, $\phi(\alpha p_i, \alpha q_i) > \phi(p_i, q_i)$.

---

$\square$ Since probabilities do not take on negative values, property 4 from Definition 1 does not apply in Definition 2.
One simple index for the upside potential of the riskier alternative $f$ relative to $g$ is given by:

$$\pi(f, g) := \left[ \max_{i \in U} \phi(p_i, q_i), \mu(x_i, y_i) \right].$$

(2)

Index (2) computes the largest upside over all columns in $F$ in favor of $f$. Similarly, a simple index for the downside risk of $f$ relative to $g$ is given by:

$$\rho(f, g) := k \left[ \max_{i \in D} \phi(p_i, q_i), \mu(x_i, y_i) \right].$$

(3)

where $k > 0$ reflects the decision maker’s sensitivity to downside risk. If $k > 1$, the decision maker is more sensitive to downside risk than to an equivalent level of upside potential. It is likely that $k > 1$, but we will nevertheless set $k = 1$ in our analysis to demonstrate that the anomalies are resolved even if we treat upside and downside risk symmetrically. All our results continue to hold for $k > 1$.

Index (3) computes the largest downside over all columns in $F$ in which $f$ is dominated by $g$. We can think of indices (2) and (3) as perceived upside potential and perceived downside risk since they depend on both the choice alternatives and the frame. Indices (2) and (3) capture the notion that the perceptual system focuses on the biggest perceived differences in attribute values in a given frame. In the following sections, we illustrate the model and state our propositions assuming indices (2) and (3) and Assumptions 1 – 4 hold.

Note that given Assumption 5 and specifications (2) and (3), the model (1) approaches everywhere risk-averse expected utility maximization as $k$ grows large, and approaches everywhere risk-seeking behavior as $k$ goes to 0 for $\pi(f, g) > 0$.

To model perceptions of differences in payoffs and probabilities one can employ the following function in Bordalo et al. (2012) which satisfies each of properties 1 – 4.
\[
\mu(x_i, y_i) = \frac{|x_i - y_i|}{|x_i| + |y_i| + \theta}
\]

where parameter $\theta > 0$. One may also use (5) for probability comparisons, where $\varepsilon > 0$:

\[
\phi(p_i, q_i) = \frac{|p_i - q_i|}{p_i + q_i + \varepsilon}.
\]

We refer to the model developed in this section as frame-dependent utility (FDU) theory.

As a concrete demonstration, we illustrate each of our resolutions to decision paradoxes with classic examples from the experimental literature, and for simplicity, we set $\theta = 1$. In this case, it can be easily verified that all of these examples are predicted by FDU for any $\varepsilon \geq 0$, and for the same pair of implicit utility functions. We subsequently provide more general conditions in which we show that when lotteries within each choice set have the same expected value, our results are predicted to hold at the choice level if $u_p$ is any convex function and $u_R$ is any concave function.

FDU explicitly models the framing of alternatives by representing lotteries in a matrix display. We then built a model of risk perception, given the frame, by specifying general properties of the perceptual system. FDU coincides with the predictions of a single-system expected utility maximizer whenever the same system is dominant for a particular set of decisions, or when both systems would make the same choice (such as a preference for first-order stochastically dominant lotteries). However, FDU also allows systematic preference reversals when the R-system is dominant in one decision, the P-system is dominant in another, and the preferences between these systems are in conflict.
Under FDU, decision making is a hierarchical process: At the *process level*, the valuation function of either the R or P system is selected to evaluate the alternatives. At the *choice level*, the decision maker reveals the preferences of the selected value function. The process level need not operate at the level of conscious awareness. Sufficient conditions are provided for resolving each anomaly in the sections to follow.

### 4 Applications to Decision Anomalies

We now apply FDU to explain some classic paradoxes for decisions under risk. We provide conditions which resolve each paradox at both the process level (restrictions on the perception of upside and downside risk) and the choice level (restrictions on utility functions).

#### 4.1 The Allais Paradox

Perhaps the best-known empirical violation of expected utility theory (EUT) is the Allais paradox (Allais, 1953). Consider the version of the paradox from Kahneman and Tversky (1979) in which a decision maker chooses separately between \( f \) and \( g \) and between \( f' \) and \( g' \) (in Figure 7). The independence axiom of EUT requires an agent with strict preferences to prefer either \( f \) and \( f' \) or \( g \) and \( g' \). However, most people chose \( g \) and \( f' \) thereby violating EUT.

![Figure 7. Illustration of the Allais Paradox](image)

<table>
<thead>
<tr>
<th></th>
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<th>((p_1, q_1))</th>
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<th>((p_3, q_3))</th>
</tr>
</thead>
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<td>2400</td>
<td>0.66</td>
<td>0</td>
<td>0.01</td>
</tr>
<tr>
<td>(g)</td>
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<td>0.33</td>
<td>2400</td>
<td>0.66</td>
<td>2400</td>
<td>0.01</td>
</tr>
</tbody>
</table>

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<th>((x_2, y_2))</th>
<th>((p_2, q_2))</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>0.67</td>
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<tr>
<td>(g')</td>
<td>2400</td>
<td>0.34</td>
<td>0</td>
<td>0.66</td>
</tr>
</tbody>
</table>

More formally, consider the general version of the paradox in Definition 3.
**Definition 3: (Allais Paradox):** Consider the frames in Figure 8, where $x > y > z$, $q > p$, and $q, p \in (0, 1)$. The *Allais paradox holds* if $g > f$ and $f' > g'$.

![Figure 8. The Allais Paradox](image)

The Allais preference pattern violates the classical independence axiom. Under FDU, the Allais paradox is resolved if condition (6) and Assumption 5 hold:

$$\mu(z, y) > \mu(x, y) > \phi(p, q)$$

(6)

**Proposition 8:** Let $E(f) = E(g)$ and consider all $x > z$, $q > p$, and $p, q \in (0, 1)$ such that (6) holds. Then Assumption 5 implies the Allais paradox holds.

**Proof:** Recall that throughout, Assumptions 1 – 4 and indices (2) and (3) are assumed to hold. If Assumption 5 and condition (6) also hold, then $f$ and $g$ are evaluated by a concave utility function and $f'$ and $g'$ are evaluated by a convex utility function. If $E(f) = E(g)$, $f$ is a mean-preserving spread of $g$, and $f'$ is a mean-preserving spread of $g'$. Thus, Assumption 5 implies $g > f$ and $f' > g'$.

---

8 Given $E(f) = E(g)$, it can also be shown that if (6) holds for all $x > z$ and $q > p$, with $p, q \in (0, 1)$, then the Allais paradox is resolved if and only if Assumption 5 holds. While the condition in Proposition 3 implying that (6) holds for some such $x, z, p$ and $q$ is very weak, the condition that it holds for all such values is very strong and unlikely to be satisfied in practice.

9 Proposition 3 (and the propositions to follow) consider pairs of lotteries with equal means. This approach highlights the role of Assumption 5 in providing a unified explanation to decision anomalies under risk. When it is not the case that $E(f) = E(g)$, each of the paradoxes is resolved at the choice level if $u_F(x)$ is sufficiently concave and $u_P(x)$ is sufficiently convex.
Condition (6) implies that the downside of receiving $z$ outweighs the upside of receiving $x$ in the choice between $f$ and $g$, but the upside of getting $x$ outweighs the downside of a lower probability of winning in the choice between $f'$ and $g'$. Ordering and diminishing absolute sensitivity imply that $u_R$ is dominant in the choice between $f$ and $g$ in Figure 6.7. Under our parametric specification ($k = 1, \theta = 1$) it can be easily verified that (6) holds for any $\varepsilon \geq 0$ in the Kahneman and Tversky example, as well as in Birnbaum’s (2007) example where $(x, y, z, p, q) = (2,000,000, 1,000,000, 2, 0.10, 0.11)$, and in the classic Allais (1953) example where $(x, y, z, p, q) = (5,000,000, 1,000,000, 0, 0.10, 0.11)$.

### 4.2 The Common Ratio Effect

Allais (1953) also proposed the common ratio effect. In a classic version due to Kahneman and Tversky (1979), respondents choose between $f$ and $g$ and between $f'$ and $g'$ in Figure 9. EUT implies that an agent with strict preferences will choose either $f$ and $f'$ or $g$ and $g'$. However, most subjects choose $g$ and $f'$.

**More formally, consider the general version of the common ratio effect below.**

**Definition 4 (Common Ratio Effect):** Consider Figure 10, where $x > y > z$, and $c, p \in (0,1)$. The common ratio effect holds if $g > f$ and $f' > g'$. 

**Figure 9. Illustration of the Common Ratio Effect**

<table>
<thead>
<tr>
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<th>$(x_1, y_1)$</th>
<th>$(p_1, q_1)$</th>
<th>$(x_2, y_2)$</th>
<th>$(p_2, q_2)$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>4000</td>
<td>0.8</td>
<td>0</td>
<td>0.2</td>
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<tr>
<td>$g$</td>
<td>3000</td>
<td>0.8</td>
<td>3000</td>
<td>0.2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$(x_1, y_1)$</th>
<th>$(p_1, q_1)$</th>
<th>$(x_2, y_2)$</th>
<th>$(p_2, q_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f'$</td>
<td>4000</td>
<td>0.20</td>
<td>0</td>
<td>0.80</td>
</tr>
<tr>
<td>$g'$</td>
<td>3000</td>
<td>0.25</td>
<td>0</td>
<td>0.75</td>
</tr>
</tbody>
</table>

**Figure 10. The Common Ratio Effect**

<table>
<thead>
<tr>
<th></th>
<th>$x$</th>
<th>$p$</th>
<th>$z$</th>
<th>$1 - p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$g$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$x$</th>
<th>$c_p$</th>
<th>$z$</th>
<th>$1 - c_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f'$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$g'$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Under FDU, at the process level, the common ratio effect requires condition (7):

\[ \mu(z, y) > \mu(x, y) > \phi(c \mathbf{p}, \mathbf{c}) \]  

(7)

**Proposition 4:** Let \( E(f) = E(g) \) and consider all \( x > z \), and \( c, p \in (0, 1) \) such that (7) holds. Then Assumption 5 implies that the Common Ratio Effect holds.

**Proof:** If Assumption 5 and condition (7) hold, then \( f \) and \( g \) are evaluated by a concave utility function and \( f' \) and \( g' \) are evaluated by a convex utility function. If \( E(f) = E(g) \), \( f \) is a mean-preserving spread of \( g \), and \( f' \) is a mean-preserving spread of \( g' \). Thus, Assumption 5 implies \( g > f \) and \( f' > g' \).

Intuitively, the paradox is resolved if the downside of receiving \( z \) outweighs the upside of receiving \( x \) in the choice between \( f \) and \( g \), and if the upside of receiving \( x \) outweighs the downside of having a lower probability of winning in the choice between \( f' \) and \( g' \). In the former case, the R-system is dominant and the decision maker is risk-averse. In the latter, the P-system is dominant yielding risk-seeking behavior. Ordering and diminishing absolute sensitivity imply \( \mu(0,3000) > \mu(4000,3000) \) without assuming a particular form for \( \mu \), and thus \( u_R \) is dominant in the choice between \( f \) and \( g \) in Figure 9 under very general conditions. Under our parametric specification employing functions (4) and (5) with \( k = 1 \), and \( \theta = 1 \), the full statement of condition (7) holds in the classic example in Figure 9 for any \( \varepsilon \geq 0 \).

### 4.3 Positive and Negative Frames

In addition to explicit assumptions such as the independence axiom, expected utility theory (EUT) makes more subtle implicit assumptions such as description invariance, which states that logically equivalent presentations of a decision problem should elicit the same
preferences (Camerer 1995). Important violations of description invariance include the gain-loss framing effect of Tversky and Kahneman (1981) and the alignment effect observed by Leland (1998), Birnbaum (2007), and Bordalo et al. (2012).

In the context of FDU, framing effects arise because changes in the frame of a decision problem systematically change the relative dominance of the R-system and P-system, or alternatively, change the decision maker’s mindset from cautious to opportunistic. Consider the study of Tversky and Kahneman (1981). In their classic example, respondents are told that the U.S. is preparing for the outbreak of an epidemic which is expected to kill 600 people. Policymakers need to choose between two disease prevention strategies: If Program A is adopted, 200 people will be saved. If Program B is adopted, there is a 1/3 probability that 600 people will be saved and 2/3 probability that no people will be saved. Respondents are asked which program they would select. The parallel frame is shown at the top of Figure 8. A different group of respondents was given the following description of the two treatment programs instead of the one above: If Program C is adopted 400 people will die. If Program D is adopted there is a 1/3 probability that nobody will die and a 2/3 probability that all 600 people will die. This frame is shown in the bottom of Figure 11.

Although Programs A and B are considered to be logically equivalent to Programs C and D, respectively, Tversky and Kahneman found that 72% of respondents chose program A, but only 22% of respondents chose C.

Figure 11: Positive and Negative Frames

<table>
<thead>
<tr>
<th></th>
<th>(x₁, y₁)</th>
<th>(p₁, q₁)</th>
<th>(x₂, y₂)</th>
<th>(p₂, q₂)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Program A</td>
<td>200 lives saved</td>
<td>1/3</td>
<td>200 lives saved</td>
<td>2/3</td>
</tr>
<tr>
<td>Program B</td>
<td>600 lives saved</td>
<td>1/3</td>
<td>0 lives saved</td>
<td>2/3</td>
</tr>
</tbody>
</table>
Under FDU, the framing effect (a preference for A and D) is predicted if (6.8) holds:

\[
\mu(200, 0) > \mu(200, 600) \text{ and } \mu(-400, 0) > \mu(-400, -600) \quad (8)
\]

Indeed, (8) holds under our parametric specification \((k = 1 \text{ and } \theta = 1)\), for any \(\varepsilon \geq 0\), in which case Assumption 5 implies the framing effect holds. In addition, ordering and diminishing absolute sensitivity imply that the opposite preference pattern cannot be observed.

### 4.4 Parallel and Minimalist Frames

FDU can also explain other violations of description invariance. Suppose, for instance, that the second pair of alternatives in the Allais paradox is presented in a parallel frame (in Figure 12). Under FDU, the downside risk comparison between 2,400 and 0 which is not cued in the minimalist frame (top of Figure 12) now yields \(\pi(f', g') < \rho(f', g')\) in the parallel frame (bottom of Figure 12), leading the R-system to be dominant. Since the R-system is also dominant in the choice between \(f\) and \(g\) in the Allais paradox, the same utility function evaluates both sets of alternatives and the classical independence axiom is predicted to hold. Indeed, in experimental tests using a presentation resembling the parallel frame in Figure 12, Birnbaum (2007) and Bordalo et al. (2012) found the Allais paradox to largely disappear. The salience model of Bordalo et al. (2012) predicts that the Allais paradox can be turned off only when lotteries are correlated. However, Birnbaum’s demonstration of the alignment effect did not involve correlated lotteries, suggesting that the preference reversal is driven instead by changes in the framing of alternatives. For further evidence on this point, see Leland (1998).
Next we formalize the ‘alignment effect’ as follows:

**Definition 5: (Alignment Effect):** Consider the two choice frames in Figure 13, for which $x > y > 0, q > p$, and $p, q ∈ (0,1)$. The Alignment Effect holds if $g > f$ in the parallel frame and $f > g$ in the minimalist frame.

![Figure 13. The Alignment Effect](image)

Although the structure of the alignment effect bears some similarity to the Allais paradox, the alignment effect differs in that it is a pure framing effect. In both choices, the decision maker chooses between $f$ and $g$, the only difference being a change from parallel to minimalist frames. Under FDU, the alignment effect requires condition (9):

$$\mu(0,y) > \mu(x,y) > \phi(p,q)$$  \hspace{1cm} (9)

**Proposition 5:** Let $E(f) = E(g)$ and consider all $x > 0, q > p$, and $p, q ∈ (0,1)$ such that (9) holds. Then Assumption 5 implies that the alignment effect holds.

---

10 We define the alignment effect with a common consequence of 0 for simplicity and to highlight the perceptual-driven nature of the effect. It can be straightforwardly extended to any common consequence.
Proof: If Assumption 5 and condition (9) hold, then \( f \) and \( g \) are evaluated by a concave utility function in the parallel frame, but are evaluated by a convex utility function in the minimalist frame. If \( E(f) = E(g) \), \( f \) is a mean-preserving spread of \( g \). Thus, Assumption 5 implies \( g > f \) in the parallel frame and \( f > g \) in the minimalist frame.

Intuitively, the paradox is resolved if the downside of receiving 0 outweighs the upside of receiving \( x \) in the parallel frame, and if the upside of receiving \( x \) outweighs the downside of a lower probability of winning in the minimalist frame. Ordering and diminishing absolute sensitivity imply that \( u_R \) is dominant in the parallel frame in Figure 12. In addition, condition (9) holds under our running parametric specification \((k = 1, \theta = 1)\) with any \( \epsilon \geq 0 \) for the example in Figure 12 from Bordalo et al. (2012) and also in the example used by Birnbaum (2007) where \( x = 2,000,000 \), \( y = 1,000,000 \), \( p = 0.10 \), and \( q = 0.11 \).

4.5 Bias toward Complexity

Analogous to our distinction between minimalist and parallel frames, individual lotteries can be presented in compressed (equivalently, ‘coalesced’) form or split form (Birnbaum, 2007). A lottery is presented in compressed form if the same outcome does not appear in the same row-vector more than once (excluding the null zero values necessary to ‘complete the frame’, as in Section 3.1). Two lotteries in compressed form generate a minimalist frame. A lottery may also be presented in split form by decomposing a single outcome-probability pair into multiple outcome-probability pairs.

As noted in Section 3.1, one novel implication of FDU is that it implies a bias toward complexity for gains and an aversion to complexity for losses. In particular, lotteries with more probability-outcome pairs are favored (for gains) since they are compared with a null zero outcome. To illustrate, consider the pair of choices in Figure 14, tested experimentally by
Birnbaum (2007). Although the two choices are logically identical, 37% of subjects chose \( f \) when \( f \) was presented in compressed form and \( g \) was in split form, but 70% chose \( f \) when \( f \) was in split form and \( g \) was in compressed form. A decision maker who chooses in this way acts as if exhibiting a preference for complexity – the agent’s preferences shift toward risk-aversion if the safer lottery \( g \) is more complex and they shift toward risk-seeking behavior if the riskier lottery \( f \) is more complex.

### Figure 14. Illustration of Bias toward Complexity

<table>
<thead>
<tr>
<th></th>
<th>( f ) (compressed)</th>
<th>( g ) (split)</th>
<th>( f ) (split)</th>
<th>( g ) (compressed)</th>
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<tbody>
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<td></td>
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<td></td>
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<td>0.85</td>
<td>100</td>
<td>0.10</td>
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<tr>
<td></td>
<td>100</td>
<td>0.85</td>
<td>50</td>
<td>0.15</td>
</tr>
</tbody>
</table>

To illustrate how a bias toward complexity is explained by FDU, consider first the case where \( f \) is compressed and \( g \) is split. Then the perceived downside is greater than the upside due to focusing on the comparison, \( \mu(\emptyset, 50) \), where we recall that null zero outcomes are computed as zeros. Thus, the R system is dominant, shifting preferences toward \( g \). When \( g \) is compressed and \( f \) is split, the perceived upside is now greater due to focusing on the comparison \( \mu(7, \emptyset) \). Hence, the P system is dominant, shifting preferences toward \( f \). At the process level, these conditions hold under our running parametric specification \( (k = 1, \theta = 1) \) for any \( \varepsilon \geq 0 \). Since a bias toward complexity occurs as a framing effect (i.e., since it occurs even when changing the perceived complexity of the same two lotteries), it cannot be explained by EUT or Cumulative Prospect Theory (Tversky and Kahneman, 1992). For losses, the zero outcomes are compared with negative payoffs and in that case FDU predicts an aversion to complexity.
**Definition 6: (Bias toward Complexity)** In Figure 15, let \( x > y > z > 0, q > p, \) and \( p, q \in (0,1) \): A bias toward complexity holds if \( g > f \) when \( g \) is split and \( f \) is compressed and \( f > g \) when \( f \) is split and \( g \) is compressed.

Under FDU, suppose the following conditions hold at the process level:

\[
egin{align*}
\pi(f, g) &< \mu(0, y) = \rho(f, g) \quad \text{when } f \text{ is compressed and } g \text{ is split} \\
\pi(f, g) &= \mu(z, 0) > \rho(f, g) \quad \text{when } g \text{ is compressed and } f \text{ is split}
\end{align*}
\]  

(10) \hspace{1cm} (11)

![Figure 15. Bias toward Complexity](image)

<table>
<thead>
<tr>
<th>( f ) (compressed)</th>
<th>( g ) (split)</th>
<th>( f ) (split)</th>
<th>( g ) (compressed)</th>
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<td>( \emptyset )</td>
<td>( 0 )</td>
</tr>
</tbody>
</table>

**Proposition 6:** Let \( E(f) = E(g) \) and suppose (10) and (11) hold. Then Assumption 5 implies a bias toward complexity (for gains).

**Proof:** If Assumption 5 and conditions (10) and (11) hold, then \( f \) and \( g \) are evaluated by a concave utility function when \( f \) is compressed and \( g \) is split but are evaluated by a convex utility function when \( g \) is compressed and \( f \) is split. If \( E(f) = E(g) \), \( f \) is a mean-preserving spread of \( g \). Thus, Assumption 5 implies \( g > f \) when \( g \) is split and \( f \) is compressed and \( f > g \) when \( f \) is split and \( g \) is compressed. □

The property of diminishing absolute sensitivity provides one reason why (10) and (11) are likely to hold in many cases: Under diminishing sensitivity, the decision maker is more sensitive to payoffs compared with zero than with any other value.
4.6 Violations of Ordinal Independence

Wu (1994) observed systematic violations of ordinal independence, a weakening of the classic independence axiom which is retained by both Rank-Dependent Utility (RDU) Theory (Quiggin, 1982) and Cumulative Prospect Theory (Tversky and Kahneman, 1992). Using the choices displayed in Figure 16, Wu found that many subjects chose A over B and D over C. However, RDU and Cumulative Prospect Theory (CPT) predict choices of either A and C or B and D (Wu, 1994).

![Figure 16. Violations of Ordinal Independence](image)

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>100</td>
<td>0.25</td>
<td>55</td>
<td>0.01</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>100</td>
<td>0.25</td>
<td>25</td>
<td>0.02</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>55</td>
<td>0.26</td>
<td>0</td>
<td>0.74</td>
<td>0</td>
</tr>
<tr>
<td>D</td>
<td>55</td>
<td>0.25</td>
<td>25</td>
<td>0.02</td>
<td>0</td>
</tr>
</tbody>
</table>

While the AD response pattern violates RDU and CPT in a manner analogous to how Allais-style choices violate EUT, this behavior is explained by FDU: In the choice between A and B, the upside $\mu(55,25)$ outweighs the one percent difference in probabilities. Thus, the P system is dominant and A is chosen over B for sufficiently convex $u_p(x)$. In the choice between C and D, the downside $\mu(0,25)$ outweighs the one percent difference in probabilities. Hence, the R system is dominant and the decision maker chooses D over C for sufficiently concave $u_r(x)$. These predictions hold at the process level under our running parametric specification ($k = 1, \theta = 1$) for any $\varepsilon \geq 0$. 
4.7 The Peanuts Effect

Markowitz (1952) first identified what Prelec and Loewenstein (1991) later dubbed the ‘peanuts’ effect – people are often risk-seeking for small stakes and risk averse for large stakes. Markowitz provided an illustration of this effect presented in Figure 17. In the figure, a decision maker prefers a 0.10 probability of winning $1 instead of a guaranteed ten cents, but prefers a guaranteed $100 instead of a 0.10 probability of winning $1000. Despite the intuitive appeal of such behavior, it is not explained by EUT with a concave utility function, nor is it explained by the most common specification of Cumulative Prospect Theory, in which the decision maker has a power value function and any probability weighting function. In contrast, at the choice level, the peanuts effect is resolved if \( u_p(x) \) is any convex function and \( u_R(x) \) is any concave function.

Figure 17. Illustration of the Peanuts Effect

| \( f \) | 1 | 0.10 | 0 | 0.90 | \( f' \) | 1000 | 0.10 | 0 | 0.90 |
| \( g \) | 0.10 | 0.10 | 0.10 | 0.90 | \( g' \) | 100 | 0.10 | 100 | 0.90 |

Definition 7 (Peanuts Effect): Consider Figure 18, where \( x > z \) and \( p \in (0,1) \). The peanuts effect holds if \( f > g \) and there is some \( c > 1 \) such that \( g' > f' \).

Figure 18. The Peanuts Effect

| \( f \) | \( x \) | \( p \) | \( z \) | \( 1 - p \) | \( f' \) | \( cx \) | \( p \) | \( cz \) | \( 1 - p \) |
| \( g \) | \( E(f) \) | \( p \) | \( E(f) \) | \( 1 - p \) | \( g' \) | \( E(f') \) | \( p \) | \( E(f') \) | \( 1 - p \) |

At the process level, the Peanuts effect holds under condition (12):

\[
\mu(x, E(f)) > \mu(z, E(f)) \quad \text{and} \quad \mu(cx, E(f')) > \mu(cz, E(f'))
\] (12)

The first inequality in condition (12), implies the perceived upside of \( f \) is greater than the perceived downside. The second inequality implies there is some \( c > 1 \), such that scaling up all payoffs makes the downside seem greater than the upside.
**Proposition 7:** For all \( x > z \) and \( p \in (0,1) \) such that (12) holds, Assumption 5 implies that the Peanuts Effect holds.

**Proof:** If Assumption 5 and condition (12) hold, then \( f \) and \( g \) are evaluated by a concave utility function and \( f' \) and \( g' \) are evaluated by a convex utility function. Since \( E(f) = E(g) \), \( f \) is a mean-preserving spread of \( g \), and \( f' \) is a mean-preserving spread of \( g' \). Thus, Assumption 5 implies \( g > f \) and \( f' > g' \).

At the process level, the example in Figure 17 is resolved under our running parametric specification (\( k = 1, \theta = 1, \varepsilon \geq 0 \)).

### 4.8 The Reflection of Risk Attitudes

A robust characteristic of risk preferences is the reflection effect over gains and losses (Kahneman and Tversky 1979). A typical example, due to Tversky and Kahneman (1992), is displayed in Figure 19 involving four choices between a degenerate lottery and a mean-preserving spread. In the figure, a decision maker with a concave utility function will always prefer \( g_i \) to \( f_i \) for all \( i = 1,2,3,4 \). In contrast, most people chose lotteries \( g_1, f_2, g_3 \), and \( f_4 \).

**Figure 19. Illustration of the Fourfold Pattern of Risk Preferences**

<table>
<thead>
<tr>
<th>( f_1 )</th>
<th>( p_1, q_1 )</th>
<th>( x_1, y_1 )</th>
<th>( p_2, q_2 )</th>
<th>( x_2, y_2 )</th>
<th>( x_1, y_1 )</th>
<th>( p_2, q_2 )</th>
<th>( x_2, y_2 )</th>
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<tbody>
<tr>
<td>100</td>
<td>0.95</td>
<td>0</td>
<td>0.05</td>
<td>95</td>
<td>0.95</td>
<td>95</td>
<td>0.05</td>
</tr>
<tr>
<td>( g_1 )</td>
<td>( p_1, q_1 )</td>
<td>( x_1, y_1 )</td>
<td>( p_2, q_2 )</td>
<td>( x_2, y_2 )</td>
<td>( x_1, y_1 )</td>
<td>( p_2, q_2 )</td>
<td>( x_2, y_2 )</td>
</tr>
<tr>
<td>5</td>
<td>0.05</td>
<td>5</td>
<td>0.95</td>
<td>100</td>
<td>0.05</td>
<td>0</td>
<td>0.95</td>
</tr>
<tr>
<td>( f_2 )</td>
<td>( p_1, q_1 )</td>
<td>( x_1, y_1 )</td>
<td>( p_2, q_2 )</td>
<td>( x_2, y_2 )</td>
<td>( x_1, y_1 )</td>
<td>( p_2, q_2 )</td>
<td>( x_2, y_2 )</td>
</tr>
<tr>
<td>-100</td>
<td>0.05</td>
<td>0</td>
<td>0.95</td>
<td>-5</td>
<td>0.05</td>
<td>-5</td>
<td>0.95</td>
</tr>
<tr>
<td>( g_3 )</td>
<td>( p_1, q_1 )</td>
<td>( x_1, y_1 )</td>
<td>( p_2, q_2 )</td>
<td>( x_2, y_2 )</td>
<td>( x_1, y_1 )</td>
<td>( p_2, q_2 )</td>
<td>( x_2, y_2 )</td>
</tr>
<tr>
<td>-95</td>
<td>0.95</td>
<td>-95</td>
<td>0.05</td>
<td>-95</td>
<td>0.95</td>
<td>-95</td>
<td>0.05</td>
</tr>
</tbody>
</table>

As noted by Tversky and Kahneman (1992), a decision maker who exhibits the fourfold pattern is risk-averse for gains of high probability and losses of low probability and risk-seeking for...
losses of high probability and gains of low probability. Note that the choice between \( f_2 \) and \( g_2 \) resembles the choice of purchasing a lottery ticket and the choice between \( f_3 \) and \( g_3 \) resembles the choice of purchasing insurance. The behavior in Figure 16 holds under our running parametric specification \((k = 1, \theta = 1, \epsilon \geq 0)\). More generally, we have the following definition and result:

**Definition 8 (Fourfold pattern):** In Figure 20, where \( x > z \geq 0 \), and \( p \in (0,1) \), the *fourfold pattern* holds if \( g > f \) for sufficiently large \( p \), \( f > g \) for sufficiently small \( p \), \( f' > g' \) for sufficiently large \( p \) and \( g' > f' \) for sufficiently small \( p \).

**Figure 20. The Fourfold Pattern of Risk Preferences**

| \( f \) | \( x \) | \( p \) | \( z \) | \( 1-p \) | \( f' \) | \( -x \) | \( p \) | \( -z \) | \( 1-p \) |
| \( g \) | \( \mathbb{E}(f) \) | \( p \) | \( \mathbb{E}(f) \) | \( 1-p \) | \( g' \) | \( \mathbb{E}(f') \) | \( p \) | \( \mathbb{E}(f') \) | \( 1-p \) |

**Proposition 8 (Fourfold pattern of risk preferences):** Let \( \mu(.) \) be any function satisfying the properties in Definition 1, and suppose conditions (13) and (14) hold. Then Assumption 5 implies that the fourfold pattern holds.

\[
\mu(\mathbb{E}(f), x) < \mu(\mathbb{E}(f), z) \text{ for sufficiently large } p \in (0,1)
\]  
(13)

\[
\mu(\mathbb{E}(f), z) < \mu(\mathbb{E}(f), x) \text{ for sufficiently small } p \in (0,1).
\]  
(14)

**Proof:** If Assumption 5 and conditions (13) and (14) hold, then \( f \) and \( g \) are evaluated by a concave utility function for sufficiently large \( p \), but are evaluated by a convex utility function for sufficiently small \( p \). Note that given (13) and (14), the reflection property in Definition 1 implies \( \mu(\mathbb{E}(f'), -x) < \mu(\mathbb{E}(f'), -z) \) for sufficiently large \( p \) and \( \mu(\mathbb{E}(f'), -z) < \mu(\mathbb{E}(f'), -x) \) for sufficiently small \( p \). By Assumption 5, \( f' \) and \( g' \) are evaluated by a convex utility function for sufficiently large \( p \), but are evaluated by a concave utility function for sufficiently small \( p \). Since
$E(f) = E(g)$, $f$ is a mean-preserving spread of $g$, and $f'$ is a mean-preserving spread of $g'$. Thus, $g > f$ for sufficiently large $p$, $f > g$ for sufficiently small $p$, $f' > g'$ for sufficiently large $p$ and $g' > f'$ for sufficiently small $p$, and the fourfold pattern holds.

Conditions (13) and (14) are very natural and state that the perceived difference between $z$ and $E(f)$ is greater than the difference between $x$ and $E(f)$ when $E(f)$ is arbitrarily close to $x$, and that the reverse holds when $E(f)$ is arbitrarily close to $z$. Conditions (13) and (14) are also satisfied by (4), and the opposite preference pattern cannot be observed:

**Proposition 9:** Under (4), FDU implies the fourfold pattern of risk attitudes.

**Proof:** In the choice between $f$ and $E(f)$, the $R$-system is dominant if and only if

$$\frac{p}{(1-p)} \geq \frac{xp + \theta}{x + xp + \theta}$$

(15)

Recall that $\theta > 0$. For any positive $x$ and $\theta$, it is clear that the above inequality always holds for any $p \in [0, 1]$, and that it can also hold for smaller values of $p$. For any such $p$ the $R$-system is dominant and $g > f$ for any concave function $u_R(x)$. Holding $x$ and $\theta$ fixed, there is also some sufficiently small $p$ such that the right-hand side of the inequality is strictly greater than the left-hand side, which implies that the $P$-system is dominant. For any such $p$, $f > g$ for any convex function $u_P(x)$. Proceeding analogously, we find that $f' > g'$ for sufficiently high $p$ and $g' > f'$ for sufficiently small $p$. ■
4.9 Summary of Results

The paradoxes explained by FDU are summarized in Table 1, which indicates the system predicted to be dominant and whether each lottery is evaluated by a concave or convex utility function. Sufficient conditions for resolving each paradox at the process and choice levels are in parentheses.

<table>
<thead>
<tr>
<th>Paradox</th>
<th>Process Level</th>
<th>Choice Level</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Allais Paradox</strong></td>
<td>(Condition 6)</td>
<td>(Assumption 5)</td>
<td></td>
</tr>
<tr>
<td>Choice Set: ((f, g))</td>
<td>R-System</td>
<td>Risk-averse</td>
<td>Allais</td>
</tr>
<tr>
<td>Choice Set: ((f', g'))</td>
<td>P-System</td>
<td>Risk-seeking</td>
<td>(1953)</td>
</tr>
<tr>
<td><strong>Common Ratio Effect</strong></td>
<td>(Condition 7)</td>
<td>(Assumption 5)</td>
<td></td>
</tr>
<tr>
<td>Choice Set: ((f, g))</td>
<td>R-System</td>
<td>Risk-averse</td>
<td>Kahneman</td>
</tr>
<tr>
<td>Choice Set: ((f', g'))</td>
<td>P-System</td>
<td>Risk-seeking</td>
<td>&amp; Tversky</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(1979)</td>
</tr>
<tr>
<td><strong>Gain - Loss Framing Effect</strong></td>
<td>(Condition 8)</td>
<td>(Assumption 5)</td>
<td>Tversky &amp;</td>
</tr>
<tr>
<td>Choice Set: Programs A, B</td>
<td>R-System</td>
<td>Risk-averse</td>
<td>Kahneman</td>
</tr>
<tr>
<td><strong>Alignment Framing Effect</strong></td>
<td>(Condition 9)</td>
<td>(Assumption 5)</td>
<td>Birnbaum</td>
</tr>
<tr>
<td>Choice Set: ((f, g)) – Parallel</td>
<td>R-System</td>
<td>Risk-averse</td>
<td></td>
</tr>
<tr>
<td>Choice Set: ((f, g)) – Minimalist</td>
<td>P-System</td>
<td>Risk-seeking</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(2007)</td>
</tr>
<tr>
<td><strong>Complexity Framing Effect</strong></td>
<td>(Conditions 10,11)</td>
<td>(Assumption 5)</td>
<td>Birnbaum</td>
</tr>
<tr>
<td>Choice Set: ((f, g)\ g (split)</td>
<td>R-System</td>
<td>Risk-averse</td>
<td></td>
</tr>
<tr>
<td>Choice Set: ((f, g)\ f (split)</td>
<td>P-System</td>
<td>Risk-seeking</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(2007)</td>
</tr>
<tr>
<td><strong>Peanuts Effect</strong></td>
<td>(Condition 12)</td>
<td>(Assumption 5)</td>
<td>Markowitz</td>
</tr>
<tr>
<td>Choice Set: ((f', g'))</td>
<td>R-System</td>
<td>Risk-averse</td>
<td></td>
</tr>
<tr>
<td>Choice Set: ((f, g))</td>
<td>P-System</td>
<td>Risk-seeking</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(1952)</td>
</tr>
<tr>
<td><strong>Fourfold Pattern</strong></td>
<td>(Conditions 13,14)</td>
<td>(Assumption 5)</td>
<td>Tversky &amp;</td>
</tr>
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<td>Choice Set: ((f_1, g_1))</td>
<td>R-System</td>
<td>Risk-averse</td>
<td>Kahneman</td>
</tr>
<tr>
<td>Choice Set: ((f_2, g_2))</td>
<td>P-System</td>
<td>Risk-seeking</td>
<td>(1992)</td>
</tr>
<tr>
<td>Choice Set: ((f_3, g_3))</td>
<td>R-System</td>
<td>Risk-averse</td>
<td></td>
</tr>
<tr>
<td>Choice Set: ((f_4, g_4))</td>
<td>P-System</td>
<td>Risk-seeking</td>
<td></td>
</tr>
</tbody>
</table>
5. Application to Experimental Data

Under Cumulative Prospect Theory (CPT) due to Tversky and Kahneman (1992), the fourfold pattern of risk attitudes arises due to the interaction between non-linear probability weighting and the prospect theory value function. In this section, we re-examine the data from Tversky and Kahneman (1992), and demonstrate that the fourfold pattern can be plausibly explained by a twofold pattern: risk-seeking behavior when the perceived reward outweighs the perceived risk and risk aversion when the perceived risk outweighs the perceived reward.

In their data\textsuperscript{11}, Tversky and Kahneman (1992) found the median responses to be risk-averse for twenty-six lotteries, risk-seeking for twenty-eight lotteries, and risk-neutral for the remaining two lotteries. Here we apply FDU to predict whether the modal response of the experimental subjects in Tversky and Kahneman (1992) is either risk-averse or risk-seeking for each of the 56 lotteries. This information can be represented as a choice between each lottery and its expected value. As noted earlier, such a choice is naturally framed in a parallel presentation since it involves a degenerate lottery. Using the function (4) due to Bordalo et al. (2012) as a simple and plausible parametric form which satisfies the properties in Definition 1, we can compute an index reflecting the perceived net upside or net downside of taking a risk. We do so by computing the index of upside potential minus the index of downside risk, given by (4). FDU predicts that when this net difference is positive, the upside exceeds the downside and the P-system will be dominant, yielding risk-seeking behavior. Conversely, when this net difference is

\textsuperscript{11} The data from Tversky and Kahneman (1992) is based on responses of 25 experimental subjects who each made choices between 56 binary lotteries and certain payoffs. Twenty-eight lotteries involved only non-negative outcomes; the remaining twenty-eight lotteries were obtained by reversing the sign of payoffs of the first twenty-eight lotteries. Tversky and Kahneman presented both the binary lottery and its expected value to each subject and asked subjects to choose between one of seven sure outcomes that they would prefer over the lottery. The subjects were subsequently asked to choose between a narrower range of certain payoffs and the same lottery to identify an indifference point.
negative, the R-system will be dominant, producing risk-aversion. The results of these calculations are depicted in Table 2. The table displays each of the 56 lotteries used by Tversky and Kahneman (1992) which have the form \((x, p; z, 1 - p)\), where \(x > z\). The payoff (degenerate lottery) with the same expected value is denoted \(y\). Note that for these choices, FDU predicts the P-system is dominant if \(\mu(x, y) - \mu(z, y) > 0\), resulting in risk-seeking behavior, and that the R-system is dominant if the inequality is reversed, producing risk aversion. The ‘choice’ columns in Table 2 note whether the median certainty equivalent indicated risk seeking behavior (labeled the ‘risky’ choice in Table 2), or whether it indicated risk aversion (labeled the ‘safe’ choice). The values for \(\mu(x, y) - \mu(z, y)\) are provided in the table, using the specification (4) with \(\theta = 1\). This value for \(\theta\) is somewhat arbitrary and chosen for rough plausibility. However, our computational experiments suggest the qualitative results hold for all \(\theta > 0\), and that the ability for FDU to predict at least 50 of the 56 modal choices is robust to values for \(\theta > 0.6\). These results at the process level are strengthened by the fact that the results hold at the choice level if \(u_R(x)\) is any concave function and \(u_P(x)\) is any convex function.

Remarkably, positive values of \(\mu(x, y) - \mu(z, y)\) correspond primarily to ‘risky’ choices and negative values correspond primarily to ‘safe’ choices as predicted by FDU, and as can be seen in Table 2. More precisely, FDU predicts 26 of the 28 risk-seeking responses (92.8%) and 24 of the 26 risk-averse responses (92.3%) for \(\theta = 1\). For \(\theta = 2\), FDU predicts 27 of the 28 risk-seeking responses (96.4%) and 25 of the 26 risk-averse responses (96.1%). That is, FDU can simultaneously explain 52 of the 56 modal choices. In contrast, EUT with a concave utility function cannot explain more than 28 of these choices since half of them were risk-seeking. When the parameters of CPT are fit to the same data, CPT explains 91% (51) of the 56 modal responses (Brandstatter et al., 2006).
Table 2: Predictions of FDU (Data from Tversky and Kahneman, 1992)

<table>
<thead>
<tr>
<th>Lottery #</th>
<th>z</th>
<th>x</th>
<th>p</th>
<th>y</th>
<th>Choice</th>
<th>μ(x,y) − μ(z,y)</th>
<th>Lottery #</th>
<th>z</th>
<th>x</th>
<th>p</th>
<th>y</th>
<th>Choice</th>
<th>μ(x,y) − μ(z,y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>50</td>
<td>0.10</td>
<td>5</td>
<td>Risky</td>
<td>-0.0298</td>
<td>29</td>
<td>-50</td>
<td>0</td>
<td>0.90</td>
<td>-5</td>
<td>Safe</td>
<td>0.0298</td>
</tr>
<tr>
<td>2</td>
<td>-50</td>
<td>0</td>
<td>0.50</td>
<td>-25</td>
<td>Risky</td>
<td>0.6326</td>
<td>30</td>
<td>0</td>
<td>50</td>
<td>0.50</td>
<td>25</td>
<td>Safe</td>
<td>-0.6326</td>
</tr>
<tr>
<td>3</td>
<td>-50</td>
<td>0</td>
<td>0.10</td>
<td>-45</td>
<td>Risky</td>
<td>0.9262</td>
<td>31</td>
<td>0</td>
<td>50</td>
<td>0.90</td>
<td>45</td>
<td>Safe</td>
<td>-0.9262</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>100</td>
<td>0.05</td>
<td>5</td>
<td>Risky</td>
<td>0.0629</td>
<td>32</td>
<td>-100</td>
<td>0</td>
<td>0.95</td>
<td>-5</td>
<td>Neutral</td>
<td>-0.0629</td>
</tr>
<tr>
<td>5</td>
<td>-100</td>
<td>0</td>
<td>0.75</td>
<td>-25</td>
<td>Risky</td>
<td>0.3663</td>
<td>33</td>
<td>0</td>
<td>100</td>
<td>0.25</td>
<td>25</td>
<td>Safe</td>
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</tr>
<tr>
<td>6</td>
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<td>0.50</td>
<td>-50</td>
<td>Risky</td>
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<td>34</td>
<td>0</td>
<td>100</td>
<td>0.50</td>
<td>50</td>
<td>Safe</td>
<td>-0.6493</td>
</tr>
<tr>
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<td>-75</td>
<td>Risky</td>
<td>0.8448</td>
<td>35</td>
<td>0</td>
<td>100</td>
<td>0.75</td>
<td>75</td>
<td>Safe</td>
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<tr>
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<td>0</td>
<td>0.05</td>
<td>-95</td>
<td>Risky</td>
<td>0.9641</td>
<td>36</td>
<td>0</td>
<td>100</td>
<td>0.95</td>
<td>95</td>
<td>Safe</td>
<td>-0.9641</td>
</tr>
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6. **Relation to Alternative Models**

The leading model for choices under risk is widely recognized to be cumulative prospect theory due to Tversky and Kahneman (1992). Under CPT, the utility function of EUT is replaced by a value function which is concave for gains and convex for losses, and probabilities are transformed by a probability weighting function. The weighting function in CPT is based on the rank-dependent utility theory (RDU) of Quiggin (1982). While CPT can explain many of the anomalies in Table 1, CPT cannot simultaneously explain the occurrence of the Allais paradox in minimalist frames and its disappearance in parallel frames, nor can it explain the experimentally observed preference for complexity or violations of ordinal independence. In addition, CPT with a power value function, the most widely used specification of CPT, cannot explain the peanuts effect under any probability weighting function (Scholten and Read, 2014). In contrast, FDU explains all paradoxes in Table 1 with preferences which are linear in probabilities.

An alternative model of decisions under risk was proposed by Bordalo et al. (2012). In that model, a decision maker forms a salience ranking over possible states of the world, and then discounts less salient states by a constant discount factor, similar to how distant time periods are discounted in models of intertemporal choice. This salience model also provides an explanation for many of the paradoxes in Table 1, although it cannot explain the bias toward complexity, discussed in Section 4.5. In addition, the salience model assumes that the decision maker forms a salience ranking over every possible pair of distinct payoff combinations that could occur, which yields a large state space, even for fairly simple lotteries. For instance, two independent lotteries each with 5 distinct outcomes require a salience ranking of 25 states. In contrast, FDU only requires lotteries to be represented as they are presented to the decision maker. The model of salience-weighted utility over presentations (SWUP) by Leland and Schneider (2015) provides
an alternative approach to modeling framing effects where larger differences in payoffs or probabilities are systematically over-weighted. However both SWUP and the salience model of Bordalo et al. (2012) can violate first-order stochastic dominance as well as transitivity. In contrast, FDU satisfies first-order stochastic dominance within any choice between two alternatives, since it reduces to EUT for any given frame. In addition, FDU satisfies a limited version of transitivity – whenever the same system is dominant for a given set of decisions, transitivity is predicted to hold.

The models most closely related to FDU are the dual system model (DSM) of Mukherjee (2010) and the model of certain versus uncertain utility in Andreoni and Sprenger (2010). The dual system model of Mukherjee assumes that decisions under risk are made through the convex combination of the value functions for an affective and a deliberative system. In contrast, we assume a simpler discrete processing form, which may also have some empirical support grounded in a winner-take-all feature of neural processing (Camerer et al., 2005). We can think of the frame as placing the decision maker in either a risk-sensitive mindset or a reward-sensitive mindset.

The DSM also explains a wide range of empirical phenomena. A limitation of the DSM is that it does not provide a systematic mechanism for how the weights placed on each system’s value function change across choice sets. Mukherjee comments that this is beyond the scope of his paper. In contrast, FDU models changes in the relative dominance of the R and P systems through changes in the framing of alternatives.

The model of certain versus uncertain utility in Andreoni and Sprenger (2010) postulates a value function of the form:
where $\mathcal{L}_D$ is the set of degenerate lotteries (i.e. lotteries offering a particular outcome with probability 1), and $\mathcal{L}_N$ is the set of non-degenerate lotteries. Thus, the model involves two implicit utility functions – one which evaluates certain payoffs and another which evaluates non-degenerate lotteries. However the model in Andreoni and Sprenger (2010) violates stochastic dominance. In the present model, we have provided a simple formal representation of preferences which bear some similarity to the preferences represented by $W(X,L)$. However, FDU does not violate dominance and can also explain framing effects, the fourfold pattern of risk attitudes, and violations of ordinal independence which are not accounted for by $W(X,L)$.

Framing effects have been somewhat elusive to model (Kreps 1988). While some framing effects can be explained by CPT, such as gain-loss framing effects, others such as the alignment effect and bias toward complexity cannot. Salant and Rubinstein (2008) model “choices with frames” in the domain of certainty and relate it to the revealed preference framework. In contrast, we explicitly model framing effects under risk.

7. Conclusion

In the preceding sections we have introduced a model in which utility is frame-dependent and we have shown that FDU simultaneously explains a variety of decision paradoxes. For the gain-loss framing effect, the alignment effect, bias toward complexity, violations of ordinal independence, the peanuts effect, the fourfold pattern, and versions of the Allais paradox and common ratio effect where lotteries within each choice pair have the same expected values, the paradoxes are each resolved at the choice level if $u_R(x)$ is any concave function and $u_p(x)$ is
any convex function. We also provided conditions for the paradoxes to hold for any functions $\mu(x, y)$ and $\phi(p, q)$. In addition, all of these conditions hold naturally for the classic experimental examples under parametric forms (4) and (5), such as for our running specification with $k = 1, \theta = 1$ and any $\varepsilon \geq 0$.

The FDU model embeds the von Neumann-Morgenstern theory of risk preference into a model which formalizes both the framing of alternatives and the decision maker’s risk perception. We applied FDU to the data supporting the fourfold pattern of risk preferences and demonstrated that the fourfold pattern is equivalent to a simpler twofold pattern of behavior. We also showed that FDU predicts a novel bias toward complexity for gains which has experimental support. For each paradox, we provided both a general characterization and the canonical example from the experimental literature. Each of these examples is explained by FDU for the same parametric specification and for the same pair of implicit utility functions. Furthermore, the same parameters used to resolve all the empirical examples in Section 4 also fit the experimental data from Tversky and Kahneman (1992) as well as CPT. We further demonstrated that FDU provides a unified explanation for many of the classical paradoxes as each emerging from the same underlying mechanism – risk-seeking behavior when the perceived reward outweighs the perceived risk and risk aversion when the perceived risk outweighs the perceived reward.
References


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Manuscript.


