Dynamic Formation of Teams: When Does Waiting for Good Matches Pay Off?

Preliminary and Incomplete

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Abstract

This paper studies the trade-off between realizing match values early and waiting for good matches that arises in a dynamic matching model with discounting. The focus is on centralized markets which we examine via a mechanism design approach. We consider heterogeneous agents that arrive stochastically over time and are to be matched to groups. Matches are irrevocable and assortative matchings are welfare enhancing. First, we derive the welfare-maximizing assignment rule depending on the parameter constellation in closed form. The optimal rule displays the subtle trade-off between realizing match values early and accumulating agents to achieve assortative matchings. Second, we study implementability of the welfare-optimal policies, when agents have private information and maximize their own match value. It is shown that the welfare-maximizing policy is implementable in a strong solution concept with contracts that satisfy natural requirements. Furthermore, we identify situations in which the designer can abstain from using monetary incentives.

Keywords: Dynamic Assignment, Mechanism Design, Dynamic Matching, Welfare Maximization.

JEL Classification: C61, C78, D61, D82.

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1 Introduction

We study a canonical situation in which agents that arrive gradually over time join forces in order to generate output. Agents are heterogeneous and when forming a group their characteristics are complements in the production function. Complementarity means that the marginal contribution to output of one agent increases in the partner’s characteristic. In a static world, when all agents are present from the beginning, it is a well established result in the matching literature that with this kind of production function positive assortative matchings are both stable and efficient.\footnote{For a study of necessary and sufficient conditions for positive assortative matchings see Legros and Newman (2002).} The dynamic arrival of agents combined with impatience, however, poses a challenge to positive assortativeness. If future outcomes are discounted, the desirability of early matches increases both from a social welfare as well as an participating individual’s perspective. This paper analyses the emerging trade-off between realizing match values early and waiting for good matches. It thus contributes to the literature on matching with frictions that originate from the dynamic nature of the problem. The growing literature on search and matching studies matching patterns using search models. Each agent from a continuous population meets random fellows one by one and then decides whether to match with that partner or to continue search. Major contributions are Shimer and Smith (2000), Smith (2006), and Atakan (2006).\footnote{See also the primary contributions by Sattinger (1995), Lu and McAfee (1996) and Burdett and Coles (1997) on two-sided matching. For a literature survey on search and matching models see Smith (2011).}

In contrast to that approach we study the centralized organization of small matching markets using a mechanism design approach. The environment is small in the sense that the arrival process is discrete, interpreted as single agents arriving. In small matching markets central organization is naturally more appropriate than decentralized search models. Applications include the formation of teams that are supposed to accomplish certain tasks but the corresponding workers get available only successively; or the grouping of entrepreneurs that arrive over time in a local market.

This paper tackles the question of assortativeness in a centralized dynamic matching market. We address both the welfare-maximizing matching procedures under complete information, and socially optimal mechanisms when agents have private information. We develop a tool that helps us to solve for the welfare-maximizing matching policy in closed form without imposing any restriction on the policy. This provides clear insights into the effects involved. Then we proof implementability of the welfare-maximizing matching policy when agents have private information about their types. Furthermore, we identify situations in which the market organizer can abstain from using monetary incentives. Finally, we address the case in which the agents can, in addition to their
private type, hide their arrival.

We consider a population of heterogeneous agents that differ in a characteristic which can, for example, be interpreted as the skill level or the profession. For the sake of simplicity we assume characteristics to be binary. Matched agents jointly produce socially valuable output according to a production function which is supermodular in the agents’ characteristics. Once matches are made, they are irrevocable. This may originate from the matched group’s need to initially make sunk investments, which make any later split economically irrational. Investments may be capital investments for example in advertisement for a newly founded company, or social investments like trust and social arrangements within a group of workers. An alternative view is that pairs simply leave the market and do not return even in case of a split. Note that a consequence of the irreversibility is that the match value associated with a match reflects the present value of the output flow that is produced by that match. The precise flow is completely flexible. Following most of the literature, we assume that matchings are pairwise.3 We consider the most natural dynamic friction to matching markets we can think of - agents dynamically arrive to the market. Agents’ types are drawn independent of the arrivals and arrivals take place according to a Poisson Process. This model is flexible with respect to four key features: The degree of complementarity of the partners’ characteristics in the output function, the relative size of absolute values of output generated by two possible matchings of similar agents, the probability distribution of arriving agents’ types and the patience represented by discounting.4

In the first part of the paper, we derive the welfare-optimal matching policy when the designer can observe both arrivals and the arriving agents’ types. As opposed to the literature on search and matching, we do not impose any restrictions on the technology the central organizer may use.5 This allows us to analyze the role of assortativeness from an efficiency perspective. The welfare-optimal policy is obtained in closed form depending on the four key characteristics. This allows for a clear-cut analysis of the dynamic friction on efficiency.

First, assume that the outputs produced by two similar agents of either kind do not differ too drastically. We refer to the type that generates the higher output when being paired with itself as ‘productive’. Depending on the remaining three key features always one of three matching policies is optimal. The Positive Assortative Policy

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3 For an exception see Ahlin (2012).
4 More precisely, the combination of the discount rate and the frequency with which arrivals are expected.
5 In the literature on search and matching there is little work on social optimality. Shimer and Smith (2001a) and Shimer and Smith (2001b) study socially optimal policies, however, the search friction is taken as given.
matches agents with similar types whenever possible and otherwise lets every agent wait. The Provident Impatient Policy matches two similar agents with priority whenever possible, but it also matches two unequal types if only those are left. Finally, the Myopic Impatient Policy always matches two productive agents. If the number of productive agents is uneven, the remaining one is then matched with priority to an ‘unproductive’ agent. Only then remaining unproductive agents are matched. The differences between the matching policies are illustrated in Figure 1. The figure depicts the matches done by the three matching policies for two different sets of agents in the market.

The relation between patience and the optimal policies is monotone in the sense that there are two cut-off levels: When discounting is weak, the Positive Assortative Policy is optimal, for intermediate levels of patience the Provident Impatient Policy maximizes welfare, and in an impatient environment it is best to apply the Myopic Impatient Policy. The intuition for this result concerning the trade-off between early matches and waiting for assortative matches is perspicuous: The stronger future payoffs are discounted, the less willing is the designer to give up immediate output for the option of realizing gains from positive assortativeness in the future. Whereas the Positive Assortative Policy always respects these options, the Myopic Impatient Policy only maximizes current payoff.

An immediate insight is that positive assortative matchings are not always welfare-maximizing. This means in particular that a fail of positive assortativeness in search models triggered by agents that are ‘too’ impatient to wait for good matches may indeed be welfare-enhancing.\footnote{Examples from the search and matching literature in which supermodular output functions are not sufficient for positive assortativeness are Shimer and Smith (2000) and Smith (2006).}

Considered from the opposite point of view, the result tells us that for little rates of discounting the efficient matching pattern resembles the standard pattern from fric-
tionless matching. The dynamic model approaches the static frictionless version when discounting gets negligible. This means in particular that the result that positive assortative matchings are efficient in static environments is robust to small dynamic frictions.

The role of the degree of complementarity of the matched agents’ characteristics is closely related to the degree of patience. The stronger the complementarity, the greater are the gains from positive assortativeness. Consequently, *ceteris paribus*, for little complementarities the Myopic Impatient Policy is optimal, for intermediate levels the Provident Impatient Policy and for strong complementarities the Positive Assortative Policy maximizes welfare.

Surprisingly, the relation between the distribution of arriving agents’ types and optimal policies may be more sophisticated. There are situations in which *ceteris paribus* the Positive Assortative Policy is optimal for intermediate probabilities of arriving agents to be productive but neither for small nor very high probabilities. Two effects are responsible for this. First, a higher probability of productive arrivals increases the likelihood that a productive agent that is present now can be matched with a peer in the near future. This increases the expected value of keeping a single productive agent instead of matching him with an unproductive agent immediately. This effect implies that the Positive Assortative Policy is increasingly attractive the higher the probability for productive agents is. However, there is an important second effect: For illustrative purposes, assume that there is one agent of either type in the market, and almost with certainty there are only productive types arriving in the future. Abstaining from the creation of mixed matches then directly implies the entire waste of the unproductive type as he will never be matched. The designer might be more willing to enforce the Positive Assignment Policy if he knows that arrivals of both types happen such that all agents get matched in near future.

Furthermore, there is a technical contribution in the paper. We solve for the welfare-maximizing matching policy using methods from dynamic programming. As the problem has discrete but infinitely many states, a guess and verify approach implies the need to check for all possible deviations on infinitely many states. We develop a tool to do what we call a ‘State Space Reduction’. This tool enables us to solve the problem via a guess and verify approach by only checking for all possible deviations on a small set of states.

Now, assume that the output produced by two unproductive agents is drastically smaller than the one generated by two productive agents. Assume further that complementarities are weak such that the Myopic Impatient Policy is strongly preferred to the Positive Assortative Policy. In that case different matching policies may be welfare-maximizing. It may become optimal to store unproductive types on the mar-
kent only for the purpose of matching productive agents with them immediately upon arrival and thereby avoiding to incur losses from letting productive agents wait.\footnote{This aspect relates to the basic thought of optimal inventory. See Arrow et al. (1951) for the fundamental thought and Whitin (1954) and Veinott Jr (1966) for early literature surveys.}

In the second part of the paper, we treat implementability of the welfare-maximizing matching policies by an authority which faces agents that have private information and care only about their own matches’ output. Match values are split equally. We follow Bergemann and Välimäki (2010) and consider mechanisms that satisfy ‘efficient exit’ and are interim incentive compatible. We show that the welfare-maximizing policy is always implementable if agents have private information about their type but the designer can observe their arrival. This holds even under the most disadvantageous information structure for implementation: Agents in equilibrium have all information about the present set of agents when arriving. While with observable arrivals the implementation of the Provident Impatient Policy and the Myopic Impatient Policy turns out to be generally unproblematic, this result is surprising concerning the Positive Assortative Policy. In static environments, positive assortativeness can be implemented using a single-crossing property with respect to each other agent’s type. By this we mean that the gain from matching with a productive agent instead of an unproductive one is higher for agents which are themselves productive as compared to unproductive agents. However, the time friction is not only a friction to efficiency but also to the incentive constraints. When the expected time until getting matched depends on the reported type, it might get more attractive for unproductive agents to report being productive than it does for productive agents. Whenever this is the case, the Positive Assortative Policy is not implementable. However, it turns out that whenever this happens, the Positive Assortative Policy is not welfare-maximizing either.

In addition, we show that if the complementarity of the match value function is sufficiently strong, the welfare-maximizing policy can be implemented with transfers that do only depend on the agent’s reported type. We further prove that whenever this is possible, there exists a splitting rule for the match value of mixed matches such that the optimal policy can be implemented without transfers.

Finally, we address implementation when the agents’ arrivals are unobservable to the principal. In this case the agents’ private information is two-dimensional: It consists of the type and the arrival time. Deviations from truthful reporting may, hence, consist of misreports about the type combined with strategic delays of the report about the arrival. We proof that even in this environment the optimal policy is always implementable. The contract that implements the Positive Assortative Policy when arrivals are observable is also incentive compatible with unobservable arrivals. Concerning the
Provident Impatient Policy and the Myopic Impatient Policy, we crucially exploit the
fact that for their implementation the authority only needs to elicit information about
arrivals and not about types.

The efficiency of positive assortativeness in a dynamic matching model is also consid-
ered in a related work by Shi (2005). The paper considers two-sided matching with
a supermodular production function where matches get split after random durations.
Shi, however, endogenizes one side’s quality choice which turns out to make the prob-
lem of our paper uninteresting. The focus of his paper is on an additionally introduced
coordination friction that cannot be overcome by the central authority either.
Dynamic matching markets that are organized by a central authority are further studied
in the growing literature on dynamic kidney exchange. Respective papers are Ünver
(2010), Ashlagi et al. (2013), Akbarpour et al. (2014) and Anderson et al. (2015).
The objective in these papers is to minimize waiting times and therefore maximize the
number of matches respecting restrictions on feasible matches that are exogenously
given on medical grounds. Opposed to this literature, we focus on maximizing total
match value in an environment in which any match is feasible.
In a broader perspective this paper adds to the literature on dynamic assignment prob-
lems. One strand of this literature considers the assignment of dynamically arriving
agents to goods which are present from the beginning. Examples are Gallien (2006),
Gershkov and Moldovanu (2009), Mierendorff (2011), Board and Skrzypacz (2013),
Gershkov and Moldovanu (2010), Dizdar et al. (2011) and Pai and Vohra (2013). An-
other strand treats the assignment of dynamically arriving goods to agents that are
queuing for these goods. Examples are Leshno (2012) and Bloch and Cantala (2014).
The housing literature combines these two strands: Agents arrive over time and are
matched with houses that get back to the market when the assigned agents have moved
out. Examples are Kurino (2009), Bloch and Houy (2012) and Bloch and Cantala
(2013). The housing literature and our paper share the property that both match-
ing partners arrive over time. There are, however, two substantial differences. First,
whereas in the housing literature the arriving stream of houses is determined by the
allocation, in our paper it is entirely exogenous. And second, in our environment there
is private information held by both matching partners.

The rest of the paper is organized as follows: Section 2 presents the setup, Section 3
derives the welfare-maximizing matching policies, Section 4 addresses implementability
of the policies under private information and Section 5 concludes.
2 Model

We consider agents that arrive over time to a matching market. Time is continuous, and the time horizon is infinite, \( t \in [0, \infty) \). Having arrived to the market, agents remain in the market until they are matched, i.e., agents are long-lived. Agents are characterized by the tuple \((\theta, a)\), where \( \theta \) is the agent’s type and \( a \in [0, \infty) \) is his arrival time. An agent’s type reflects his productivity; he is either productive \( H \) or unproductive \( L \), \( H > L > 0 \).

Arrivals are described by a Poisson process \((N_t)_{t \geq 0}\). The random variable \( N_t \) describes the number of arrivals up to time \( t \). A Poisson process is a counting processes and thus describes discrete arrivals. Let \( t_n \) be the random variable describing the time of the \( n \)-th arrival. Agents’ types are drawn independent of the process \((N_t)_{t \geq 0}\) and i.i.d. across time; we denote the probability of the productive type by \( p \in (0, 1) \). We refer to the process induced by \((N_t)_{t \geq 0}\) joint with the Bernoulli distribution associated with \( p \) as arrival process.

There exists a central authority, the designer, which organizes the market. Once an agent arrives, the designer may assign him to another agent that is in the market. After being assigned a partner, an agent cannot be reassigned. This could be, for example, because agents leave the market after forming a group and are thus no longer available for the designer or because they make sunk investments that are too costly to forfeit. Together agents produce a match value depending on the pair’s types. Formally,

\[
m: \mathbb{R}_{>0} \times \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0},
\]

\[
\theta_1 \times \theta_2 \mapsto m(\theta_1, \theta_2).
\]

In accordance with the literature, the match value function \( m \) is assumed to be symmetric and strictly increasing in both arguments. Given the binary type space, \( m \) can be described by three match values \( m_{LL}, m_{HL}, \) and \( m_{HH} \), where \( m_{\theta_1, \theta_2} := m(\theta_1, \theta_2) \).\(^8\)

We refer to pairs where both agents have the same type as homogeneous matches; pairs of agents with different types are termed mixed matches. We assume that the match value function is supermodular, which in our setup boils down to \( 2m_{HL} \leq m_{HH} + m_{LL} \). Supermodularity implies that in a static model, where all agents are present simultaneously, positive assortative matching maximizes the sum of match values.\(^9\) Alternatively, if we regard the two match partners as contributing to the match value, the types, inter-

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\(^8\) As will become clear from the analysis below, one could normalize only one of these three values.

\(^9\) A positive assortative matching is a pairing of all agents in the market, in which productive types pair with productive types and unproductive types pair with unproductive types.
Agents derive utility from their share of the match value. In the first part of the paper, the precise share and the way it is determined, endogenously or exogenously, may be arbitrary.

Future payoffs are discounted with a common discount rate, \( r \in (0, \infty) \). We denote the expected discount factor until the arrival of the next agent by \( \delta, \delta = \mathbb{E}[e^{-rt}] \).

The designer seeks to maximize the expected discounted sum of match values, i.e., the expected sum of discounted utilities. If we assume that all agents are present from \( t = 0 \) but only enter the market at their arrival time \( a \), then our objective corresponds to maximizing the expected sum of utilities. In yet another, less benevolent, interpretation the designer maximizes output. For a formal description of the designer’s objective we define histories \((h, t) := \{(a, \theta), t \mid a \leq t\}\) and denote by \( H \times \mathbb{R}_{\geq 0} \) the set of all histories. A history is a complete list of those agents’ types and arrival times that have entered the market so far. A matching policy \( \rho \) describes the pairs that are created at any point in time after any history, i.e.,

\[
\rho : H \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{N}^3, \\
(h, t) \mapsto (\rho_{HH}(h, t), \rho_{HL}(h, t), \rho_{LL}(h, t)),
\]

where \( \rho_{\theta_1\theta_2}(h, t) \) denotes the number of \( \theta_1\theta_2 \)-groups that are formed at time \( t \). Denote by \( x \) and \( y \) the number of productive and unproductive types that are still available in the market given history \((h, t)\) and policy \( \rho \). Formally, \( x \) and \( y \) are given by

\[
x := \#\{(a, \theta) \mid a \leq t, \ \theta = H\} - \sum_{t' = (h, t') > 0}^{t' \leq t} (2\rho_{HH}(h, t) + \rho_{HL}(h, t)),
\]

and

\[
y := \#\{(a, \theta) \mid a \leq t, \ \theta = L\} - \sum_{t' = (h, t') > 0}^{t' \leq t} (2\rho_{LL}(h, t) + \rho_{HL}(h, t)),
\]

where

\[
v(h, t) = \rho_{HH}(h, t) m_{HH} + \rho_{HL}(h, t) m_{HL} + \rho_{LL}(h, t) m_{LL}.
\]
We call a matching policy $\rho$ feasible if it never matches more agents than available in the market, i.e., for all $(h,t)$,

$$
2\rho_{HH}(h,t) + \rho_{HL}(h,t) \leq x(h,t),
$$

$$
2\rho_{LL}(h,t) + \rho_{HL}(h,t) \leq y(h,t).
$$

Denoting by $\mathcal{P}_F$ the set of feasible matching policies, we can state the designer’s objective as

$$
\sup_{\rho \in \mathcal{P}_F} \mathbb{E} \left[ \sum_{t: v(h,t) > 0} e^{-rt} v(h,t) \right].
$$

(1)

**Recursive Formulation.** As a first step to solve (1), we identify the points in time at which an optimal policy forms groups.

**Lemma 1** In any optimal policy matches occur only at arrival times of agents.

We sketch the proof of Lemma 1. All formal proofs are relegated to the Appendix. Given that the inter-arrival times of the Poisson process are independent, it is sufficient to consider policies that condition only on $(x, y)$ and the time since the last arrival. The memorylessness of the exponentially distributed inter-arrival times implies that the probability of the next arrival does not depend on the time since the last arrival. Consider two points in time, firstly, the time of the last arrival, secondly, an intermediate time after the last arrival but before the next arrival. As the probability of the next arrival does not change in the time since the last arrival, there is no difference in the trade-off between matching and waiting at these two points in time. Hence, if the designer chooses to wait with a set of agents after the arrival of an agent, he will continue waiting until the next arrival.

Lemma 1 allows us to restrict attention to feasible policies that condition solely on $x$ and $y$. Denote this subset of policies by $\mathcal{P}_S$ and the state space $\{(x, y) \in \mathbb{N}_0^2 | x \geq 0, y \geq 0, x + y > 0\}$ by $S$. We define $\rho(x,y)$ as $\rho(h,t)$ with $x$ productive and $y$ unproductive agents remaining in the market. Building upon the insight of Lemma 1, we conclude the section by reformulating (1) in recursive form:

$$
V(x,y) = \sup_{\rho \in \mathcal{P}_S} \{ \rho_{HH}m_{HH} + \rho_{HL}m_{HL} + \rho_{LL}m_{LL}
$$

$$
+ \mathbb{E} \left[ e^{-rt} (pV(x - 2\rho_{HH} - \rho_{HL} + 1, y - 2\rho_{LL} - \rho_{HL}) + (1-p)V(x - 2\rho_{HH} - \rho_{HL}, y - 2\rho_{LL} - \rho_{HL} + 1)) \right] \}.
$$

(2)
3 Optimal Policy

We start by considering the welfare maximization problem under complete information. This means at the point in time of an agent’s arrival the authority observes both the agent’s arrival and his type. As we do not impose any technological restrictions on feasible policies, the problem is to maximize objective (1) without restrictions. Incentive constraints arising from the agents’ informational advantage are added in Section 4.

There are two major driving forces when it comes to welfare maximization. We refer to the first one as the gain of assortative matching. The value of matching two productive agents and two unproductive agents is higher than the value of two mixed matches. In order to achieve positive assortative matchings, it might be necessary to accumulate agents. This is the case whenever pairing agents in the order of their arrivals induces creating mixed pairs. Hence, the gain of assortative matching is effective in favor of waiting with agents in the market. The second force is the loss from deferring matches.

Having agents waiting in the market is costly due to discounting. This force is effective in favor of creating match values early; this may include creating mixed pairs.

The two driving forces suggest that, whenever present in the market, two productive agents are always paired immediately: In this case the two forces are not in conflict - the highest match value gets realized immediately and the match does not oppose positive assortativeness. When considering whether to create a mixed match, the two forces obviously are in conflict. At first glance surprisingly, it is not clear either whether it is optimal to always match two unproductive agents. Consider the example when there are exactly one productive and two unproductive agents in the market. Strong discounting might cause the designer to create the mixed match even though the gain of assortative matching pushes the authority to pair the two unproductive agents. Even when there are only unproductive agents in the market, it might still be welfare-maximizing to store them and not match anybody in order to pair incoming productive types immediately.

3.1 The Regular Case

By means of the assumption $m_{HH} \leq 3m_{LL}$ we have guaranteed that the match value of a pair of productive agents is not extremely higher than the output generated by two unproductive agents. In this case there are three important policies to be considered, which are portrayed in the following.

Definition 1 Positive Assortative Policy

The Positive Assortative Policy creates in each state $(x,y)$ the maximal number of
homogeneous pairs of both kinds of agents. Mixed matches are never created.

The matching pattern produced by this policy is positive assortative. In the absence of discounting, this policy maximizes the overall match value. Whenever there is exactly one agent of either kind left in the market, the policy induces both agents to wait despite waiting costs.

**Definition 2 Provident Impatient Policy**
The Provident Impatient Policy creates in each state \((x, y)\) the maximal number of homogeneous pairs of both kinds of agents. If both \(x\) and \(y\) are uneven, one mixed match is created in addition.

The matching pattern produced by this policy is not positive assortative but contains a relatively large number of homogeneous matches. The policy can be easiest understood as follows: In any state it creates the maximal number of homogeneous matches and if afterwards there is exactly one agent of either type left, they are matched as well. In that sense homogeneous matches are given priority over mixed matches. However, the policy never induces two agents to wait.

**Definition 3 Myopic Impatient Policy**
The Myopic Impatient Policy creates in each state \((x, y)\) the maximal number of pairs of productive agents. If \(x\) is uneven and \(y \geq 1\), one mixed match is created. The maximal number of pairs from the pool of remaining unproductive agents formed.

The matching pattern produced by this policy is not positive assortative and contains a relatively little number of homogeneous matches. Again, the policy first creates the maximal number of productive matches. If afterwards there is a productive agent left, it is matched to an unproductive one with priority. Only then pairs of unproductive agents are formed. In that sense productive agents are given priority over homogeneous matches. The policy acts entirely myopic in the sense that it maximizes the sum of immediate match values.

Note that matching policies are defined on the entire state space \(S\). Given that at the very beginning, when a certain policy is being introduced, there is an arbitrary set of agents present in the market, the entire definition is necessary in order to evaluate the welfare induced by a matching policy. Once one of the three policies defined above has been installed, the set of states that is actually reached in later periods is limited. When starting in one of these states, the same welfare is achieved by policies that differ from the initial policy on states which are never reached anyways. Concerning the three policies defined above, the Provident Impatient Policy and the Myopic Impatient
Policy do not differ on states which are reached once one of the two policies is established. Once they are installed, the total number of agents in the market never exceeds two and both policies form a pair whenever possible, which is each second arrival. We call the set of policies which always form a pair as soon as two agents are present the set of *Impatient Policies*. We denote a policy that maximizes the discounted sum of match values on each state that can be reached when starting with \((0, 1)\) or \((1, 0)\) as a *welfare-maximizing policy*.

In this section we will consider matching policies which maximize the discounted sum of match values starting from any state \(\{(x, y) | x, y \in \mathbb{N}_0\}\) and refer to such policies as *optimal policies*. A rationale behind this is that at the point in time the policy is established, an arbitrary set of agents may be present in the market. Of course, an optimal policy is a welfare-maximizing policy. The primary reason for considering optimal policies is to clearly expose the trade-offs connected to welfare maximization and their resolutions, which is the major objective of this section. Finally, solving the problem for optimal policies is indispensable even when only caring about welfare-maximizing policies.

In order to state the main theorem, a detailed description of optimal policies, we introduce two definitions, which are used to formalize a partitioning of the space of parameter constellations \((p, \delta, m_{HH}, m_{LL}, m_{HL})\). Denote

\[
m_{HL}^1 := m_{HH} \frac{\delta p}{1 - \delta(1 - 2p)} + m_{LL} \frac{\delta(1 - p)}{1 + \delta(1 - 2p)}
\]  

and

\[
m_{HL}^2 := m_{HH} \frac{\delta p}{1 - \delta(1 - 2p)} + m_{LL} \frac{1 - \delta(1 - p)}{1 - \delta(1 - 2p)}.
\]  

Note that for any \(m_{HH}, m_{LL}, m_{HL}, \delta\) and \(p\) it holds that \(m_{HL}^1 < m_{HL}^2\).

**Theorem 1** Depending on the parameter constellation \((p, \delta, m_{HH}, m_{LL}, m_{HL})\), one of three matching policies is optimal:

- If \(m_{HL} \leq m_{HL}^1\), the Positive Assortative Policy is optimal.
- If \(m_{HL} \in [m_{HL}^1, m_{HL}^2]\), the Provident Impatient Policy is optimal.
- If \(m_{HL} \geq m_{HL}^2\), the Myopic Impatient Policy is optimal.

The optimal policy is unique if \(m_{HL} \notin \{m_{HL}^1, m_{HL}^2\}\).

In the following we first elaborate on the statement of Theorem 1. Then we provide an economic intuition for the result and the underlying trade-offs. The treatment of the theorem is completed with an outline of the proof.
Immediate implications. Notice that $m_{LL} < m_{HL}^1 < m_{HL}^2 < 1/2(m_{LL} + m_{HH})$. Consequently, each of the three policies is generic in the following sense: When the parameter constellation is drawn from a joint distribution that has positive density on the entire parameter space, then the probability for each of the policies to be optimal is strictly positive. Even for arbitrarily given match values, each of the three policies is generic: The probability for each of the policies to be optimal is strictly positive when drawing $p$ and $\delta$ from the previously mentioned distribution conditional on the realization of the match values.

In particular, the theorem states that optimality never lets two unproductive agents wait when there is nobody else in the market. The rationale behind not pairing two unproductive agents would be the following: Consider an environment in which the Myopic Impatient Policy is optimal. If the relative importance of productive agents is even higher, the desire to match productive agents directly upon arrival increases even more. In that situation it may be optimal to always keep unproductive agents on stock to avoid that productive agents arrive and no match partner is available. The theorem states that the desire to match productive agents immediately upon arrival is never strong enough to outweigh the cost of waiting with two unproductive agents. As will be shown after the discussion of the theorem, this crucially hinges on the assumption $3m_{LL} \geq m_{HH}$.

Intuition. We first comment on the influence of complementarity of the match value function on optimal policies. Given all other parameters, the match value of a mixed pair characterizes the degree of supermodularity of the match value function in the agents’ types. The larger $m_{HL}$, the more are the partners’ types substitutes and perfect substitutability is achieved at the upper bound $m_{HL} = 1/2(m_{HH} + m_{LL})$. From the theorem it can be seen directly that the relation between the match value of a mixed pair and the optimal policies is monotone in the sense that there are two cut-off levels: When complementarity is strong, the Positive Assortative Policy is optimal, for intermediate levels the Provident Impatient Policy is optimal, and for high degrees of substitutability it is optimal to apply the Myopic Impatient Policy. Concerning the trade-off between early matches and waiting for assortative matches, the intuition is perspicuous: The higher the value of mixed matches, the smaller is the gain from assortative matching. As all other parameters are kept fixed, the loss from deferring matches is unchanged. Hence, with increasing match values for mixed pairs the trade-off between gains from positive assortativeness and losses from deferring matches increasingly respects the second aspect. Whereas the Positive Assortative Policy fully realizes gains from positive assortativeness, the Myopic Impatient Policy only prevents losses from deferring matches.
Second, for given match values we discuss how the choice of optimal policies depends on the impatience, represented by the discount factor $\delta$. Recall that $\delta$ is a compound expression of the arrival rate and the discounting rate. We make use of the following consequence of Theorem 1:

**Corollary 1** For any given parameters $(p,m_{HH},m_{LL},m_{HL})$, there exist two cut-off levels $\delta^1, \delta^2 \in [0,1]$ with $\delta^1 > \delta^2$ such that:

If $\delta \geq \delta^1$, the Positive Assortative Policy is optimal.
If $\delta \in [\delta^2, \delta^1]$, the Provident Impatient Policy is optimal.
If $\delta \leq \delta^2$, the Myopic Impatient Policy is optimal.

The existence of two cut-off levels follows from showing that $\partial m_{HL}^1/\partial \delta \geq 0$ and $\partial m_{HL}^2/\partial \delta \geq 0$ independent of the specific choice of parameters. This means that when fixing all parameters but $\delta$, the monotonicity in $m_{HL}$ carries over to monotonicity in $\delta$. $1 \geq \delta^1 \geq \delta^2 \geq 0$ implies that given any parameter constellation $(p,m_{HH},m_{LL},m_{HL})$, for each of the three policies there exists a $\delta$ such that the policy is optimal.

Implication 1: The relation between patience and the optimal policies is monotone as well: When discounting is weak, the Positive Assortative Policy is optimal, for intermediate levels of patience it is optimal to apply the Provident Impatient Policy, and in an impatient environment the Impatient Policy is optimal. The stronger the discounting, the greater are the losses from deferring matches. Hence, the stronger future payoffs are discounted, the less willing is the designer to give up immediate output for the option of realizing gains from positive assortativeness in the future.

Implication 2: The result tells us that for little rates of discounting the efficient matching pattern resembles the standard pattern from frictionless matching. More precisely, note that for $\delta = 1$ holds $m_{HL}^1 = 1/2(m_{HH} + m_{LL})$, which is the maximum value for $m_{HL}$, and whenever $m_{HL}^1$ is interior, then $\delta^1$ is interior as well. $m_{HL}^1$ is interior for all $p, \delta \neq \{0,1\}$. From an economic perspective, the dynamic model approaches the static frictionless version when discounting gets negligible. Combining the above statements, this implies that the result that positive assortative matchings are efficient in static environments is robust to small dynamic frictions.

Third, we examine the role of the distribution of arriving agents’ types represented by $p$. Surprisingly, the relation between that distribution and optimal policies is more sophisticated. There are situations in which *ceteris paribus* the Positive Assortative Policy is optimal for intermediate values of $p$ but neither for small nor very high probabilities. This non-monotonicity arises from the presence of two different effects. First, a higher probability of productive arrivals decreases the expected time until the next productive
arrival. For a single productive agent that is present in the market this means that the likelihood that he can be matched with a productive peer in the near future increases. This raises the expected value of keeping a single productive agent instead of matching him with an unproductive agent immediately. Hence, the first effects implies that the Positive Assortative Policy is more attractive the higher the probability of productive agents. Hence, the cut-off value of patience $\delta^1$ should decrease in $p$. However, there is an important second effect. For high levels of $p$ not creating mixed matches induces a high loss from an unproductive agent if he is present in the market. For illustrative purposes, assume that there is one agent of either type in the market, and $p$ is close to one such that almost with certainty there are only productive types arriving in the future. Abstaining from the creation of mixed matches implies the entire waste of the unproductive type as he will never be matched. The designer might be more willing to enforce the Positive Assignment Policy and therefore let the unproductive agent wait if he knows that arrivals of both types happen regularly such that all agents get matched in near future. Considering the second effect alone, the cut-off value of patience $\delta^1$ as a function of $p$ should have its minimum at some interior value, decrease for small $p$, and increase for high values of $p$. In particular, if the values $m_{LL}$ and $m_{HH}$ do not differ much, the second effect is strong enough such that as $p$ approaches one, $\delta^1$ increases in $p$.

The dependence of the optimal matching policy on $p$ and $\delta$ is graphically illustrated for match values that represent three canonical match value functions: The case of perfect complements (Figure 2), the multiplicative case (Figure 3), and the case of (almost) perfect substitutes (Figure 4). The red line depicts $\delta_1$, the boundary between the parameter regions in which the Positive Assortative Policy and the Provident Impatient Policy are optimal. The blue line depicts $\delta^2$, the boundary between the Provident Impatient Policy and the Myopic Impatient Policy.

In the case of perfect complements, Figure 2, the match value of a mixed match equals the match value of a homogeneous match of unproductive agents. As the match value function is ‘very’ supermodular, the gain of assortative matching is high, and the parameter region in which the Positive Assortative Policy is optimal is large. Furthermore, the Provident Impatient Policy dominates the Myopic Impatient Policy in the sense that the expected welfare from the Provident Impatient Policy is weakly higher on each possible state. The reason is that, starting on a given state, both policies generate the same sum of match values in the first period using the same number of agents. However, the Myopic Impatient policy uses weakly more productive agents for it than the Provident Impatient Policy.\textsuperscript{11}

\textsuperscript{11}Consider, for example, the state $(1, 2)$. The Provident Impatient Policy matches the two unproductive agents, whereas the Myopic Impatient Policy creates a mixed match. Both match values are equal, but the Provident Impatient Policy leaves a productive agent over whereas the Myopic
Figure 2: Perfect complements: \( m(H, L) = \min\{H, L\} \); Here \( H = 3 \) and \( L = 1 \). The red line depicts \( \delta^1 \); the blue line depicts \( \delta^2 \).

Figure 3: Product: \( m(H, L) = H \cdot L \); Here \( H = \sqrt{3} \) and \( L = 1 \).
Figure 4: Almost perfect substitutes: Almost $m(H, L) = H + L$; Here $H = 1.5$, $L = 0.5$ and $m(H, L) = 1.98 < H + L$.

In the product case, Figure 3, complementarities have an intermediate strength. The case of multiplication is regularly used to model complementarities in the match value function. The parameter region in which the Positive Assortative Policy is optimal hence has a smaller, but positive size. Note that for $p$ close to one, the Positive Assortative Policy is never optimal - the designer always wants to avoid wasting unproductive agents. Furthermore, there is a region in which the Myopic Impatient Policy is optimal, because the match value of a mixed pair exceeds the output generated by a pair of unproductive agents.

In the case of perfect substitutes, approximated by Figure 4, there are no complementarities in the match value function. From $\partial m_{1HL}/\partial \delta \geq 0$ and $\partial m_{2HL}/\partial \delta \geq 0$ joint with $m_{1HL} = m_{2HL} = 1/2(m(H, H) + m(L, L))$ at $\delta = 1$ follows that when $m(H, L)$ approaches $1/2(m(H, H) + m(L, L))$, $m(H, L) > m_{1HL}, m_{2HL}$ for almost all $\delta$. Hence, the parameter regions, in which the Positive Assortative Policy and the Provident Impatient Policy are optimal, vanish.

**Outline of the proof.** In the following, we sketch the approach that is used to proof Theorem 1. Along the way, we explain why finding the optimal matching policies is necessary to find the welfare-maximizing policies.

The problem is solved using a Guess & Verify method. Motivated by the introductory thoughts and further structural considerations, we guess candidate matching policies, which turn out to be optimal on some subset of the parameter space. As a side product of the verification, we obtain both the precise parameter region where the verified policy is optimal and its (generic) uniqueness on that parameter region. It turns out Impatient Policy leaves an unproductive agent in the market.
that the respective parameter regions of the three matching policies constitute a partitioning of parameter space. The challenging step in the procedure is the verification, as it involves checking deviations on a discrete but infinite state space. We cope with the situation by developing a procedure we call ‘State Space Reduction’.

*Guess.* Besides using the thoughts presented in the introduction to this section, the range of candidates for optimal policies can be restricted by showing that certain properties of a matching policy are necessary for optimality. These characterizing properties are proved by contradiction. Consider, for example, the argument for matching at least one homogeneous pair of productive agents whenever two productive agents are in the market. Assume not: Then these two productive agents are matched later, possibly with unproductive agents, or not at all. Irrespective of the future stream of arriving agents, pairs can be rearranged profitably: Match the two productive agents immediately and form a match of their ‘old’ partners as soon as possible.

Joint with the arrival process, a matching policy defines a Markov chain over future states \((x, y) \in \mathbb{N}_0^2\). Note, that the state represents the set of agents in the market including the new arrival. As matching policies are deterministic, there are two possible successors for each state with transition probabilities \(p\) and \(1 - p\). The matching policy further determines a payoff associated with each state that is achieved.

For all relevant candidates, the Positive Assortative Policy, The Provident Impatient Policy, and the Myopic Impatient Policy, the induced Markov chains jumps to a finite recurrent set after the first period. This means that (apart from the initial state) only a finite number of states realize. For the Provident Impatient Policy and the Myopic Impatient Policy only the five states \((1, 0), (0, 1), (0, 2), (2, 0), \text{ and } (1, 1)\) can occur. For the Positive Assortative Policy the recurrent set is \{\((1, 0), (0, 1), (1, 1), (0, 2), (2, 0), (2, 1), (1, 2)\}\.

Given that these sets are finite, the value function of each policy at each element of the respective recurrent set can be computed as the solution to a finite system of equations. For illustrative reasons, we show here the system of equations for the Myopic Impatient Policy. Denote the corresponding values by \(V_{MIP}\):

\[
\begin{align*}
V_{MIP}(0, 1) &= \delta [pV_{MIP}(1, 1) + (1 - p)V_{MIP}(0, 2)] \\
V_{MIP}(1, 0) &= \delta [pV_{MIP}(2, 0) + (1 - p)V_{MIP}(1, 1)] \\
V_{MIP}(0, 2) &= m_{HH} + \delta [pV_{MIP}(1, 0) + (1 - p)V_{MIP}(0, 1)] \\
V_{MIP}(2, 0) &= m_{LL} + \delta [pV_{MIP}(1, 0) + (1 - p)V_{MIP}(0, 1)] \\
V_{MIP}(1, 1) &= m_{HL} + \delta [pV_{MIP}(1, 0) + (1 - p)V_{MIP}(0, 1)].
\end{align*}
\]

\(^{12}\)Whenever the optimal policy is unique, there exists no stochastic policy which could improve.

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The value function at an element in the recurrent set only depends on payoffs generated from states in that set and the transition probabilities. An immediate consequence is that policies that differ only on states outside the recurrent set have the same value function at states inside the recurrent set. The recurrent sets are the same as well. The economic intuition is that the policies do the same on any state that can occur and hence have the same value. In particular, the values of the Provident Impatient Policy \(V_{PIP}\) equal those of the Myopic Impatient Policy on the recurrent set. Denote \(V_{PAP}\) the value under the Positive Assortative Policy.

The value functions on states outside the respective recurrent sets can then easily be computed as they only differ from the ones on the recurrent set by the payoff generated in the starting period. This is illustrated by a short example.

**Example 1** The example shows how the value function of the Myopic Impatient Policy at state \((6, 5)\) can be written as a function of the immediate payoff and the value at a state in the recurrent set:

\[
V_{MIP}(6, 5) = 3m_{HH} + 2m_{LL} + \delta[pV_{MIP}(1, 1) + (1 - p)V_{MIP}(0, 2)]
\]

\[
= 3m_{HH} + 2m_{LL} + \delta pV_{MIP}(0, 1).
\]

Only for the length of this passage, exclude the possibility that the policy could ‘start’ outside the recurrent set, but assume it starts with the arrival of the first agent. As the start lies in the recurrent set, only the values at states in the recurrent set are of importance. Any policies that differ only on states outside the recurrent set have the same value function on the states of the recurrent set. Hence, if our guesses turn out to be optimal on a particular parameter specification - which means in particular that it is welfare-maximizing - , any policy that differs outside of the recurrent set is welfare-maximizing as well.\(^{13}\) As will become clear from the next paragraph, guessing optimal policies on the entire state space is necessary even if the aim is to find welfare-maximizing policies, which are pinned down only on the states that actually occur.

**Verification.** The verification of the optimality of the three candidates can potentially be cumbersome. We describe the procedure that we apply to all three candidate policies. Fix one candidate. In the previous step, we have determined the value on each state. The verification consists of showing for any state that following the policy is better than deviating on the particular state and then subsequently following the candidate solution. By the principle of optimality, when all conditions are checked, we have shown both the matching policy, defined as a course of actions, and the associated

\(^{13}\) For the difference of optimality and welfare maximization see the introductory remark to this section.
value function to be optimal. When following the candidate is strictly better than any deviation on any state, the optimal policy is unique.

The difficulty in the verification is that there is a large number of potential deviations to be checked: First, the number of states on which deviations have to be checked is (countably) infinite; and second, on states with many agents in the market there is a large amount of deviations possible. At first glance, one might think that this problem can be avoided by restricting attention to welfare-maximizing policies that start with the very first arrival. The reasoning behind this thought is that, given the recurrent set associated with the candidate is finite and contains the state of the initial arrival, welfare maximization only demands for optimality on a finite set of states. However, this thought is misleading. Even though in the latter case welfare maximization demands for optimality only on a finite set of states, the verification still demands for guessing the optimal policy on each possible state. The reasoning is as follows: On each state in the recurrent set, applying the candidate policy must be better than any alternative action. These alternative actions might, however, lead to states outside the recurrent set.

**Example 2** The example illustrates how checking deviations from the Myopic Impatient Policy at state \((1, 1)\), a state in the respective recurrent state, involves the value on state \((2, 1)\) and \((2, 1)\), which are outside-states for that candidate. The (only possible) deviation considered here is to not create the mixed pair:

\[
V_{MIP}(1, 1) = m_{HL} + \delta[pV_{MIP}(1, 0) + (1 - p)V_{MIP}(0, 1)]
\]

\[
> \delta[pV_{MIP}(2, 1) + (1 - p)V_{MIP}(1, 2)].
\]

In order to evaluate whether the value associated with a deviation is strictly lower than the candidate course of action, we must make statements about the upper bound of the value when being in this ‘outside’ state. Especially when we want to determine for which parameter configurations precisely the candidate policy is optimal, we need to know exactly the maximal value of the ‘outside’ state. Therefore we need to know what the optimal policy does on that state. In order to find the optimal policy on that ‘outside’ state we guess it and again check deviations, which may lead to more ‘outside’ states. As one deviation always is to not match anything, any state in the state space is reached by the procedure. Hence, it is necessary to set up a candidate that matches optimally on the entire state space and verify it.

We tackle the problem of checking the large amount of deviations by a proof strategy that involves what we call ‘State Space Reduction’. The State Space Reduction is the crucial step of the proof. Given a matching policy, it shows that checking all deviations
on a finite set of small states, which it identifies, is sufficient to proof optimality. The inequalities on small states are then checked by hand for each policy. Both the finite set of small states and the particular inequalities that are to be checked on each of those states depend on the candidate policy.

The denotation ‘small’ refers to the number of agents in the market. The following definition states precisely what is meant by one state being smaller than another.

**Definition 4** State \((x, y)\) is smaller than state \((x', y')\) iff \((x, y) < (x', y')\). \((x, y) < (x', y')\) iff \(x \leq x'\) and \(y \leq y'\) with at least one inequality being strict.

The State Space Reduction works as follows: Instead of checking deviations on each state, we set up a number of general statements. On an arbitrary state, these statements identify deviations, which are not optimal, given the candidate policy will be continued in the following states and given the candidate policy is optimal on smaller states. As we consider this procedure in a constructive way, we call these statements ‘principles’ that specify which kind of deviations do not have to be considered. Then we identify states on which these principles capture each possible deviation. We show that there is only a finite number of states including the smallest ones, on which the principles do not capture each possible deviation. When showing by hand that on this finite set there are no profitable deviations, we have shown that there are no profitable deviations at all. Note, that this set of states does not equal the recurrent set and also differs between the Provident Impatient Policy and the Myopic Impatient Policy. In order to state the principles, we use one further definition.

**Definition 5** A policy \(\rho\) is consistent iff for any state \((x, y) \in S\) and any \((\theta_1, \theta_2) \in \Theta^2\) holds: \(\rho_{\theta_1, \theta_2}(x, y) > 0 \Rightarrow \rho_{\theta_1, \theta_2}(x - 1_{\{\theta_1 = L\}} - 1_{\{\theta_2 = L\}}, y - 1_{\{\theta_1 = H\}} - 1_{\{\theta_2 = H\}}) = \rho_{\theta_1, \theta_2}(x, y) - 1\) and \(\rho_{\theta_1, \theta_2}(x - 1_{\{\theta_1 = L\}} - 1_{\{\theta_2 = L\}}, y - 1_{\{\theta_1 = H\}} - 1_{\{\theta_2 = H\}}) = \rho_{\theta_1, \theta_2}(x, y)\) for all \((\theta_3, \theta_4) \notin \{(\theta_1, \theta_2), (\theta_2, \theta_1)\}\).

The meaning of consistency is best illustrated by an example: Suppose on a given state the policy creates matches with at least one pair of productive agents amongst them, \(\theta_1 = \theta_2 = H\). Then, on the state with two productive agents less, the policy creates the same matches except for one pair of productive agents less. This definition is used in order to formulate the first principle.

**Lemma 2 (Principle 1)** Assume that the candidate policy is consistent. Then in every state, deviations that form a pair that is also formed under the candidate policy, do not have to be checked.

For example, consider a consistent candidate policy and a state on which it forms a homogeneous pair of productive types. On that state no deviations have to be
considered that also match two productive agents. The same holds for homogeneous pairs of unproductive agents and mixed pairs.

The reason why these deviations do not have to be considered is based on two observations. First, note that the match value of the pair that is created under the candidate and the deviation can be canceled out from the inequality that corresponds to checking the deviation. Second, due to consistency the candidate policy creates the same matches except for one pair of productive agents less on the state with two productive agents less. Given these two observations, the deviation is not profitable given there was no profitable deviation on the state with two less agents of the kind which is matched in both the candidate course of action and the deviation. The following example illustrates this point:

Example 3 The example shows how a deviation from the Positive Assortative Policy as in Principle 2 on state \((6, 5)\) can be traced back to a deviation on a smaller set. The deviation considered matches exactly one pair of productive agents:

\[
V_{PAP}(6, 5) = 3m_{HH} + 2m_{LL} + \delta[pV_{PAP}(1, 1) + (1 - p)V_{PAP}(0, 2)] > m_{HH} + \delta[pV_{PAP}(5, 5) + (1 - p)V_{PAP}(4, 6)]
\]

\[
\Leftrightarrow 2m_{HH} + 2m_{LL} + \delta[pV_{PAP}(1, 1) + (1 - p)V_{PAP}(0, 2)] > \delta[pV_{PAP}(5, 5) + (1 - p)V_{PAP}(4, 6)]
\]

\[
\Leftrightarrow V_{PAP}(4, 5) > \delta[pV_{PAP}(5, 5) + (1 - p)V_{PAP}(4, 6)].
\]

Lemma 3 (Principle 2) Assume that the candidate policy is consistent. In every state, deviations that leave two agents in the market of whose types will be created a match after the next arrival independent of the arriving type, do not have to be checked.

For example, consider a deviation which lets two productive types in the market. When, given this deviation, in the next period (upon the next arrival) independently of the type of arrival a match of two productive agents is created, the deviation does not have to be considered. The reason is that there can be constructed a better deviation. It suffices to check against the best deviation instead of all deviations. There exists a more profitable deviation that resembles the excluded one except that the match, which would be made later for sure, is created instantaneously. Because of discounting, this deviation has a higher value.

Lemma 4 (Principle 3) In every state, deviations that create more than one mixed pair do not have to be considered.

The reason is similar to the one of Principle 2. There exists another deviation which is more profitable. The more profitable deviation exploits the supermodularity of the
Figure 5: This graph represents the state space. It visualizes Lemmas 5, 6 and 7. Deviations on states below the lines do not have to be verified. The red line corresponds to Lemma 5 and 6, the green line to Lemma 7.

match value function and creates one homogeneous pair of either type instead of two mixed pairs.

In the next step, we apply the principles to all three candidate matching policies. For each candidate we identify the set of states for which all deviations can be excluded.

**Lemma 5** When the Positive Assortative Policy is the candidate, there is no need to verify deviations on all states that contain more than two agents of the same type.

First, apply Principle 1 to the Positive Assortative Policy: Deviations, that create a homogeneous pair do not have to be checked. This is immediate, as the Positive Assortative Policy creates the maximal number of homogeneous matches.

Second, apply Principle 2 to the Positive Assortative Policy: Deviations, that leave more than one agent of the same type in the market do not have to be checked. If two or more productive agents stay unmatched, there is a match of two productive agents in the next period (when the next agent arrives) when following the Positive Assortative Policy. This is independent of whether the arriving agent is productive or not. The same holds for unproductive agents.

Third, the application of Principle 3 to the Positive Assortative Policy is straightforward.

Finally, we combine the applications of the principles: Consider a state with three productive types. Principle 1 implies that there is no need to consider deviations that match two productive types. Principle 2 states that we do not need to treat deviations that leave two or more productive types unmatched. The only deviations left to consider match two or more productive types with unproductive ones. For those deviations, Principle 2 applies. The proof for unproductive agents is analogous.

**Lemma 6** When the Provident Impatient Policy is the candidate, there is no need to verify deviations on all states that contain more than two agents of the same type.
The applications of the three principles to the Provident Impatient Policy follow similar thoughts, even though the application of the principles is not straight forward. There are states in which the candidates do not create homogeneous matches even though this is possible and in addition there are states in which mixed matches are created.

**Lemma 7** When the Myopic Impatient Policy is the candidate, there is no need to verify deviations on all states that contain more than two productive agents or more than three unproductive agents.

When the Myopic Impatient Policy is the candidate solution, the sets of remaining states that are to be verified one by one is slightly larger than for the other two candidates. The reason is that if two unproductive agents stay in the market, it might happen that one of them is matched with a productive arrival in the next period. Hence, the statement that two unproductive agents get matched in the next period anyways if they are not matched, now does not hold in this case.

The deviations on the remaining states are verified one by one. There are some deviations, which are only unprofitable under certain conditions on parameters. These conditions define the region of the parameter space in which a candidate is optimal. It turns out that these regions constitute a partitioning of the entire parameter space. In particular $m_{HL} \leq m_{HL}^1$ is the condition that ensures creating the mixed match in $(1,1)$ not to be profitable given the guess of the Positive Assortative Policy. $m_{HL} \geq m_{HL}^1$ ensures that creating the mixed match in $(1,1)$ is profitable given the guess is the Provident Impatient Policy. $m_{HL}^2$ takes the corresponding role for the question whether to create the homogeneous match or the mixed match in $(1,2)$. Given the candidate is the Provident Impatient Policy, the condition $m_{HL} \leq m_{HL}^2$ ensures that the homogeneous match is optimal and given the candidate is the Myopic Impatient Policy, the condition $m_{HL} \geq m_{HL}^2$ ensures that the mixed match is value-maximizing.

### 3.2 Extension

In this extension we drop the assumption $m_{HH} \leq 3m_{LL}$. This means we allow for extreme differences in the productivity of the agents. When the value of unproductive agents is very low compared to the value of productive agents, new matching policies can be optimal. Optimality sometimes requires two unproductive agents to wait in the market, when there is nobody else in the market. Therefore we need to define a new class of matching policies.

**Definition 6 Matching Policy $P_k$**

The Matching Policy $P_k$ creates in each state $(x,y)$ the maximal number of pairs of
productive agents. If \( x \) is uneven and \( y \geq 1 \), one mixed matches is created. If a mixed match is created, the maximal number of pairs from the pool of \( y - 1 - k \) unproductive agents is formed. If no mixed match is created, the maximal number of pairs from the pool of \( y - k \) unproductive agents is formed.

Policy \( \mathcal{P}_k \) has similarities to the Myopic Impatient Policy. The difference is that not the maximum number of homogeneous matches of unproductive agents is created. When there are only unproductive agents in the market, the policy always keeps at least \( k \) of them in the market. For example, Policy \( \mathcal{P}_1 \) creates no match in state \( (0, 2) \).

**Proposition 1** For any given \( m_{LL}, m_{HH} \) such that \( m_{HH} > 3m_{LL} \), there exist parameter constellations \( (p, \delta, m_{HL}) \) for which matching policy \( \mathcal{P}_1 \) is optimal.

In order to prove Proposition 1 we apply the same strategy as for proving Theorem 1. An application of the three principles reduces the verification to a finite set of states on which deviations are verified by hand.

To gain intuition for the result, consider a situation in which complementarities in the match value function are weak and discounting is intermediate. In this environment it is optimal to match a single remaining productive type with an unproductive agent whenever possible: The foregone gain of assortative matching is little; however, because the value of a productive agent is very high, the loss from deferring the match of the productive type is considerable. When complementarities are very weak and the value of productive agents is very high compared to unproductive agents, the availability of an unproductive agent that can be matched with a productive agent is desirable. If the value of this availability exceeds the cost of deferring the match of two unproductive agents, it is optimal to keep an unproductive agent as stock in the market. The reasoning is that the possibility to match a potentially arriving productive agent immediately is worth the cost of waiting with the unproductive agents instead of matching them with each other earlier.

The trade-off between the reduction of waiting cost that arises when accumulating unproductive agents and the gain from having unproductive agents available when an productive agent arrives is depicted in Figure 6. In Figure 6 we fix match values such that for each possible \( (p, \delta) \) one of the policies listed in Theorem 1 or the Policy \( \mathcal{P}_1 \) is optimal. The figure illustrates the respective parameter regions \( (p, \delta) \). The red and the blue line are \( \delta^1 \) and \( \delta^2 \) as before. The black line depicts the boundary between the parameter regions on which the Myopic Impatient Policy and Matching Policy \( \mathcal{P}_1 \) are optimal. Matching Policy \( \mathcal{P}_1 \) has no boundary to the Provident Impatient Policy. Policy \( \mathcal{P}_1 \) is optimal for large values of \( p \). The larger \( p \), the more likely is the arrival
Figure 6: Parameter choice: \( m(H, H) = 10, m(L, L) = 1, m(H, L) = 4.8 \). The black line depicts the boundary between the parameter regions on which the Myopic Impatient Policy and Matching Policy \( \mathcal{P}_1 \) are optimal. Matching Policy \( \mathcal{P}_1 \) has no boundary to the Provident Impatient Policy.

of a productive agent and hence the higher is the expected profit from stocking unproductive agents. The second observation is that Policy \( \mathcal{P}_1 \) is optimal for intermediate values of \( \delta \). This reflects the underlying trade-off. On the one hand, the smaller \( \delta \), the higher is the value of having unproductive agents available when an productive agent arrives. On the other hand, the accumulation of unproductive agents is less costly the larger \( \delta \). When discounting is very little, there is no desire to create mixed matches anyways and clearly Policy \( \mathcal{P}_1 \) is not optimal. When discounting is very strong, Policy \( \mathcal{P}_1 \) is not optimal either: The designer does not care about the option to match productive agents earlier, because the option can only increase payoffs in the future. The designer only maximizes current payoff, which means he minimizes the cost of delay for unproductive agents.

As the outline of the intuition has pointed out, the Policy \( \mathcal{P}_1 \) is optimal for low levels of complementarity. The following corollary to Proposition 1 shows that the monotonicity of optimal policies with respect to \( m_{HL} \) extends to Matching Policy \( \mathcal{P}_1 \).

**Corollary 2** If \( m_{HH} > 3m_{LL} \), there exist cut-off levels \( m^3_{HL} < 1/2(m_{LL} + m_{HH}) \) and \( m^4_{HL} \) with \( m^2_{HL} \leq m^3_{HL} \leq m^4_{HL} \) such that:

- If \( m_{HL} \leq m^1_{HL} \), the Positive Assortative Policy is optimal.
- If \( m_{HL} \in [m^1_{HL}, m^2_{HL}] \), the Provident Impatient Policy is optimal.
- If \( m_{HL} \in [m^2_{HL}, m^3_{HL}] \), the Myopic Impatient Policy is optimal.
- If \( m_{HL} \in [m^3_{HL}, \min\{m^4_{HL}, 1/2(m_{LL} + m_{HH})\}] \), the Matching Policy \( \mathcal{P}_1 \) is optimal.
A consequence of Corollary 2 is that the condition $m_{HH} \leq 3m_{LL}$ is not only sufficient for the statement of Theorem 1 to hold but also necessary. Whenever $m_{HH} > 3m_{LL}$, there exist parameters $(p, \delta)$ such that none of the three matching policies from Theorem 1 is optimal if $m_{HL} = 1/2(m_{LL} + m_{HH})$.

Note, that for some parameter constellations $m_{HL}^4 < 1/2(m_{LL} + m_{HH})$. This may happen for some values of $(p, \delta)$ if the ratio $m_{HH}/m_{LL}$ is extremely high. In that case none of the four matching policies is optimal if $m_{HL}$ is close to its upper limit. Following the logic presented in this extension, our guess is that in these special cases it would be optimal to hold even more than one unproductive agent on stock to prepare for the case that several productive agents arrive to the market in row. Proposition 2 describes an extreme case in which the this stock is even infinite.

**Proposition 2** Policy $\mathcal{P}_\infty$ is optimal only if $m_{LL} = 0$.

Policy $\mathcal{P}_\infty$ never matches two unproductive agents. It would potentially accumulate an unbounded stock of unproductive agents. When two unproductive agents generate no value, there is no loss of deferring homogeneous matches of unproductive agents. If in addition the creation of mixed pairs is strictly profitable when a single productive agent is in the market, Policy $\mathcal{P}_\infty$ is uniquely optimal.

We, however, show that apart from the extreme case $m_{LL} = 0$, Policy $\mathcal{P}_\infty$ is never optimal: This means that generically Policy $\mathcal{P}_\infty$ is not optimal. Hence, if a policy that keeps unproductive agents on stock is optimal and there is a positive cost of waiting with the agents, there is a maximum number of unproductive agents above which two of them get matched. The reason for keeping $k$ unproductive agents on stock is to prepare for the event that $k$ productive agents arrive in row. The probability for this event is exponentially decreasing in $k$; the cost of holding an additional agent on stock is, however, not decreasing in $k$. Therefore, at some number of agents the additional cost from accumulating a larger stock exceeds the additional expected profit.

The previous propositions have identified parameter constellations on which none of the three initially introduced policies is optimal. Recall that we interpret the three match values as describing the possible outcomes of a function that maps tuples of types $(\theta_1, \theta_2)$ into match values $m(\theta_1, \theta_2)$. Despite the results of this extension, there is a large number of natural match value functions for which on each parameter constellation either the Positive Assortative Policy, the Provident Impatient Policy, or the Myopic Impatient Policy is optimal. An important functional form, which is regularly used to represent complementarities in matching, is the product case.
Proposition 3 Assume \( m(\theta_1, \theta_2) = \theta_1 \cdot \theta_2 \). For any values of \( H, L, p \) and \( \delta \) one of the following matching policies is optimal: The Positive Assortative Policy, the Provident Impatient Policy, or the Myopic Impatient Policy.

Note, that this result holds irrespective of the ratio \( H/L \). This means that even when the ration is large such that \( m_{HH} > 3m_{LL} \) unproductive agents are never accumulated.

4 Implementation

In our model agents are characterized by their productivity \( \theta \) and their arrival time \( a \). In deriving the welfare-maximizing policies in Section 3, we assumed that the designer can observe agents’ characteristics. We now consider situations in which the designer cannot observe agents’ entry into the market or their productivity, which are thus private information to the agents. Therefore, the designer needs to elicit private information from agents. As the designer’s and agents’ interests are not aligned, e.g. either type of agent wants to be assigned a productive partner, the presence of private information gives rise to an incentive problem. In this section we analyze ways of implementing the welfare-maximizing policies under various information structures.

Henceforth, we assume that the match value is divided equally among the two partners. This splitting-rule can be justified as the Nash Bargaining Solution: A distinctive feature of our model is that the designer has the ability to assign two agents to a pair. Once the match is formed, both agents leave the market and cannot return. This implies that once they are matched, both partners have an outside option of zero. The surplus of the cooperation, hence, equals the match value. We allow the designer to use monetary transfers and assume that agents have quasilinear utility. Thus, they maximize (half of the) match value minus payments. We study the market beginning with the first arrival. Therefore, initially the market is in the recurrent set of all optimal policies, and we may focus more generally on the implementation of welfare-maximizing policies. In particular, we only have to distinguish between the Positive Assortative Policy and the Impatient Policy.

4.1 Observable Arrivals

First, we analyze the case in which both, the designer and agents, observe arrivals whereas types are private information to agents. We focus attention on direct mechanisms which truthfully implement the welfare-maximizing policies. Agents report their type upon arrival. Upon arrival, an agent observes the past reports of all other agents
in the market. Consider an agent that arrives to the market. With a slight abuse of notation we denote by $\mathcal{S}$ the set of agents that are already in the market and by $\Theta_S$ the vector containing their types. We adopt the convention to denote reported types with hats. We call the vector $\hat{\Theta}_S$ market report.

**Properties of the Mechanism.** We concentrate on mechanisms that support efficient exit meaning that an agent who stops to affect current and future matches also stops to receive and pay transfers.\(^{14}\) Particularly, payments do not condition on realized match values and agents cannot reveal their partner’s type to the designer after being matched. This is in accordance with our interpretation that agents leave the market after forming a group. Thus, we focus on payments $\tau^{\Theta_S}(\theta)$ that are charged upon arrival and depend on the market report and the agent’s type.

We study direct mechanisms that have an equilibrium in which welfare is maximized. Agents arrive to the market, observe past reports of all agents in the market, form Bayesian expectations with respect to the future, and maximize their utility given that all other agents report truthfully. Observe that our notion of incentive compatibility coincides with interim incentive compatibility in Bergemann and Välimäki (2010). Given policy $\rho$, market report $\hat{\Theta}_S$, and an agent entering the market with type $\theta$, we denote the (random) variable describing the type of that agent’s partner by $\rho(\hat{\Theta}_S, \theta)$ and the random variable describing the time when the agent will be matched by $t_{\rho(\hat{\Theta}_S, \theta)}$. With this notation in hand, we can formulate the incentive compatibility constraints:

$$
\frac{1}{2}E[e^{-rt_{\rho(\theta_S, \hat{\theta})}}m(\rho(\Theta_S, \theta), \theta)] - \tau^{\Theta_S}(\theta) \geq \frac{1}{2}E[e^{-rt_{\rho(\hat{\theta}_S, \hat{\theta})}}m(\rho(\Theta_S, \hat{\theta}), \theta)] - \tau^{\Theta_S}(\hat{\theta}), \forall \theta, \hat{\theta}, \Theta_S,
$$

(6)

where the expectation is taken with respect to the partner’s type and the matching time.\(^{15}\) There are other perceivable specifications of the agents’ information structure in which agents observe only the number of reports, i.e. the number of agents in the market, or do not observe reports at all. Our assumption complicates the designer’s problem because the more information agents have, the more incentive constraints the designer has to take into account.\(^{16}\) In particular, if the designer can implement the welfare-maximizing policies under this information structure, he can implement the welfare-maximizing policies under any information structure in which agents have less

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\(^{14}\)See also Bergemann and Välimäki (2010). In the dynamic assignment literature an analogous condition is the requirement that mechanisms are online, cf. Gershkov and Moldovanu (2010).

\(^{15}\)As types do not change over time, (6) implies that the mechanism is periodic ex post incentive compatible in the sense of Bergemann and Välimäki (2010).

\(^{16}\)See also Myerson (1986).
information.\footnote{For example, the designer can reveal any information that is missing to agents and use the original mechanism.}

As agents participate in the mechanism voluntarily, additionally, the following \textit{individual rationality} constraints have to be satisfied

\[
\frac{1}{2} \mathbb{E}[e^{-rt(\Theta_S, \theta)} m(\rho(\Theta_S, \theta), \theta)] - \tau^{\Theta_S}(\theta) \geq 0, \ \forall \theta, \Theta_S. \tag{7}
\]

Observe that (7) entails a strong notion of individual rationality because, in addition to observing his type, an agent also observes the market report before he decides whether to participate. Individual rationality requires that the expected utility from participating has to be at least as large as the expected utility the agent would derive from the mechanism if he would not participate, namely, zero.\footnote{As will become clear from the analysis below the outside option could be any sufficiently small positive value.}

As last condition, we impose that the mechanism requires no external injection of money:

\[
\tau^{\Theta_S}(\theta) \geq 0, \ \forall \theta, \Theta_S \tag{8}
\]

i.e., all payments are positive, which implies that the mechanism runs a \textit{balanced budget} at any point in time.

\textbf{Impatient Policy.} If the designer observes arrivals, implementation of the Impatient Policy is straightforward. As the policy does not condition on the type but matches every two consecutive agents, irrespective of their types, there is no need to elicit agents’ private information. Hence, it is possible to set all payments equal to zero. This is individual rational, budget balanced, and satisfies efficient exit.

\textbf{Positive Assortative Policy.} In case of the Positive Assortative Policy the situation is more intricate. The agents’ report affects their match: an agent that reports a productive type will be assigned a productive partner; an agent that reports an unproductive type will be assigned an unproductive partner. It is useful to aggregate the market report $\Theta_S$ into a tuple, similarly, as we aggregated agents’ types in Section 3 in order to describe states. Under the Positive Assortative Policy there can be either no agent $(0,0)$, an agent with a productive report $(1,0)$, an agent with an unproductive report $(0,1)$, or two agents with one productive and one unproductive report $(1,1)$ in the market. Consider an agent that arrives to a market in $(1,1)$. This situation resembles
the static model where all agents are present simultaneously, because, independently of
the third agent’s report, he will be matched immediately with a partner whose report
coincides with his report. Either type of arriving agent would like to form a group with
the productive agent. Supermodularity of the match value implies increasing differences,
\( m_{HH} - m_{HL} \geq m_{HL} - m_{LL} \), which is the discrete analogon of the single-crossing
property in mechanism design with continuous types. Intuitively, increasing differences
means that a productive type values a match with a productive type relative to a match
with an unproductive more, compared to an unproductive type. This gap allows the
designer to construct a payments that makes truthful revelation incentive compatible.
More precisely, the designer can construct payments such that the productive agent
reports the productive type and the unproductive agent reports the unproductive type.
If the unproductive agent would have a larger incentive to report the productive type
than the productive agent himself, separation would also be possible but would induce
both types of agents to lie.

Now consider an agent that arrives to a market in which there is one unproductive
agent, \((0, 1)\). Reporting the productive type is less attractive because of the waiting
costs associated with the time until the arrival of the next productive agent. Therefore,
in this scenario, the increasing differences property, which guaranteed truthful revelation for \((1, 1)\), is challenged by the time constraint. The following Theorem establishes
that, despite these time constraints, the Positive Assortative Policy is implementable
whenever it is welfare-maximizing.

**Theorem 2** There exist payments such that the implementation of the welfare-maximizing
policies is incentive compatible, individual rational, budget-balanced, and supports effi-
cient exit.

We provide a sketch of the proof of Theorem 2. The Impatient Policy is implementable
by setting \( \tau^\theta_s(\theta) = 0 \), for all \( \Theta_s, \theta \).

Consider the Positive Assortative Policy. We start to outline the problem by consid-
ering the incentive constraints. Denote by \( \Delta^\theta(p, r) = \mathbb{E}[e^{-r\theta}] \) the expected discount
factor until the next arrival of type \( \theta \). In accordance with our intuition, \( \Delta^\theta(p, r) \) is
decreasing in \( r \), \( \Delta^L(p, r) \) is decreasing in \( p \), and \( \Delta^H(p, r) \) is increasing in \( p \).

The incentive constraint for the productive type given market report \((0, 1)\) can be
written as

\[
\Delta^H(p, r)m_{HH} - \tau^{(0,1)}(H) \geq m_{HL} - \tau^{(0,1)}(L).
\]  

(9)

Analogously, the incentive constraint for the unproductive type in \((0, 1)\) is

\[
\Delta^H(p, r)m_{HL} - \tau^{(0,1)}(H) \leq m_{LL} - \tau^{(0,1)}(L).
\]

(10)
Rearranging yields the following condition on the payment difference

\[ \Delta^H(p, r)m_{HH} - m_{HL} \geq \tau^{(0,1)}(H) - \tau^{(0,1)}(L) \geq \Delta^H(p, r)m_{HL} - m_{LL}. \]  

(11)

Therefore,

\[ \Delta^H(p, r)m_{HH} - m_{HL} \geq \Delta^H(p, r)m_{HL} - m_{LL} \]  

(12)

is a necessary and sufficient condition for the existence of an incentive compatible payment pair in \((0, 1)\).

Observe that the left side of (12) decreases quicker than the right side as the expected discount factor decreases. Especially for high discount rates \(r\) or low values of \(p\), (12) might be violated. If (12) and the corresponding conditions for the other market reports \(\hat{\Theta}_S\) hold, incentive compatibility and efficient exit are satisfied. The corresponding conditions for market reports \((0, 0), (1, 0), (1, 1)\) are respectively:

\[ \Delta^H(p, r)m_{HH} - \Delta^L(p, r)m_{HL} \geq \Delta^H(p, r)m_{HL} - \Delta^L(p, r)m_{LL}, \]  

(13)

\[ m_{HH} - \Delta^L(p, r)m_{HL} \geq m_{HL} - \Delta^L(p, r)m_{LL}, \]  

(14)

\[ m_{HH} - m_{HL} \geq m_{HL} - m_{LL}. \]  

(15)

As a consequence of increasing differences and \(\Delta^\theta(p, r) \leq 1\), (14) and (15) are always satisfied. Yet, for market reports \((0, 1)\) and \((0, 0)\), there exist parameters \(p\) and \(r\) for which (12) and (13) do not hold. Comparing (12) and (13) to the boundary of the Positive Assortative Policy (3), shows, however, that for all parameters for which the Positive Assortative Policy is welfare-maximizing (12) and (13) hold. Thus, there exists an incentive compatible payment difference for all market reports.

The proof proceeds by charging the unproductive type the maximum individual rational payment for all possible market reports,

\[ \tau^{(0,0)}(L) = \tau^{(1,0)}(L) = \Delta^L(p, r)m_{LL} \]  

(16)

\[ \tau^{(0,1)}(L) = \tau^{(1,1)}(L) = m_{LL} \]  

(17)

Given the unproductive type’s payment, we choose the maximal payment for the productive type such that the payment pair is incentive compatible, cf. e.g. (11). Explic-
Individually, we obtain

\[
\tau^{(0,0)}(H) = \Delta^L(p,r)m_{LL} + \Delta^H(p,r)m_{HH} - \Delta^L(p,r)m_{HL},
\]

\[
\tau^{(0,1)}(H) = m_{LL} + \Delta^H(p,r)m_{HH} - m_{HL},
\]

\[
\tau^{(1,0)}(H) = \Delta^L(p,r)m_{LL} + m_{HH} - \Delta^L(p,r)m_{HL},
\]

\[
\tau^{(1,1)}(H) = m_{LL} + m_{HH} - m_{HL}.
\]

Individual rationality of the unproductive type plus incentive compatibility of the productive type yield individual rationality for the productive type. The proof concludes by verifying that all payments are positive.

The proof of Theorem 2 reveals that whenever the Positive Assortative Policy is welfare-maximizing, (12) holds. Put differently, if positive assortative matching fails to be incentive compatible, it also fails to be welfare-maximizing. Intuitively, if (12) is violated, the incentive for an unproductive agent to report the productive type is stronger than the incentive for a productive agent. This means that an unproductive agent’s gain of being matched with a productive agent instead of being matched with an unproductive agent outweighs the respective loss for a productive agent.

Note that the mechanism constructed in the proof of Theorem 2 generates revenues. Firstly, we set the unproductive type’s expected utility to zero for all market reports by charging the highest payment that is individual rational and, secondly, choose the maximal incentive compatible payment for the productive type. Either by redistributing collected revenues, independently of the type, or by reducing the payments in the first place, the mechanism could account for more lucrative outside options or the constraint of having a budget exactly equal zero.

Observe that the payments of the unproductive type in Theorem 2 depend on the market report only through the presence or absence of an agent in the market whose type coincides with the agent’s reported type, i.e., the payments are equal for market reports (0,0), (1,0) and for market reports (0,1), (1,1). This resembles the fact that under the Positive Assortative Policy an agent’s report fixes his (future) partner’s type and the expected time until the match.

4.2 Extension: Simple Payments

The payments that implement the Positive Assortative Policy in Theorem 2 depend on the market report. In applications it is often desirable to use simple mechanisms that condition on as few parameters as possible. In this section, we examine under which conditions the Positive Assortative Policy is implementable with transfers that depend
solely on the reported type but not on the market report. In addition, this sheds light on the relation of implementation of positive assortative policy in our dynamic model and in the static analogon where all agents are in the market at the same time. As the market reports are a consequence of the time friction and do not exist in the static model, our analysis investigates when positive assortative matching can be implemented in the dynamic model with ‘static’ transfers, i.e., transfers that depend only on the type.

**Proposition 4** The Positive Assortative Policy is implementable with payments that depend only on the type if

\[
m_{HL} \leq \Delta^H(p,r)\frac{m_{HH}}{2} + \Delta^L(p,r)\frac{m_{LL}}{2}. \tag{22}
\]

This parameter region is a strict subset of the parameter region where the Positive Assortative Policy is optimal.

In Theorem 2 we proved that the conditions ensuring incentive compatibility hold, cf. (12)-(15). Note, however, that the conditions differ across market reports. Therefore, the main issue is to find a single payment pair \((\tau(H), \tau(L))\) that is incentive compatible for all possible market reports. To this end, it is instructive to consider Figure 7. Rows correspond to market reports. The green part of each line marks the region where payment differences are incentive compatible, whereas payment differences that lie within the red region are not incentive compatible. We are looking for a payment difference \(\tau(H) - \tau(L)\) which lies in the green interval across all market reports. As the boundaries vary significantly with \(p\) and \(r\), existence of such a payment difference is not guaranteed. From Figure 7 we infer that the left boundary of market report \((1, 0)\) and the right boundary of market report \((0, 1)\) are most restrictive. Intuitively, \((1, 0)\) is the most attractive market report for reporting the productive type and thus defines a lower bound on \(\tau(H) - \tau(L)\), whereas \((0, 1)\) is the most attractive market report for reporting the unproductive type and defines an upper bound on \(\tau(H) - \tau(L)\). Combining these two conditions, yields (22). Individual rationality and budget-balancedness are readily checked.

Observe that (4) holds as \(r\) goes to zero, that is, the time friction vanishes. Recall that increasing differences, \(m_{HH} - m_{HL} \geq m_{HL} - m_{LL}\), guarantees the existence of an incentive compatible payment difference in the static model. Above, we argued that the time friction challenges increasing differences, cf. (12). If the time constraints fades, however, we can draw on static payments to implement the Positive Assortative Policy. Similarly, an increase in complementarities, i.e. a decrease in \(m_{HL}\), strengthens...
Figure 7: Incentive compatible payment differences

the increasing differences property and thus allows for an implementation with static payments.

4.3 Extension: Asymmetric Match Value Splits

Hitherto, we assumed that partners share their match value equally. While this seems intuitive if partners are homogeneous, i.e. have the same type, one can imagine other sharing rules in case of mixed pairs. Also, an appropriate sharing rule might alleviate the incentive problem. This section investigates under which conditions there exists a sharing rule which induces truthful revelation of types without further intervention, that is to say, without incentivizing agents with payments. These would be the conditions we would expect to prevail in small unregulated markets. In the following, denote by \( \alpha \) the productive type’s share of the match value when he forms a group with an unproductive type.

**Proposition 5** There exists a share \( \alpha \) such that the welfare-maximizing policies are implementable without payments if and only if the welfare-maximizing policies are implementable with static payments.

**Outline of the Proof.** Recall that we implemented the Impatient Policy in Theorem 2 without payments. Hence, we may concentrate on the Positive Assortative Policy. Consider the incentive constraints for market report \( (0, 1) \). The productive agent reports truthfully if

\[
\Delta^H(p, r) \frac{m_{HH}}{2} \geq \alpha m_{HL}.
\]  

(23)

Analogously, the unproductive agent reports truthfully if

\[
\frac{m_{LL}}{2} \geq (1 - \alpha) \Delta^H(p, r) m_{HL}.
\]  

(24)
Observe that the incentive constraint of the productive agent gives an upper bound on \( \alpha \), whereas the incentive constraint of the unproductive agent gives a lower bound on \( \alpha \). We proceed similar for the remaining market reports. Again, the most restrictive conditions arise in \((1, 0)\) and \((0, 1)\). In particular, (23) yields the lowest upper bound on \( \alpha \), whereas the incentive constraint of the unproductive type for market report \((1, 0)\)

\[ \Delta^L(p, r) \frac{m_{LL}}{2} \geq (1 - \alpha)m_{HL}. \]  

(25)
yields the highest lower bound on \( \alpha \). Thus, an incentive compatible sharing rule exists if

\[ \frac{1}{2} \Delta^H(p, r)m_{HH} \geq \frac{m_{HL}}{2} - \Delta^L(p, r) \frac{m_{LL}}{2}. \]  

(26)

Reformulating (26) shows that it coincides with (22), which concludes the proof.

To get some intuition for Proposition 5, reformulate the crucial incentive constraints (23) and (25) as if agents in the mixed group would share the match value equally and interpret the difference between the equal split and the \( \alpha \) split as a substitute for payments:

\[ \Delta^H(p, r) \frac{m_{HH}}{2} \geq \frac{m_{HL}}{2} + \left( \alpha m_{HL} - \frac{m_{HL}}{2} \right), \]  

(27)

\[ \Delta^L(p, r) \frac{m_{LL}}{2} \geq \frac{m_{HL}}{2} - \left( \alpha m_{HL} - \frac{m_{HL}}{2} \right). \]  

(28)

Recall that the boundaries on the difference of incentive compatible, static payments in the proof of Proposition 4 are determined by exactly the same incentive constraints:

\[ \Delta^H(p, r) \frac{m_{HH}}{2} \geq \frac{m_{HL}}{2} + (\tau(H) - \tau(L)), \]  

(29)

\[ \Delta^L(p, r) \frac{m_{LL}}{2} \geq \frac{m_{HL}}{2} - (\tau(H) - \tau(L)). \]  

(30)

Because \( \alpha \) is contained in \([0, 1]\), (27) and (28) provide less flexibility than (29) and (30). Intuitively, the value that can be redistributed through a sharing rule is bounded by the total match value that is generated in the mixed match, whereas there is no bound on the payment difference. Hence, if there exists an incentive compatible sharing rule, we can also find an incentive compatible, static payment difference. Proposition 5 shows, however, that the converse is true as well. Therefore, we infer that for any incentive compatible pair of static payments, the payment difference never exceeds the total match value. Put differently, incentive compatibility does not require extreme
transfer differences.

Observe that (25) implies that for any incentive compatible sharing rule $\alpha$, it holds $\alpha > \frac{1}{2}$, i.e., the productive type has to receive a larger share of the match value when he forms a group with an unproductive type. In the Positive Assortative Policy the mixed group never occurs on path. Thus, changes in the sharing rule only affect the attractiveness of deviations. Particularly, distorting the sharing rule in favor of the productive type deters the unproductive type from misreporting. This effect turns out to be more important than deterring a misreport of the productive agent.

4.4 Unobservable Arrivals

Depending on the organizational details of the market, the designer might not observe agents' arrivals to the market. Instead, agents report their arrival to the designer. Given the welfare-maximizing policies, agents may want to exploit this additional source of private information by strategically delaying their arrival report. We maintain the assumption of private types. This renders implementation of the welfare-maximizing policy a multidimensional screening problem. The current section examines conditions under which the designer can overcome this additional challenge and implement the welfare-maximizing policies.

We focus on the case where arriving agents observe all previous reports, i.e., reports of types and reports of arrival times. As in Section 4.1, we focus on incentive compatible, individual rational, budget balanced, direct mechanisms. Consider an agent that arrives to the market. Denote by $\hat{S}$ the set of agents in the market who reported their arrival, by $\Theta_{\hat{S}}$ the vector of their types, and by $\hat{\Theta}_{\hat{S}}$ the corresponding market report. Payments only depend on the reported type and the market report and thus satisfy efficient exit. For an agent with characteristics $(\theta, a)$, we denote the reported arrival time by $\hat{a}$, $\hat{a} \geq a$. Only agents who have reported their arrival to the market may report their type.

Because of substantially different issues, we discuss the Positive Assortative Policy and the Impatient Policy separately.

**Positive Assortative Policy.** When type spaces have more than one dimension, incentive constraints pose a severe challenge to the design of incentive compatible mechanisms as one has to account for double deviations, i.e., deviations in several dimensions at the same time. Surprisingly, the mechanism constructed in the proof of
Theorem 2 also implements the Positive Assortative Policy with unobservable arrivals.

**Proposition 6** The Positive Assortative Policy is implementable, when both, arrivals and types, are private information to the agents.

To prove Proposition 6 we show that agents report their type truthfully and also the timing of the report remains unchanged, which means that agents reveal their arrival immediately, i.e. truthfully.

Payments are designed in a way closely related to the ones described in (16)-(21): When an agent reveals his arrival and his type the same time, payments are as described in (16)-(21). Whenever an agent reveals his arrival strictly before his type, he is punished by a flat payment of $m_{HH}$.

To tackle the issue of double deviations in the framework of our model, we divide the problem of showing incentive compatibility into two steps:

(i) First, we show that whenever agents report their type, they report truthfully.

(ii) Second, we argue that given agents report their type truthfully, agents report their arrival time truthfully.

By the memorylessness of the Poisson process, the incentive problem faced by an agent at an arbitrary point in time is the same as the incentive problem at the time of the last arrival. This latter problem, however, resembles the incentive problem with observable arrivals. As the payments solve the incentive problem with observable arrivals, we deduce that (i) holds.

We now argue for (ii). Recall that the payments constructed in the proof of Theorem 2 are constructed such that the unproductive type’s expected utility is zero for all market reports, and that given the unproductive type’s payments the productive type’s payments are chosen maximal under the constraint that the payment pair is incentive compatible with observable arrivals. Given our specification of payments, agents report arrival time and type simultaneously. By the first step, agents report their type truthfully. Under the Positive Assortative Policy an agent’s report fixes his match partner’s type. The agent’s partner is, depending on the market report, either the next agent who arrives to the market and whose type coincides with the agent’s report or an agent with the same type that is already in the market. It remains to be shown that agents want to report their arrival as early as possible. By memorylessness of the Poisson process, delaying an arrival may only be profitable for an agent if the market
report changes compared to the market report at the arrival time. By our choice of payments, the unproductive type receives zero expected utility upon arrival for every market report. Therefore, it is an optimal strategy for the unproductive agent to report his arrival time truthfully. Denote by $U_{\Theta}^{\Theta_S}$ the expected utility of the productive type at the point of his arrival if he arrives to a market with market report $\Theta_S$. Given positive assortative matching and our payments, we obtain for the productive type:

$$U_{H}^{0,0} = U_{H}^{1,0} = \Delta^L(p,r)(m_{HL} - m_{LL}), \quad U_{H}^{0,1} = U_{H}^{1,1} = m_{HL} - m_{LL}. \quad (31)$$

From (31) we see that the productive agent’s expected utility is highest if an unproductive agent is already in the market. Thus, if the productive agent arrives in $(1,1)$ or $(0,1)$, he reports his arrival immediately. On the other hand, if the productive agent arrives in $(0,0)$ or $(1,0)$, he might consider waiting for the arrival of an unproductive agent before he reports his arrival to get a higher level of expected utility. Yet, the waiting time until the next arrival of an unproductive agent discounts future payoffs with an expected discount factor of at least $\Delta^L(p,r)$ thereby mitigating the advantage of waiting. Hence, also in $(0,0)$ and $(1,0)$ it is unprofitable for the productive type to delay his arrival report. This concludes the proof of Step (ii).

Jointly, (i) and (ii) imply that the Positive Assortative Policy together with our payments is incentive compatible even when arrivals are unobservable. Observe that also individual rationality, budget-balancedness, and efficient exit remain satisfied, completing the construction of the mechanism.

**Impatient Policy.** Implementing the Impatient Policy in a market where the designer can observe arrivals turns out to be straightforward. As the designer may ignore agents’ private information to implement the Impatient Policy, he can abstain from using incentivizing payments. Yet, if the designer cannot observe agents’ arrivals, which is information relevant for implementing the welfare-maximizing policies, implementation of the Impatient Policy becomes more difficult.

In contrast to the Positive Assortative Policy, in the Impatient Policy the agent’s reported type does not fix his match partner’s type. The agent’s partner is, depending on the status of the market, either the only agent that is present in the market or the next agent that arrives to the market, irrespective of his type. If the designer asks

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19 To avoid issues with large states that occur because several agents report their arrival simultaneously, we punish agents reporting the same arrival time with a sufficiently high payment, say, $m_{HH}$. In equilibrium this entails no welfare loss. The deviations checked in the outline of the proof are, thus, an upper bound for the most profitable deviation.
an agent for his type, future agents may condition their reporting strategy on that report. Consider, for example, a productive agent that arrives to a market in which an unproductive agent is present. If the productive agent reveals his arrival immediately, he will form a group with the unproductive agent. On the other hand if the productive agent delays his arrival report until after the next arrival, he has the opportunity to be matched with a productive agent. Therefore, depending on the parameter constellation, it might be profitable for the productive agent to delay his arrival report.

The designer can circumvent this problem by separating the agents’ arrival report from their type report. To implement the Impatient Policy, the designer only needs agents’ arrival times but not their types. If the designer asks agents only for their arrival time, future agents only observe arrival reports. Given that agents only observe arrival reports, it is optimal for the agents to report their arrival as early as possible, i.e., truthfully. Therefore, anticipating the agents’ informational advantage from reported types, the designer strategically chooses not to ask the agents for their type in order to implement the Impatient Policy.

Combining the insights of the last two sections, we find that even if the designer does not observe arrivals to the market, the welfare-maximizing policies are implementable.

4.5 Concluding Remarks

Remark 1. When treating the implementation of an Impatient Policy in the case with unobservable arrivals, we have shown that strategically not receiving reports from agents can be beneficial. When transferring this thought, we can construct another contract which implements the Positive Assortative Policy when arrivals are observable. In general, the Positive Assortative Policy uses information about the agents’ types. Hence, the designer needs to ask for reports about the types upon arrival. However, if (and only if) an agent arrives to an empty market, the designer has no need to obtain this information immediately, but only upon arrival of a second agent. The reason is that there is no decision to be taken when only one agent is present. The designer could, hence, set up a contract, in which the agent that arrives to an empty market does not immediately report his type. When the second agent arrives, both agents report simultaneously. The difference to the contract studied in detail in Section 4.1 is that the agent arriving second does not know the type of the agent that is present. The most critical situation for implementability in Section 4.1 was when the market consists of one unproductive agent and small values of $p$. A similar situation arises in the modified contract when $p$ is small. The only difference is that the second agent
does not know that the first agent is unproductive but he attaches a high probability to this event.

Remark 2. Observe that throughout Section 4, we did not use the assumption of Section 3 that the value of the productive pair is not too large compared to the value of the unproductive pair. Hence, our implementation results carry over to the case \( m_{HH} > 3m_{LL} \) whenever the Positive Assortative Policy and the Impatient Policy are welfare-maximizing.

5 Conclusion

This paper studies a small dynamic matching market organized by a central authority. Agents of different types that arrive to the market according to a discrete process are matched by a social planner. The model is flexible with respect to four key features: The degree of complementarity of the partners’ characteristics in the match value function, the relative size of absolute values of output generated by the two possible homogeneous matchings, the probability distribution of arriving agents’ types and the patience represented by discounting. We first address the optimal matching policies under complete information. We develop a tool that helps us to solve for the optimal matching policy in closed form without imposing any restriction on the policy. Whenever the agents’ productivities do not differ too much, one of three policies is optimal: The Positive Assortative Policy, the Provident Impatient Policy, or the Myopic Impatient Policy. The social planner is more willing to abstain from creating mixed matches in order to wait for positive assortative matchings when discounting is little or complementarities are strong. This has two immediate implications: a) The optimality of possitive assortative matchings in static matching is robust to small discounting frictions. b) When due to impatience mixed matches are created in models of search and matching, this might be welfare-enhancing. The role of the distribution of arriving agents’ types is more sophisticated: The designer might abstain from mixed matches only for intermediate probabilities of productive arrivals. When the match value of two productive agents exceeds the match value of the unproductive counterpart by far, it is sometimes optimal to stock unproductive agents in the market in order to insure that arriving productive agents can get paired immediately. In the second part, we consider implementability of the optimal policy in the presence of private information. We prove implementability of the optimal matching policy when agents have private information about their types and can hide their arrival to the market. Finally, we identify situations in which the market organizer can abstain from using monetary in-
centives.
The simple structure of our model helps to expose the trade-off between accumulating agents to achieve positive assortative matchings and matching agents early in order to avoid waiting costs. Analyzing a model with a continuum of types, but discrete arrivals, would allow for a more detailed comparison between the centralized matching market and decentralized search and matching models. This is an interesting question to be examined.
References


A Appendix

Proof of Lemma 1:

The proof of Theorem 1 uses Lemma 2 to 7, which are, hence, proven first.

Proof of Lemma 2:

Fix an arbitrary state \((x, y) \in S\). Denote the candidate policy by \(\rho\). Consider a deviation that matches on state \((x, y)\) \(d_{HH}, d_{HL}\), and \(d_{LL}\) many homogeneous pairs of productive agents, mixed pairs, and homogeneous pairs of unproductive agents, respectively.

Assume that \(\rho_{HH}(x, y) > 0\) and \(d_{HH} > 0\). The proof is analogous for the two other cases \(\rho_{HL}(x, y), d_{HL} > 0\) and \(\rho_{LL}(x, y), d_{LL} > 0\). The condition for the deviation to be unprofitable is

\[
V(x, y) = \rho_{HH}(x, y)m_{HH} + \rho_{HL}(x, y)m_{HL} + \rho_{LL}(x, y)m_{LL} \\
+ \delta[pV(x' + 1, y') + (1 - p)V(x', y' + 1)] \\
\geq d_{HH}m_{HH} + d_{HL}m_{HL} + d_{LL}m_{LL} \\
+ \delta[pV(x'' + 1, y'') + (1 - p)V(x'', y'' + 1)].
\]

with \(x' = x - 2\rho_{HH}(x, y) - \rho_{HL}(x, y), \quad y' = y - 2d_{LL}(x, y) - d_{HL}(x, y)\)

and \(x'' = x - 2d_{HH}(x, y) - d_{HL}(x, y), \quad y'' = y - 2d_{LL}(x, y) - d_{HL}(x, y)\).
Cancelling \( m_{HH} \) (a pair that is always formed) on both sides gives

\[
(\rho_{HH}(x, y) - 1)m_{HH} + \rho_{HL}(x, y)m_{HL} + \rho_{LL}(x, y)m_{LL} + V(x', y') \\
\geq (d_{HH} - 1)m_{HH} + d_{HL}m_{HL} + d_{LL}m_{LL} + \delta[pV(x'' + 1, y'') + (1 - p)V(x'', y'' + 1)].
\]

By consistency, \((\rho_{HH}(x, y) - 1)m_{HH} + \rho_{HL}(x, y)m_{HL} + \rho_{LL}(x, y)m_{LL} + V(x', y') = V(x - 2, y)\) such that the inequality becomes

\[
V(x - 2, y) \\
\geq (d_{HH} - 1)m_{HH} + d_{HL}m_{HL} + d_{LL}m_{LL} + \delta[pV(x'' + 1, y'') + (1 - p)V(x'', y'' + 1)].
\]

The deviation \(d_{HH} - 1, d_{HL}\) and \(d_{LL}\) is feasible on \((x - 2, y)\), which is a smaller state than \((x, y)\). Hence, given that on the smaller state no deviation is profitable, the inequality holds and deviation \((d_{HH}, d_{HL}, d_{LL})\) is not profitable on \((x, y)\).

\[
\blacksquare
\]

**Proof of Lemma 3:**

Fix an arbitrary state \((x, y) \in S\). Denote the candidate policy by \(\rho\). Consider a deviation that matches on state \((x, y)\) \(d_{HH}, d_{HL}\), and \(d_{LL}\) many homogeneous pairs of productive agents, mixed pairs, and homogeneous pairs of unproductive agents, respectively. The value of the one-period deviation \((d_{HH}, d_{HL}, d_{LL})\) is

\[
V_d(x, y) = d_{HH}m_{HH} + d_{HL}m_{HL} + d_{LL}m_{LL} + \delta[pV(x'' + 1, y'') + (1 - p)V(x'', y'' + 1)]
\]

with \(x'' = x - 2d_{HH}(x, y) - d_{HL}(x, y)\) and \(y'' = y - 2d_{LL}(x, y) - d_{HL}(x, y)\).

Suppose the deviation \((d_{HH}, d_{HL}, d_{LL})\) is designed such that both \(\rho_{HH}(x'' + 1, y'') > 0\) and \(\rho_{HH}(x'' + 1, y'' + 1) > 0\). The two other cases \(\rho_{HL}(x'' + 1, y''), \rho_{HL}(x'', y'' + 1) > 0\) and \(\rho_{LL}(x'' + 1, y''), \rho_{LL}(x'', y'' + 1) > 0\) are analogous.

The proof consists of constructing another deviation \((d'_{HH}, d'_{HL}, d'_{LL})\) on \((x, y)\) that has a higher value than deviation \((d_{HH}, d_{HL}, d_{LL})\). If deviation \((d'_{HH}, d'_{HL}, d'_{LL})\) has a higher value and is not profitable, deviation \((d_{HH}, d_{HL}, d_{LL})\) is not profitable either. As all deviations - including deviation \((d'_{HH}, d'_{HL}, d'_{LL})\) - have to be non-profitable, it is not necessary to verify the non-profitability of deviation \((d_{HH}, d_{HL}, d_{LL})\).
Consider \((d'_{HH}, d'_{HL}, d'_{LL}) = (d_{HH} + 1, d_{HL}, d_{LL})\). The constructed deviation is feasible, as deviation \((d_{HH}, d_{HL}, d_{LL})\) is feasible and \(\rho_{HH}(x'', y'' + 1) > 0\).

Note that by consistency of \(\rho\), \(\rho_{HH}(x'' + 1, y'') > 0\) and \(\rho_{HH}(x'', y'' + 1) > 0\) follows \(V(x'' + 1, y'') = m_{HH} + V(x'' - 1, y'')\) and \(V(x'', y'' + 1) = m_{HH} + V(x'' - 2, y'' + 1)\). Using this, \(\delta < 1\), and the definition of \((d'_{HH}, d'_{HL}, d'_{LL})\) gives

\[
V_d(x, y) = d_{HH}m_{HH} + d_{HL}m_{HL} + d_{LL}m_{LL} + \delta m_{HH}
\]

\[
< (d_{HH} + 1)m_{HH} + d_{HL}m_{HL} + d_{LL}m_{LL}
\]

\[
+ \delta[pV(x'' - 1, y'') + (1 - p)V(x'' - 2, y'' + 1)]
\]

\[
= d_{HH}'m_{HH} + d_{HL}'m_{HL} + d_{LL}'m_{LL}
\]

\[
+ \delta[pV(x'' - 1, y'') + (1 - p)V(x'' - 2, y'' + 1)].
\]

The latter expression is the value of deviation \((d'_{HH}, d'_{HL}, d'_{LL})\).

\[\Box\]

**Proof of Lemma 4:**

Fix an arbitrary state \((x, y) \in S\). Denote the candidate policy by \(\rho\). Consider a deviation that matches on state \((x, y) d_{HH}, d_{HL} > 2\), and \(d_{LL}\) many homogeneous pairs of productive agents, mixed pairs, and homogeneous pairs of unproductive agents, respectively. The proof strategy corresponds to the strategy of the proof of Lemma 3:

We construct another deviation \((d'_{HH}, d'_{HL}, d'_{LL})\) on \((x, y)\) that has a higher value than deviation \((d_{HH}, d_{HL}, d_{LL})\).

Consider \((d'_{HH}, d'_{HL}, d'_{LL}) = (d_{HH} + 1, d_{HL} - 2, d_{LL} + 1)\). The constructed deviation is feasible, as deviation \((d_{HH}, d_{HL}, d_{LL})\) is feasible. Denote the value of the one-period deviation \((d_{HH}, d_{HL}, d_{LL})\) by \(V_d(x, y)\), and the value of the deviation \((d'_{HH}, d'_{HL}, d'_{LL})\) by \(V_{d'}(x, y)\). The difference between the values is given by

\[
V_{d'}(x, y) - V_d(x, y) = m_{HH} + m_{LL} - 2m_{HL} > 0.
\]

The difference is positive by supermodularity of the match value function.

\[\Box\]

**Proof of Lemma 5:**
Denote the Positive Assortative Policy by $\rho^{PAP}$. By construction, $\rho^{PAP}$ is consistent. Consider an arbitrary state $(x, y)$ with $x \geq 3$. Denote a deviation on $(x, y)$ by $(d_{HH}, d_{HL}, d_{LL})$.

By construction, $\rho^{PAP}_{HH}(x, y) > 0$. By Lemma 2, no deviation with $d_{HH} > 0$ has to be verified. When following deviation $(d_{HH}, d_{HL}, d_{LL})$ in state $(x, y)$, the minimum number of productive agents after the next arrival is $x - 2d_{HH} - d_{HL}$. As $\rho^{PAP}_{HH}(x', y') > 0$, $\forall x' \geq 2$, deviations with $x - 2d_{HH} - d_{HL} \geq 2$ do not have to be checked by Lemma 3. Deviations with $d_{HL} \geq 2$ do not have to be checked by Lemma 4. As $x \geq 3$, the set of deviations $(d_{HH}, d_{HL}, d_{LL})$ on $(x, y)$ that remain to be checked is empty. By an analogous proof, no deviation on $(x, y)$ with $x \geq 3$ has to be checked.

**Proof of Lemma 6:**

Denote the Provident Impatient Policy by $\rho^{PIP}$. By construction, $\rho^{PIP}$ is consistent. The proof of Lemma 6 is analogous to the proof of Lemma 5: Consider an arbitrary state $(x, y)$ with $x \geq 3$. Denote a deviation on $(x, y)$ by $(d_{HH}, d_{HL}, d_{LL})$.

By construction, $\rho^{PIP}_{HH}(x, y) > 0$. By Lemma 2, no deviation with $d_{HH} > 0$ has to be verified. When following deviation $(d_{HH}, d_{HL}, d_{LL})$ in state $(x, y)$, the minimum number of productive agents after the next arrival is $x - 2d_{HH} - d_{HL}$. As $\rho^{PIP}_{HH}(x', y') > 0$, $\forall x' \geq 2$, deviations with $x - 2d_{HH} - d_{HL} \geq 2$ do not have to be checked by Lemma 3. Deviations with $d_{HL} \geq 2$ do not have to be checked by Lemma 4. As $x \geq 3$, the set of deviations $(d_{HH}, d_{HL}, d_{LL})$ on $(x, y)$ that remain to be checked is empty. By an analogous proof, no deviation on $(x, y)$ with $y \geq 3$ has to be checked.

**Proof of Lemma 7:**

Denote the Myopic Impatient Policy by $\rho^{MIP}$. By construction, $\rho^{MIP}$ is consistent.
Consider an arbitrary state \((x, y)\) with \(x \geq 3\). Denote a deviation on \((x, y)\) by 
\((d_{HH}, d_{HL}, d_{LL})\).

For this case, the proof is again analogous to Lemma 4: By construction, \(\rho_{HH}^{MIP}(x, y) > 0\). By Lemma 2, no deviation with \(d_{HH} > 0\) has to be verified. When following deviation \((d_{HH}, d_{HL}, d_{LL})\) in state \((x, y)\), the minimum number of productive agents after the next arrival is \(x - 2d_{HH} - d_{HL}\). As \(\rho_{HH}^{MIP}(x', y') > 0\), \(\forall x' \geq 2\), deviations with \(x - 2d_{HH} - d_{HL} \geq 2\) do not have to be checked by Lemma 3. Deviations with \(d_{HL} \geq 2\) do not have to be checked by Lemma 4. As \(x \geq 3\), the set of deviations that remain to be checked is empty.

Consider an arbitrary state \((x'', y'')\) with \(y'' \geq 4\). By construction, \(\rho_{LL}^{MIP}(x'', y'') > 0\).

By Lemma 2, no deviation with \(d_{LL} > 0\) has to be verified. When following deviation \((d_{HH}, d_{HL}, d_{LL})\) in state \((x'', y'')\), the minimum number of unproductive agents after the next arrival is \(y'' - 2d_{LL} - d_{HL}\). As \(\rho_{LL}^{MIP}(x', y') > 0\), \(\forall y' \geq 3\), deviations with \(y'' - 2d_{LL} - d_{HL} \geq 3\) do not have to be checked by Lemma 3. Deviations with \(d_{HL} \geq 2\) do not have to be checked by Lemma 4. All deviations that need to be verified, hence, satisfy

\[
\begin{align*}
    d_{LL} &= 0, \\
    y'' - 2d_{LL} - d_{HL} &< 3, \\
    d_{HL} &< 2.
\end{align*}
\]

As \(y'' \geq 4\), the set of deviations \((d_{HH}, d_{HL}, d_{LL})\) that satisfy (35) to (37) is empty. Hence, no deviation on \((x'', y'')\) with \(y'' \geq 4\) has to be checked.

\[\blacksquare\]

**Proof of Theorem 1:**

For each candidate policy, Lemmas 5 to 7 identify the set of states on which each possible deviation has to be verified for its unprofitability by hand. For all parameter constellations \((p, \delta, m_{HH}, m_{LL}, m_{HL})\) such that \(m_{HL} \notin \{m_{HL}^1, m_{HL}^2\}\) deviations from the respective candidate policies give a strictly lower payoff, as is shown in the following. This implies uniqueness.

**Claim 1:** The Positive Assortative Policy is optimal for all parameter constellations \((p, \delta, m_{HH}, m_{LL}, m_{HL})\) such that \(m_{HL} \leq m_{HL}^1\).
The value function $V_{PAP}$ at states in the respective recurrent set is given by the following equations:

\[
\begin{align*}
V_{PAP}(1, 0) &= \delta[pV_{PAP}(2, 0) + (1 - p)V_{PAP}(1, 1)], \\
V_{PAP}(0, 1) &= \delta[pV_{PAP}(1, 1) + (1 - p)V_{PAP}(0, 2)], \\
V_{PAP}(1, 1) &= \delta[pV_{PAP}(2, 1) + (1 - p)V_{PAP}(1, 2)], \\
V_{PAP}(2, 0) &= m_{HH} + \delta[pV_{PAP}(1, 0) + (1 - p)V_{PAP}(0, 1)], \\
V_{PAP}(0, 2) &= m_{LL} + \delta[pV_{PAP}(1, 0) + (1 - p)V_{PAP}(0, 1)], \\
V_{PAP}(2, 1) &= m_{HH} + \delta[pV_{PAP}(1, 1) + (1 - p)V_{PAP}(0, 2)], \\
V_{PAP}(1, 2) &= m_{LL} + \delta[pV_{PAP}(2, 0) + (1 - p)V_{PAP}(1, 1)].
\end{align*}
\]

Define $V_{PAP}(0, 0) := \delta[pV_{PAP}(1, 0) + (1 - p)V_{PAP}(0, 1)]$. The value at any state $(x, y)$ is obtained as follows:

\[
\begin{align*}
V_{PAP}(x, y) &= \rho_{HH}^{PAP}(x, y) + \rho_{LL}^{PAP}(x, y) + \rho_{HL}^{PAP}(x, y) + V_{PAP}(x', y') \\
\text{with} \quad x' &= x - 2\rho_{HH}^{PAP}(x, y) - \rho_{HL}^{PAP}(x, y) < 2 \\
\text{and} \quad y' &= y - 2\rho_{LL}^{PAP}(x, y) - \rho_{HL}^{PAP}(x, y) < 2.
\end{align*}
\]

Before verifying the Positive Assortative Policy, we derive a couple of useful relationships. Using the definition of $V_{PAP}(0, 0)$ and inserting $V_{PAP}(2, 0)$, $V_{PAP}(0, 2)$, $V_{PAP}(2, 1)$, and $V_{PAP}(1, 2)$, the system (38) - (44) can be reformulated to

\[
\begin{align*}
V_{PAP}(0, 0) &= \delta[pV_{PAP}(1, 0) + (1 - p)V_{PAP}(0, 1)], \\
V_{PAP}(1, 0) &= \delta[p(m_{HH} + V_{PAP}(0, 0)) + (1 - p)V_{PAP}(1, 1)], \\
V_{PAP}(0, 1) &= \delta[pV_{PAP}(1, 1) + (1 - p)(m_{LL} + V_{PAP}(0, 1))], \\
V_{PAP}(1, 1) &= \delta[p(m_{HH} + V_{PAP}(0, 1)) + (1 - p)(m_{LL} + V_{PAP}(1, 0))].
\end{align*}
\]

Firstly, consider $V_{PAP}(1, 0) - V_{PAP}(0, 1)$. By (46) and (47), we obtain

\[
\begin{align*}
V_{PAP}(1, 0) - V_{PAP}(0, 1) &= \\
&= \delta[p(m_{HH} + V_{PAP}(0, 0) - V_{PAP}(1, 1)) + (1 - p)(-m_{LL} + V_{PAP}(1, 1) - V_{PAP}(0, 0))],
\end{align*}
\]

51
which yields, inserting (45) and (48), the equation

\[
V_{PAP}(1, 0) - V_{PAP}(0, 1) = \delta p [m_{HH} + \delta p (V_{PAP}(1, 0) - m_{HH} - V_{PAP}(0, 1)) \\
+ \delta (1 - p) (V_{PAP}(0, 1) - m_{LL} + V_{PAP}(1, 0))] \\
+ \delta (1 - p) [-m_{LL} + \delta p (m_{HH} + V_{PAP}(0, 1) - V_{PAP}(1, 0)) \\
+ \delta (1 - p) (m_{LL} + V_{PAP}(1, 0) - V_{PAP}(0, 1))].
\]

Solving for \(V_{PAP}(1, 0) - V_{PAP}(0, 1)\) gives

\[
V_{PAP}(1, 0) - V_{PAP}(0, 1) = \frac{\delta [pm_{HH}(1 - 2p\delta + \delta) - (1 - p)m_{LL}(1 - \delta + 2p\delta)]}{1 - \delta^2(1 - 2p)^2}. \quad (49)
\]

Similarly, by (48) and (45), we obtain

\[
V_{PAP}(1, 1) - V_{PAP}(0, 0) = \\
\delta [p(m_{HH} + V_{PAP}(0, 1) - V_{PAP}(1, 0)) + (1 - p)(m_{LL} + V_{PAP}(1, 0) - V_{PAP}(0, 1))],
\]

which gives, inserting (46) and (47), the equation

\[
V_{PAP}(1, 1) - V_{PAP}(0, 0) = \delta p [m_{HH} + \delta p (V_{PAP}(1, 1) - m_{HH} - V_{PAP}(0, 0)) \\
+ \delta (1 - p) (m_{LL} + V_{PAP}(0, 0) - V_{PAP}(1, 1))] \\
+ \delta (1 - p) [m_{LL} + \delta p (m_{HH} + V_{PAP}(0, 0) - V_{PAP}(1, 1)) \\
+ \delta (1 - p) (V_{PAP}(1, 1) - m_{LL} - V_{PAP}(0, 0))].
\]

Solving for \(V_{PAP}(1, 1) - V_{PAP}(0, 0)\) yields

\[
V_{PAP}(1, 1) - V_{PAP}(0, 0) = \frac{\delta [pm_{HH}(1 - 2p\delta + \delta) + (1 - p)m_{LL}(1 - \delta + 2p\delta)]}{1 - \delta^2(1 - 2p)^2}. \quad (50)
\]

Comparing (50) to (49), observe that

\[
V_{PAP}(1, 1) - V_{PAP}(0, 0) = V_{PAP}(1, 0) - V_{PAP}(0, 1) + m_{LL} \cdot A \quad (51)
\]

with \(A := \frac{2\delta (1 - p)}{1 + \delta - 2p\delta}.\)

Note that

\[
0 < A < 1. \quad (52)
\]

The set of states on which the unprofitability of deviations has to be verified is \(\{(x, y)|x \leq 2, y \leq 2, x + y \geq 1\}\).
On the states (1,0) and (0,1) there is no possible deviation and hence no profitable deviation.

On the states (2,0) and (0,2) the only possible deviation is \((d_{HH}, d_{HL}, d_{LL}) = (0, 0, 0)\). In both states this deviation does not have to be considered by Lemma 3.

On the state (1,2) there are two possible deviations: (0, 0, 0) and (0, 1, 0). Deviation (0, 0, 0) does not have to be considered by Lemma 3. Deviation (0, 1, 0) is not profitable either if

\[
V_{PAP}(1,2) = m_{LL} + V_{PAP}(1,0) \geq m_{HL} + V_{PAP}(0,1). \tag{53}
\]

On the state (2,1) there are two possible deviations: (0, 0, 0) and (0, 1, 0). Deviation (0, 0, 0) does not have to be considered by Lemma 3. Deviation (0, 1, 0) is not profitable either if

\[
V_{PAP}(2,1) = m_{HH} + V_{PAP}(0,1) \geq m_{HL} + V_{PAP}(1,0). \tag{54}
\]

On the state (2,2) there are five possible deviations: (0, 0, 0), (0, 1, 0), (1, 0, 0), (0, 0, 1) and (0, 2, 0). Deviations (0, 0, 0), (1, 0, 0) and (0, 0, 1) do not have to be considered by Lemma 3. Deviation (0, 2, 0) does not have to be considered by Lemma 4. Deviation (0, 1, 0) is not profitable either if

\[
V_{PAP}(2,2) = m_{HH} + m_{LL} + V_{PAP}(0,0) \geq m_{HL} + V_{PAP}(1,1). \tag{55}
\]

On the state (1,1) there is one possible deviation, which is (0, 1, 0). The condition for deviation (0, 1, 0) to be not profitable is

\[
V_{PAP}(1,2) = m_{LL} + V_{PAP}(1,0) \geq m_{HL} + V_{PAP}(0,1). \tag{56}
\]

The final step is to show that inequalities (53) to (56) hold if and only if \(m_{HL} \leq m_{HL}^1\). Using (51) and (52), inequalities (53) to (56) can be reformulated to

\[
V_{PAP}(1,0) - V_{PAP}(0,1) \geq m_{HL} - m_{LL}A, \tag{57}
\]
\[
V_{PAP}(1,0) - V_{PAP}(0,1) \geq m_{HL} - m_{LL}, \tag{58}
\]
\[
m_{HH} + m_{LL} - m_{HL} \geq V_{PAP}(1,0) - V_{PAP}(0,1) + m_{LL}A, \tag{59}
\]
\[
m_{HH} - m_{HL} \geq V_{PAP}(1,0) - V_{PAP}(0,1). \tag{60}
\]

It is immediate that (57) implies (58) and (60) implies (59). Furthermore, (57) implies (60): Inserting (49), (57) can be written as
\[ m_{HL} \leq m_{HH} \frac{\delta p}{1 - \delta + 2p\delta} + m_{LL} \frac{\delta(1 - p)}{1 + \delta - 2p\delta}. \]  

(61)

Inserting (49), (60) can be written as

\[ m_{HL} \leq m_{HH} \left(1 - \frac{\delta p}{1 - \delta + 2p\delta}\right) + m_{LL} \frac{\delta(1 - p)}{1 + \delta - 2p\delta}. \]  

(62)

Thus (57) implies (60) if

\[ \frac{2\delta p}{1 - \delta + 2p\delta} \leq 1 \quad \iff \quad \delta \leq 1, \]  

which holds.

Finally, as (61) reveals, (57) corresponds exactly to the condition \( m_{HL} \leq m_{1HL} \).

**Claim 2:** The Provident Impatient Policy is optimal for all parameter constellations \((p, \delta, m_{HH}, m_{LL}, m_{HL})\) such that \( m_{1HL} \leq m_{HL} \leq m_{2HL} \).

The value function \( V_{PIP} \) at states in the respective recurrent set is given by the following equations:

\[ V_{PIP}(1, 0) = \delta[pV_{PIP}(2, 0) + (1 - p)V_{PIP}(1, 1)], \]  

(64)

\[ V_{PIP}(0, 1) = \delta[pV_{PIP}(1, 1) + (1 - p)V_{PIP}(0, 2)], \]  

(65)

\[ V_{PIP}(1, 1) = m_{HL} + \delta[pV_{PIP}(1, 0) + (1 - p)V_{PIP}(0, 1)], \]  

(66)

\[ V_{PIP}(2, 0) = m_{HH} + \delta[pV_{PIP}(1, 0) + (1 - p)V_{PIP}(0, 1)], \]  

(67)

\[ V_{PIP}(0, 2) = m_{LL} + \delta[pV_{PIP}(1, 0) + (1 - p)V_{PIP}(0, 1)]. \]  

(68)

Define \( V_{PIP}(0, 0) := \delta[pV_{PIP}(1, 0) + (1 - p)V_{PIP}(0, 1)] \). The value at any state \((x, y)\) is obtained as follows:

\[ V_{PIP}(x, y) = \rho_{HH}^{PIP}(x, y) + \rho_{LL}^{PIP}(x, y) + \rho_{HL}^{PIP}(x, y) + V_{PIP}(x', y'), \]

with \( x' = x - 2\rho_{HH}^{PIP}(x, y) - \rho_{HL}^{PIP}(x, y) < 2 \)

and \( y' = y - 2\rho_{LL}^{PIP}(x, y) - \rho_{HL}^{PIP}(x, y) < 2 \).
Solving the system gives

\[ V_{\text{PIP}}(0, 0) = \frac{\delta^2}{1 - \delta^2}[(p^2 + (1 - p)^2)m_{HL} + (1 - p)p(m_{HH} + m_{LL})], \]  
(69)

\[ V_{\text{PIP}}(1, 0) = \delta V_{\text{PIP}}(0, 0) + \delta(pm_{HH} + (1 - p)m_{HL}), \]  
(70)

\[ V_{\text{PIP}}(0, 1) = \delta V_{\text{PIP}}(0, 0) + \delta(pm_{HL} + (1 - p)m_{LL}), \]  
(71)

\[ V_{\text{PIP}}(1, 1) = m_{HL} + \delta V_{\text{PIP}}(0, 0), \]  
(72)

\[ V_{\text{PIP}}(2, 0) = m_{HH} + \delta V_{\text{PIP}}(0, 0), \]  
(73)

\[ V_{\text{PIP}}(0, 2) = m_{LL} + \delta V_{\text{PIP}}(0, 0). \]  
(74)

The set of states on which the unprofitability of deviations has to be verified is \(\{(x, y)|x \leq 2, y \leq 2, x + y \geq 1\}\).

On the states (1, 0) and (0, 1) there is no possible deviation and hence no profitable deviation.

On the states (2, 0) and (0, 2) the only possible deviation is \((d_{HH}, d_{HL}, d_{LL}) = (0, 0, 0)\).

In both states this deviation does not have to be considered by Lemma 3.

On the state (2, 1) there are two possible deviations: (0, 0, 0) and (0, 1, 0). Deviation (0, 0, 0) does not have to be considered by Lemma 3. Deviation (0, 1, 0) is not profitable either: The corresponding inequality is

\[ V_{\text{PIP}}(2, 1) = m_{HH} + V_{\text{PIP}}(0, 1) \geq m_{HL} + V_{\text{PIP}}(1, 0). \]  
(75)

By rearranging terms to isolate \(V_{\text{PIP}}(1, 0) - V_{\text{PIP}}(0, 1)\) and inserting the expression for \(V_{\text{PIP}}(1, 0) - V_{\text{PIP}}(0, 1)\), (75) can be rewritten as

\[ m_{HH} - m_{HL} > \delta[pm_{HH} + (1 - 2p)m_{HL} - (1 - p)m_{LL}]. \]  
(76)

If (76) holds for \(\delta = 1\), it holds for any \(\delta\). Setting \(\delta = 1\) and rearranging terms yields \(m_{HH} + m_{LL} > 2m_{HL}\) which is satisfied by assumption.

On the state (2, 2) there are five possible deviations: (0, 0, 0), (0, 1, 0), (1, 0, 0), (0, 0, 1) and (0, 2, 0). Deviations (0, 0, 0), (1, 0, 0) and (0, 0, 1) do not have to be considered by Lemma 3. Deviation (0, 2, 0) does not have to be considered by Lemma 4. Deviation (0, 1, 0) is not profitable if

\[ m_{HH} + m_{LL} + V_{\text{PIP}}(0, 0) \geq m_{HL} + \delta[pV_{\text{PIP}}(2, 1) + (1 - p)V_{\text{PIP}}(1, 2)]. \]  
(77)

On the state (1, 1) there is one possible deviation, which is (0, 0, 0). The condition for
deviation \((0, 0, 0)\) to be not profitable is

\[
V_{\text{PIP}}(1, 1) = m_{HL} + V_{\text{PIP}}(0, 0) \geq \delta[pV_{\text{PIP}}(2, 1) + (1 - p)V_{\text{PIP}}(1, 2)].
\] (78)

On the state \((1, 2)\) there are two possible deviations: \((0, 0, 0)\) and \((0, 1, 0)\). Deviation \((0, 0, 0)\) does not have to be considered by Lemma 3. Deviation \((0, 1, 0)\) is not profitable either if

\[
V_{\text{PIP}}(1, 2) = m_{LL} + V_{\text{PIP}}(1, 0) \geq m_{HL} + V_{\text{PIP}}(0, 1).
\] (79)

The final step is to show that inequalities (77) to (79) hold if and only if \(m_{1HL} \leq m_{HL} \leq m_{2HL}\). Supermodularity and (78) imply (77). (78) corresponds exactly to the condition \(m_{HL} \geq m_{1HL}\); this results when inserting the explicit solutions for \(V_{\text{PIP}}\) into (78) and solving for \(m_{HL}\). (79) corresponds exactly to the condition \(m_{HL} \leq m_{2HL}\); this results when inserting the explicit solutions for \(V_{\text{PIP}}\) into (79) and solving for \(m_{HL}\).

**Claim 3:** The Myopic Impatient Policy is optimal for all parameter constellations \((p, \delta, m_{HH}, m_{LL}, m_{HL})\) such that \(m_{HL} \geq m_{2HL}\).

The Myopic Impatient Policy has the same recurrent set as the Provident Impatient Policy, \(R := \{(x, y)|1 \leq x+y \leq 2\}\). By definition, \(\rho_{\text{MIP}}(x, y) = \rho_{\text{PIP}}(x, y), \forall (x, y) \in R\).

An immediate consequence is \(V_{\text{MIP}}(x, y) = V_{\text{PIP}}(x, y), \forall (x, y) \in R\).

Define \(V_{\text{MIP}}(0, 0) := \delta[pV_{\text{MIP}}(1, 0) + (1 - p)V_{\text{MIP}}(0, 1)]\). The values at states in the recurrent set are given by (69) to (74). The value at any state \((x, y)\) is

\[
V_{\text{MIP}}(x, y) = \rho_{\text{MIP}}^{HH}(x, y) + \rho_{\text{MIP}}^{LL}(x, y) + \rho_{\text{MIP}}^{HL}(x, y) + V_{\text{MIP}}(x', y')
\]

with \(x' = x - 2\rho_{\text{MIP}}^{HH}(x, y) - \rho_{\text{MIP}}^{HL}(x, y) < 2\) and \(y' = y - 2\rho_{\text{MIP}}^{LL}(x, y) - \rho_{\text{MIP}}^{HL}(x, y) < 2\).

The set of states on which the unprofitability of deviations has to be verified is \(\{(x, y)|x \leq 2, y \leq 3, x + y \geq 1\}\).

On the states \((1, 0)\) and \((0, 1)\) there is no possible deviation and hence no profitable deviation.

On the state \((2, 0)\) the only possible deviation is \((d_{HH}, d_{HL}, d_{LL}) = (0, 0, 0)\). This deviation does not have to be considered by Lemma 3.

On the state \((2, 1)\) there are two possible deviations: \((0, 0, 0)\) and \((0, 1, 0)\). Deviation \((0, 0, 0)\) does not have to be considered by Lemma 3. Deviation \((0, 1, 0)\) is not
profitable either: The corresponding inequality is

\[ V_{MIP}(2, 1) = m_{HH} + V_{MIP}(0, 1) \geq m_{HL} + V_{MIP}(1, 0). \] (80)

As \( V_{MIP}(0, 1) = V_{PIP}(0, 1) \) and \( V_{MIP}(1, 0) = V_{PIP}(1, 0) \), (80) equals (75) and hence holds.

On the state \((1, 1)\) there is one possible deviation, which is \((0, 0, 0)\). The condition for deviation \((0, 0, 0)\) to be not profitable is

\[ V_{MIP}(1, 1) = m_{HL} + V_{MIP}(0, 0) \geq \delta[pV_{MIP}(2, 1) + (1 - p)V_{MIP}(1, 2)]. \] (81)

On the state \((2, 2)\) there are five possible deviations: \((0, 0, 0), (0, 1, 0), (1, 0, 0), (0, 0, 1)\) and \((0, 2, 0)\). Deviations \((0, 0, 0)\), and \((1, 0, 0)\) do not have to be considered by Lemma 3. Deviation \((0, 2, 0)\) does not have to be considered by Lemma 4. Deviation \((0, 0, 1)\) does not have to be considered by Lemma 2. Deviation \((0, 1, 0)\) is not profitable if

\[ m_{HH} + m_{LL} + V_{MIP}(0, 0) \geq m_{HL} + \delta[pV_{MIP}(2, 1) + (1 - p)V_{MIP}(1, 2)]. \] (82)

On the state \((0, 3)\) there is one possible deviation, which is \((0, 0, 0)\). This deviation does not have to be considered by Lemma 3.

On the state \((1, 3)\) there are two possible deviations: \((0, 0, 0), (0, 0, 1)\) and \((0, 1, 0)\). Deviations \((0, 0, 1)\) and \((0, 1, 0)\) do not have to be considered by Lemma 2. Deviation \((0, 0, 0)\) does not have to be considered by Lemma 3.

On the state \((2, 3)\) there are five possible deviations: \((0, 0, 0), (0, 1, 0), (1, 0, 0), (0, 0, 1)\) and \((0, 2, 0)\). Deviations \((0, 0, 0)\), and \((1, 0, 0)\) do not have to be considered by Lemma 3. Deviation \((0, 2, 0)\) does not have to be considered by Lemma 4. Deviation \((0, 0, 1)\) does not have to be considered by Lemma 2. Deviation \((0, 1, 0)\) is not profitable either if

\[ m_{HH} + m_{LL} + V_{MIP}(0, 1) \geq m_{HL} + \delta[pV_{MIP}(2, 2) + (1 - p)V_{MIP}(1, 3)]. \] (83)

On the state \((0, 2)\) there is one possible deviation, which is \((0, 0, 0)\). Deviation \((0, 1, 0)\) is not profitable if

\[ V_{MIP}(0, 2) = m_{LL} + V_{MIP}(0, 0) \geq \delta[pV_{MIP}(1, 2) + (1 - p)V_{MIP}(0, 3)]. \] (84)

On the state \((1, 2)\) there are two possible deviations: \((0, 0, 0)\) and \((0, 0, 1)\). Deviation
(0, 0, 1) is not profitable if
\[ m_{HL} + V_{MIP}(0, 1) \geq m_{LL} + V_{MIP}(1, 0). \]

Deviation (0, 0, 0) is not profitable if
\[ m_{HL} + V_{MIP}(0, 1) \geq \delta[pV_{MIP}(2, 2) + (1 - p)V_{MIP}(1, 3)]. \]

(85) corresponds exactly to the condition \( m_{HL} \geq m_{2HL} \); this results when inserting the explicit solutions for \( V_{MIP} \) into (85) and solving for \( m_{HL} \). Then we show that (86) is implied by (85):

\[
\begin{align*}
\delta[pV_{MIP}(2, 2) + (1 - p)V_{MIP}(1, 3)] & = \delta[p(m_{HH} + m_{LL} + V_{MIP}(0, 0)) + (1 - p)(m_{HL} + m_{LL} + V_{MIP}(0, 0))] \\
& = \delta m_{LL} + V_{MIP}(1, 0) \\
& < m_{LL} + V_{MIP}(1, 0) \\
& \leq m_{HL} + V_{MIP}(0, 1),
\end{align*}
\]

where the last inequality follows from (85).

(83) is shown using supermodularity and the fact that (0, 1, 0) is optimal on state (1, 2):
\[
\begin{align*}
m_{HL} + \delta[pV_{MIP}(2, 2)] + (1 - p)V_{MIP}(1, 3)] & \leq m_{HL} + V_{MIP}(1, 2) \\
& = 2m_{HL} + V_{MIP}(0, 1) \\
& \leq m_{HH} + m_{LL} + V_{MIP}(0, 1).
\end{align*}
\]

(82) is shown as well using supermodularity and the fact that (0, 1, 0) is optimal on state (1, 1):
\[
\begin{align*}
m_{HL} + \delta[pV_{MIP}(2, 1)] + (1 - p)V_{MIP}(1, 2)] & \leq m_{HL} + V_{MIP}(1, 2) \\
& = 2m_{HL} + V_{MIP}(0, 0) \\
& \leq m_{HH} + m_{LL} + V_{MIP}(0, 0).
\end{align*}
\]

(81) holds as a consequence of (84) and (85): (81) is equivalent to
\[
\delta V_{MIP}(0, 1) - V_{MIP}(0, 0) \leq m_{HL} - \delta pm_{HH} - \delta(1 - p)m_{HL}
\]

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and (84) is equivalent to
\[ \delta V_{MIP}(0,1) - V_{MIP}(0,0) \leq m_{LL} - \delta pm_{HL} - \delta(1-p)m_{LL}. \]

A sufficient condition for (81) to hold is
\[ m_{LL} - \delta pm_{HL} - \delta(1-p)m_{LL} \leq m_{HL} - \delta pm_{HH} - \delta(1-p)m_{HL}, \]
which is equivalent to \( m_{HL} \geq m_{HL}^2 \).

(84) corresponds exactly to the condition \( m_{HL} \leq m_{HL}^3 \); this results when inserting the explicit solutions for \( V_{MIP} \) into (84) and solving for \( m_{HL} \).

The final step of the proof is to show that \( m_{HL} \leq m_{HL}^3 \) for all \( p, \delta, m_{HH}, m_{LL}, m_{HL} \) such that \( m_{HH} \leq 3m_{LL} \). Then the parameter regions on which the three candidates are optimal span the entire parameter space. First, note that whenever there exists a \( m_{HL} \) such that \( m_{HL} > m_{HL}^3 \), then \( \frac{1}{2}m_{HH} + \frac{1}{2}m_{LL} > m_{HL}^3 \). Reformulating \( m_{HL} \leq m_{HL}^3 \) \( \forall p, \delta, m_{HH}, m_{LL} \) gives
\[ m_{HL} - m_{LL} \leq \frac{1-\delta}{\delta p} m_{LL} + \delta[p(m_{HH} - m_{HL}) + (1-p)(m_{HL} - m_{LL})], \quad \forall p, \delta, m_{HH}, m_{LL}. \]

Inserting \( m_{HL} = \frac{1}{2}m_{HH} + \frac{1}{2}m_{LL} \) and rearranging terms yields
\[ \frac{1}{2}(m_{HH} - m_{LL}) \leq \frac{m_{LL}}{\delta p}, \quad \forall p, \delta, m_{HH}, m_{LL}. \]

This holds if and if
\[ \frac{1}{2}(m_{HH} - m_{LL}) \leq m_{LL}, \quad \forall m_{HH}, m_{LL}. \]

This holds if and if
\[ m_{HH} \leq 3m_{LL}, \quad \forall m_{HH}, m_{LL}. \]

\[ \blacksquare \]

**Proof of Corollary 1:**

The existence of two cut-off levels follows from showing that \( \partial m_{HL}^1/\partial \delta \geq 0 \) and \( \partial m_{HL}^2/\partial \delta \geq 0 \), independent of the specific choice of parameters. Taking the defini-
tions of $m_{HL}^1$ and $m_{HL}^2$ from (3) and (4) we obtain
\[
\frac{\partial m_{HL}^1}{\partial \delta} = m_{HH} \frac{p}{[1 - \delta(1 - 2p)]^2} + m_{LL} \frac{1 - p}{[1 + \delta(1 - 2p)]^2} > 0
\]

and
\[
\frac{\partial m_{HL}^2}{\partial \delta} = m_{HH} \frac{p}{[1 - \delta(1 - 2p)]^2} + m_{LL} \frac{-p}{[1 + \delta(1 - 2p)]^2} > m_{HH} \frac{p}{[1 - \delta(1 - 2p)]^2} + m_{HH} \frac{-p}{[1 + \delta(1 - 2p)]^2} = 0.
\]

Finally, $m_{HL}^1 < m_{HL}^2$ implies $\delta^1 > \delta^2$.

**Proof of Proposition 1:**

Denote the Matching Policy $\mathcal{P}_1$ by $\rho^{P1}$. By construction, $\rho^{P1}$ is consistent.

*Claim 1: To verify candidate policy $\rho^{P1}$ it is sufficient to verify deviations on \{$(x, y) | x \leq 2, y \leq 4, x + y \geq 1$\}.*

Consider an arbitrary state $(x, y)$ with $x \geq 3$. Denote a deviation on $(x, y)$ by $(d_{HH}, d_{HL}, d_{LL})$.

For this case, the proof is again analogous to Lemma 4: By construction, $\rho_{HH}^{P1}(x, y) > 0$. By Lemma 2, no deviation with $d_{HH} > 0$ has to be verified. When following deviation $(d_{HH}, d_{HL}, d_{LL})$ in state $(x, y)$, the minimum number of productive agents after the next arrival is $x - 2d_{HH} - d_{HL}$. As $\rho_{HH}^{P1}(x', y') > 0$, $\forall x' \geq 2$, deviations with $x - 2d_{HH} - d_{HL} \geq 2$ do not have to be checked by Lemma 3. Deviations with $d_{HL} \geq 2$ do not have to be checked by Lemma 4. All deviations that need to be verified, hence,
satisfy

\[ d_{LL} = 0, \quad (87) \]

\[ y'' - 2d_{LL} - d_{HL} < 4, \quad (88) \]

\[ d_{HL} < 2. \quad (89) \]

As \( y'' \geq 5 \), the set of deviations \((d_{HH}, d_{HL}, d_{LL})\) that satisfy (87) to (89) is empty. Hence, no deviation on \((x'', y'')\) with \( y'' \geq 5 \) has to be checked.

Claim 2: There exists a parameter region on which there is no profitable deviation from \(\rho^{P1}\) on \(\{(x, y) | x \leq 2, y \leq 4, x + y \geq 1\}\).

Define \(V_{P1}(0, 0) := \delta[pV_{P1}(1, 0) + (1-p)V_{P1}(0, 1)]\). The values at states in the recurrent set are given by

\[
\begin{align*}
V_{P1}(1, 0) &= \delta[pV_{P1}(2, 0) + (1-p)V_{P1}(1, 1)], \\
V_{P1}(0, 1) &= \delta[pV_{P1}(1, 1) + (1-p)V_{P1}(0, 2)] \\
V_{P1}(1, 1) &= m_{HL} + \delta[pV_{P1}(1, 0) + (1-p)V_{P1}(0, 1)], \\
V_{P1}(2, 0) &= m_{HH} + \delta[pV_{P1}(1, 0) + (1-p)V_{P1}(0, 1)], \\
V_{P1}(0, 2) &= \delta[pV_{P1}(1, 2) + (1-p)V_{P1}(0, 3)], \\
V_{P1}(0, 3) &= m_{LL} + \delta[pV_{P1}(1, 1) + (1-p)V_{P1}(0, 2)], \\
V_{P1}(1, 2) &= m_{HL} + \delta[pV_{P1}(1, 2) + (1-p)V_{P1}(0, 3)].
\end{align*}
\]

The value at any state \((x, y)\) is

\[
V_{P1}(x, y) = \rho_{HH}^{P1}(x, y) + \rho_{LL}^{P1}(x, y) + \rho_{HL}^{P1}(x, y) + V_{P1}(x', y')
\]

with

\[
x' = x - 2\rho_{HH}^{P1}(x, y) - \rho_{HL}^{P1}(x, y) < 2
\]

and

\[
y' = y - 2\rho_{LL}^{P1}(x, y) - \rho_{HL}^{P1}(x, y) < 3.
\]

On the states \((1, 0)\) and \((0, 1)\) there is no possible deviation and hence no profitable deviation.

On the state \((2, 0)\) the only possible deviation is \((d_{HH}, d_{HL}, d_{LL}) = (0, 0, 0)\). This deviation does not have to be considered by Lemma 3.

On the state \((2, 1)\) there are two possible deviations: \((0, 0, 0)\) and \((0, 1, 0)\). Deviation \((0, 0, 0)\) does not have to be considered by Lemma 3. Deviation \((0, 1, 0)\) is not
profitable if

\[ V_{P1}(2, 1) = m_{HH} + V_{P1}(0, 1) \geq m_{HL} + V_{P1}(1, 0). \]  \tag{97} 

On the state \((1,1)\) there is one possible deviation, which is \((0, 0, 0)\). The condition for deviation \((0, 0, 0)\) to be not profitable is

\[ V_{P1}(1, 1) = m_{HL} + V_{P1}(0, 0) \geq \delta[pV_{P1}(2, 1) + (1 - p)V_{P1}(1, 2)]. \]  \tag{98} 

On the state \((2,2)\) there are five possible deviations: \((0, 0, 0)\), \((0, 1, 0)\), \((1, 0, 1)\), \((0, 0, 1)\) and \((0, 2, 0)\). Deviation \((0, 0, 0)\) does not have to be considered by Lemma 3. Deviation \((0, 2, 0)\) does not have to be considered by Lemma 4. Deviation \((0, 0, 1)\) and \((1, 0, 1)\) do not have to be considered by Lemma 2. Deviation \((0, 1, 0)\) is not profitable if

\[ m_{HH} + m_{LL} + V_{P1}(0, 0) \geq m_{HL} + \delta[pV_{P1}(2, 1) + (1 - p)V_{P1}(1, 2)]. \]  \tag{99} 

On the state \((0,3)\) there is one possible deviation, which is \((0, 0, 0)\). The condition for deviation \((0, 0, 0)\) to be not profitable is

\[ V_{P1}(0, 3) = m_{LL} + V_{P1}(0, 1) \geq \delta[pV_{P1}(1, 3) + (1 - p)V_{P1}(0, 4)]. \]  \tag{100} 

On the state \((1,3)\) there are two possible deviations: \((0, 0, 0)\), \((0, 0, 1)\) and \((0, 1, 1)\). Deviation \((0, 1, 1)\) does not have to be considered by Lemma 2. Deviation \((0, 0, 0)\) does not have to be considered by Lemma 3. Deviation \((0, 0, 1)\) is not profitable if

\[ m_{HL} + V_{P1}(0, 2) \geq m_{LL} + \delta[pV_{P1}(2, 1) + (1 - p)V_{P1}(1, 2)]. \]  \tag{101} 

On the state \((0,2)\) there is one possible deviation, which is \((0, 0, 1)\). Deviation \((0, 0, 1)\) is not profitable if

\[ V_{P1}(0, 2) = \delta[pV_{P1}(1, 2) + (1 - p)V_{P1}(0, 3)] \geq m_{LL} + V_{P1}(0, 0). \]  \tag{102} 

On the state \((1,2)\) there are two possible deviations: \((0, 0, 0)\) and \((0, 0, 1)\). Deviation \((0, 0, 1)\) is not profitable if

\[ m_{HL} + V_{P1}(1, 0) \geq m_{LL} + V_{P1}(1, 0). \]  \tag{103}
Deviation (0, 0, 0) is not profitable if
\[ m_{HL} + V_{P1}(0, 1) \geq \delta[pV_{P1}(2, 2) + (1 - p)V_{P1}(1, 3)]. \] (104)

On the state (2, 3) there are five possible deviations: (0, 0, 0), (0, 1, 0), (1, 0, 0), (0, 0, 1) and (0, 2, 0). Deviation (0, 0, 0) does not have to be considered by Lemma 3. Deviation (0, 2, 0) does not have to be considered by Lemma 4. Deviations (1, 0, 0) and (0, 0, 1) do not have to be considered by Lemma 2. Deviation (0, 1, 0) is not profitable if
\[ m_{HH} + m_{LL} + V_{P1}(0, 1) \geq m_{HL} + \delta[pV_{P1}(2, 2) + (1 - p)V_{P1}(1, 3)]. \] (105)

On the state (0, 4) there are two possible deviations: (0, 0, 0) and (0, 0, 2). Deviation (0, 0, 0) does not have to be considered by Lemma 3. Deviation (0, 0, 2) does not have to be considered by Lemma 2.

On the state (1, 4) there are four possible deviations: (0, 0, 0), (0, 1, 0), (0, 0, 1) and (0, 0, 2). Deviations (0, 0, 0) and (0, 1, 0) do not have to be considered by Lemma 3. Deviations (0, 0, 1) and (0, 0, 2) do not have to be considered by Lemma 2.

On the state (2, 4) there are nine possible deviations: (0, 0, 0), (0, 1, 0), (1, 0, 0), (0, 1, 1), (0, 0, 2), (0, 2, 1), (1, 0, 2), (0, 0, 1) and (0, 2, 0). Deviations (0, 0, 0) and (0, 1, 0) do not have to be considered by Lemma 3. Deviations (0, 0, 1) and (0, 1, 1), (0, 2, 1) and (0, 0, 1) do not have to be considered by Lemma 2.

The next step is to show that inequalities (97) to (105) hold if and only if \( m_{HL}^3 \leq m_{HL} \leq m_{HL}^4 \) with
\[ m_{HL}^4 = \frac{1}{1 - \delta(1 - 2p)} \left[ m_{HH} \delta p + m_{LL} \left( 2 - \delta(2 - p) + \frac{1 - \delta + \delta p(1 - \delta)^2}{\delta^2 p^2} \right) \right]. \] (106)

It is useful to rewrite (90) - (96).

\[ V_{P1}(0, 0) = \delta[pV_{P1}(1, 0) + (1 - p)V_{P1}(0, 1)], \] (107)
\[ V_{P1}(1, 0) = \delta[p(m_{HH} + V_{P1}(0, 0)) + (1 - p)(m_{HL} + V_{P1}(0, 0))], \] (108)
\[ V_{P1}(0, 1) = \delta[p(m_{HL} + V_{P1}(0, 0)) + (1 - p)V_{P1}(0, 2)], \] (109)
\[ V_{P1}(0, 2) = \delta[p(m_{HL} + V_{P1}(0, 1)) + (1 - p)(m_{LL} + V_{P1}(0, 1))]. \] (110)

We start with some preliminary calculations. First, we derive a closed-form expression for \( V_{P1}(0, 1) - \delta V_{P1}(0, 2) \). Plugging (108) into (107) gives
\[ V_{P_1}(0, 0) = \delta p \left( \delta V_{P_1}(0, 0) + \delta(pm_{HH} + (1 - p)m_{LL}) \right) + \delta(1 - p)V_{P_1}(0, 1). \quad (111) \]

For \( V_{P_1}(0, 1) - \delta V_{P_1}(0, 2) \) we obtain, inserting (109) and (110),
\[
V_{P_1}(0, 1) - \delta V_{P_1}(0, 2) = \delta p(m_{HL} + V_{P_1}(0, 0) - \delta m_{HL} - \delta V_{P_1}(0, 1)) \\
+ \delta(1 - p)(V_{P_1}(0, 2) - \delta m_{LL} - \delta V_{P_1}(0, 1)). \quad (112)
\]

Rearranging (110) we see that
\[
V_{P_1}(0, 2) - \delta V_{P_1}(0, 1) = \delta(pm_{HL} + (1 - p)m_{LL}). \quad (113)
\]

Inserting (109), (111), and (113) into (112), we can solve for \( V_{P_1}(0, 1) - \delta V_{P_1}(0, 2) \) which is explicitly given by
\[
V_{P_1}(0, 1) - \delta V_{P_1}(0, 2) = \frac{\delta p [m_{HL} - \delta m_{LL} + m_{HH}p^2\delta^2 - m_{HL}p^2\delta^2 + m_{LL}p\delta - m_{HLP}\delta]}{1 + p^2\delta^2 - p\delta^2}. \quad (114)
\]

Second, we derive a closed-form expression for \( V_{P_1}(0, 0) - \delta V_{P_1}(0, 1) \). To this end, note that
\[
V_{P_1}(0, 0) - \delta V_{P_1}(0, 1) = \delta p(V_{P_1}(1, 0) - \delta m_{HL} - \delta V_{P_1}(0, 0)) \\
+ \delta(1 - p)(V_{P_1}(0, 1) - \delta V_{P_1}(0, 2)). \quad (115)
\]

Furthermore by (108)
\[
V_{P_1}(1, 0) - \delta V_{P_1}(0, 0) = \delta(pm_{HH} + (1 - p)m_{HL}). \quad (116)
\]

Inserting (116) and (114) into (115) gives
\[
V_{P_1}(0, 0) - \delta V_{P_1}(0, 1) = \frac{\rho\delta^2 [m_{HH}p + m_{HL}(1 - 2p - p\delta + p^2\delta) + m_{LL}(-\delta + 2p\delta - p^2\delta)]}{1 - \rho^2 - p^2\delta^2}. \quad (117)
\]

Now, we verify the remaining inequalities.
We start by showing that (100) holds if and only if \( m_{HL} \leq m_{HL}^4 \).
Rearranging terms in (100) yields
\[
V_{P_1}(0, 1) - \delta V_{P_1}(0, 2) \geq \delta [pm_{HL} + (1 - p)m_{LL}] - m_{LL}. \quad (118)
\]

Plugging (114) into (118) and rewriting (118) as a condition on \( m_{HL} \), we obtain
\[ m_{HL} \leq \frac{1}{1 - \delta(1 - 2p)} \left[ m_{HH}\delta p + m_{LL} \left( 2 - \delta(2 - p) + \frac{1 - \delta + \delta p(1 - \delta)^2}{\delta^2 p^2} \right) \right]. \quad (119) \]

The term on the right side of (119) is \( m_{HL}^4 \).

Then we show that (102) holds if and only if \( m_{HL} \geq m_{HL}^3 \). Inserting \( V_{P1}(1, 2) \) and \( V_{P1}(0, 3) \) in (102) gives

\[
\delta \left[ p(m_{HL} + V_{P1}(0, 1)) + (1 - p)(m_{LL} + V_{P1}(0, 1)) \right] \geq m_{LL} + V_{P1}(0, 0).
\]

Rearranging terms yields

\[
\delta [p m_{HL} + (1 - p)m_{LL}] - m_{LL} \geq V_{P1}(0, 0) - \delta V_{P1}(0, 1).
\]

Plugging (117) into (120) and some algebra yields

\[
m_{HL} \geq m_{HH} \frac{\delta p}{1 - \delta(1 - 2p)} + m_{LL} \frac{1 - \delta(1 - p) + \frac{1 - \delta}{\delta p}}{1 - \delta(1 - 2p)}.
\]

(121)

Observe that the right side of (121) coincides with \( m_{HL}^3 \).

Now we show that (100) implies (104). Reformulating (100) gives

\[
m_{LL} - \delta pm_{HL} - \delta(1 - p)m_{LL} \geq \delta V_{P1}(0, 2) - V_{P1}(0, 1) \quad (122)
\]

and (104) gives

\[
m_{HL} - \delta pm_{HH} - \delta(1 - p)m_{HL} \geq \delta V_{P1}(0, 2) - V_{P1}(0, 1). \quad (123)
\]

(122) implies (123) if

\[
m_{HL} - m_{LL} \geq \delta p(m_{HH} - m_{HL}) + \delta(1 - p)(m_{HL} - m_{LL}). \quad (124)
\]

Note that if (124) holds with equality then it coincides with \( m_{HL} \geq m_{HL}^2 \). As (102) requires \( m_{HL} \geq m_{HL}^3 \) and \( m_{HL}^3 \geq m_{HL}^2 \), (124) is implied by (102).

Now we argue that (102) implies (98). Note that we can rewrite (102) as

\[
V_{P1}(0, 0) - \delta V_{P1}(0, 1) \leq \delta (pm_{HL} + (1 - p)m_{LL}) - m_{LL}, \quad (125)
\]

and (98) as
\[ V_{P_1}(0, 0) - \delta V_{P_1}(0, 1) \geq \delta (p m_{HH} + (1 - p)m_{HL}) - m_{HL}. \]  

(126)

Plugging (117) into (125) and (126) yields, after some algebra, for (125)

\[
\delta^2 p^2 (m_{HH} - m_{HL}) + m_{LL}(1 - \delta) \leq \delta^2 p^2 (m_{HL} - m_{LL}) + (1 - \delta)p\delta(m_{HL} - m_{LL}).
\]  

(127)

and for (126)

\[
\delta p(m_{HH} - m_{HL})(1 - \delta p - \delta^2 p(1 - p)) + (1 - p)p^2 \delta^3(m_{HL} - m_{LL}) \leq m_{HL}(1 - \delta) + (1 - p)p\delta^3(m_{HL} - m_{LL}).
\]  

(128)

It is sufficient for (125) implies (126), i.e. (127) implies (128), if it holds that

\[
(\delta p(m_{HL} - m_{LL}) + (1 - \delta)(m_{HL} - m_{LL}))(1 - \delta p - \delta^2 p(1 - p)) \leq m_{HL}(1 - \delta) + (1 - p)^2 p\delta^3(m_{HL} - m_{LL}).
\]  

(129)

For (129) in turn it is sufficient if

\[
(\delta p + (1 - \delta))(1 - \delta p - \delta^2 p(1 - p)) \leq (1 - \delta) + (1 - p)^2 p\delta^3.
\]  

(130)

Simplifying (130) shows that the term on the left side coincides with the term on the right side.

Next we argue that (102) implies (97).

Inserting (108) and (109) into (97) and rearranging terms yields

\[
m_{HH} - m_{HL} \geq \delta [p(m_{HH} - m_{HL}) + (1 - p)(m_{HL} + V_{P_1}(0, 0) - V_{P_1}(0, 2))].
\]  

(131)

By (102) it is sufficient for (131) to hold that

\[
m_{HH} - m_{HL} \geq \delta [p m_{HH} + (1 - 2p)m_{HL} - (1 - p)m_{LL}].
\]  

(132)

Reformulating yields

\[
\frac{m_{HH} - m_{HL}}{m_{HL} - m_{LL}} \geq \frac{\delta(1 - p)}{1 - \delta p}.
\]  

(133)

which holds as by supermodularity the left side is larger than one, whereas the right side is smaller than one. Therefore (102) implies (97).
(105) is implied by (103), (104) and supermodularity: Given (103) and (104), from the optimality on state \((1,2)\) we know

\[ m_{HL} + \delta [pV_{P1}(2,2) + (1-p)V_{P1}(1,3)] \leq 2m_{HL} + V_{P1}(0,1) \]  

(134)

and from supermodularity follows

\[ 2m_{HL} + V_{P1}(0,1) \leq m_{HH} + m_{LL} + V_{P1}(0,1). \]  

(135)

Combining (134) and (135) gives (105).

(99) is implied by (98) and supermodularity: Given (98), from the optimality on state \((1,1)\) we know

\[ m_{HL} + \delta [pV_{P1}(2,1) + (1-p)V_{P1}(1,2)] \leq 2m_{HL} + V_{P1}(0,0) \]  

(136)

and from supermodularity follows

\[ 2m_{HL} + V_{P1}(0,0) \leq m_{HH} + m_{LL} + V_{P1}(0,0). \]  

(137)

Combining (136) and (137) gives (99).

(101) is implied by (102), (98) and supermodularity: Given (102), from the optimality on state \((0,2)\) we know

\[ m_{HL} + V_{P1}(0,2) \geq m_{HL} + m_{LL} + V_{P1}(0,0) \]  

(138)

and given (98), from the optimality on state \((1,1)\) follows

\[ m_{HH} + m_{LL} + V_{P1}(0,0) \leq m_{LL} + \delta [pV_{P1}(2,1) + (1-p)V_{P1}(1,2)]. \]  

(139)

Combining (138) and (139) gives (101).

Next, we show that (102) implies (103) by showing that ‘not (103)’ implies ‘not (102)’. Applying ‘not (103)’ and (102) leads to a contradiction:

\[
\begin{align*}
m_{LL} + V_{P1}(0,0) & \leq \delta [p(m_{HL} + V_{P1}(0,1)) + (1-p)(m_{LL} + V_{P1}(0,1))] \\
& \leq \delta [p(m_{LL} + V_{P1}(1,0)) + (1-p)(m_{LL} + V_{P1}(0,1))] \\
& = \delta m_{LL} + V_{P1}(0,0).
\end{align*}
\]

The proof is finished by showing that Policy \(P_1\) actually arises. First, we show that \(m^3_{HL} \leq m^4_{HL}\). Note that the terms in front of \(m_{HH}\) in the definition of \(m^4_{HL}\) and \(m^3_{HL}\)
coincide. Thus \( m_{HL}^3 \leq m_{HL}^4 \) holds iff

\[
2(1 - \delta) + \delta p + \frac{1 - \delta}{\delta^2 p^2} + \frac{(1 - \delta)^2}{\delta p} \geq 1 - \delta + \delta p + \frac{1 - \delta}{\delta p}.
\] (140)

As \( \delta, p \in (0, 1) \), (140) is true and hence \( m_{HL}^3 \leq m_{HL}^4 \).

Second, we show that if \( m_{HH} > 3m_{LL} \), there exist \( p, \delta, m_{HL} \) such that \( m_{HL}^3 < \frac{1}{2} m_{HH} + \frac{1}{2} m_{LL} \). This statement is proven in the last part of the proof to Theorem 1.

Proof of Corollary 2:

The corollary immediately follows from Theorem 1 and the proof of Proposition 1.

Proof of Proposition 2:

Assume that there exists an optimal policy \( \rho \) that never matches two unproductive agents and denote its value function given state \((x, y)\) by \( V_{\rho}(x, y) \). The proof by contradiction consists of deriving a lower bound \( a \) and an upper bound \( \bar{a} \) for \( V_{\rho}(x, y) \) such that \( a > \bar{a} \). Observe that by optimality

\[
V_{\rho}(0, k) \geq \left\lfloor \frac{k}{2} \right\rfloor m_{LL} + V_{\rho}(0, k - 2 \left\lfloor \frac{k}{2} \right\rfloor) \geq \left\lfloor \frac{k}{2} \right\rfloor m_{LL} = a,
\] (141)

where the second inequality holds because \( V_{\rho}(0, 0), V_{\rho}(0, 1) \geq 0 \).

As \( \rho \) never matches two unproductive agents, we can derive an upper bound on the number of matches created in each period when starting in state \((0, k)\) and following policy \( \rho \). When being in state \((0, k)\) in period \( t \), the maximal number of matches in period \( t+s \) is bounded from above by \( s \) for any \( s \in \mathbb{N} \). \( V_{\rho}(0, k) \) is, hence, bounded above by the value generated from creating the highest match value the most often and as early as possible, which is matching two productive agents, \( m_{HH} \), in every subsequent period, i.e.,

\[
V_{\rho}(0, k) \leq m_{HH} \frac{1}{1 - \delta} = \bar{a}.
\]

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For every parameter constellation \((p, \delta, m_{HH}, m_{LL}, m_{HL})\) there exists a \(k\) such that \(a > \bar{a}\), which is a contradiction.

**Proof of Proposition 3:**

It is sufficient to proof that if \(m(\theta_1, \theta_2) = \theta_1 \cdot \theta_2\) then

\[
m_{HL} \leq m_{HL}^3 = m_{HH} \frac{\delta p}{1 - \delta(1 - 2p)} + m_{LL} \frac{1 - \delta(1 - p) + \frac{1 - \delta}{\delta p}}{1 - \delta(1 - 2p)}, \quad \forall H, L, p, \delta
\]

with \(H > L > 0\). Inserting \(m(\theta_1, \theta_2) = \theta_1 \cdot \theta_2\), dividing by \((L)^2\), and rearranging terms yields

\[
0 \leq \left(\frac{H}{L}\right)^2 \delta p - \frac{H}{L} (1 - \delta(1 - 2p)) + \left(1 - \delta(1 - p) + \frac{1 - \delta}{\delta p}\right).
\]

The parabola in \(\frac{H}{L}\) on the right side of (142) is minimized at

\[
\frac{H}{L} = 1 + \frac{1 - \delta}{2p\delta}.
\]

Plugging (143) into (142) gives

\[
0 \leq 3 - 2\delta - \delta^2,
\]

which holds true as \(\delta < 1\). This completes the proof.

**Proof of Theorem 2:**

The Impatient Policy is implementable by setting \(\tau^{\Theta_S}(\theta) = 0\), for all \(\Theta_S, \theta\). The implementation of the Positive Assortative Policy requires a proof. Start by observing that

\[
\Delta^H(p, r) = \frac{\delta p}{1 - \delta(1 - p)}, \quad \Delta^L(p, r) = \frac{\delta(1 - p)}{1 - \delta p}.
\]

The incentive constraint for the productive and the unproductive type are, for market report \((0,0)\),

\[
\Delta^H(p, r)m_{HH} - \tau^{(0,0)}(H) \geq \Delta^L(p, r)m_{HL} - \tau^{(0,0)}(L), \quad \Delta^H(p, r)m_{HL} - \tau^{(0,0)}(H) \leq \Delta^L(p, r)m_{LL} - \tau^{(0,0)}(L),
\]
for market report (1,1),

\[ m_{HH} - \tau^{(1,1)}(H) \geq m_{HL} - \tau^{(1,1)}(L), \]
\[ m_{HL} - \tau^{(1,1)}(H) \leq m_{LL} - \tau^{(1,1)}(L), \]

for market report (1,0),

\[ m_{HH} - \tau^{(1,0)}(H) \geq \Delta^L(p,r)m_{HL} - \tau^{(1,0)}(L), \]
\[ m_{HL} - \tau^{(1,0)}(H) \leq \Delta^L(p,r)m_{LL} - \tau^{(1,0)}(L), \]

and for market report (0,1),

\[ \Delta^H(p,r)m_{HH} - \tau^{(0,1)}(H) \geq m_{HL} - \tau^{(0,1)}(L), \]
\[ \Delta^H(p,r)m_{HL} - \tau^{(0,1)}(H) \leq m_{LL} - \tau^{(0,1)}(L). \]

For every market report \( \Theta_S \), combining the incentive constraint of the productive type with the incentive constraint of the unproductive type yields the following conditions on the payment differences:

\[ \Delta^H(p,r)m_{HH} - \Delta^L(p,r)m_{HL} \geq \tau^{(0,0)}(H) - \tau^{(0,0)}(L) \geq \Delta^H(p,r)m_{HL} - \Delta^L(p,r)m_{LL}, \]  
(145)

\[ m_{HH} - m_{HL} \geq \tau^{(1,1)}(H) - \tau^{(1,1)}(L) \geq m_{HL} - m_{LL}, \]  
(146)

\[ m_{HH} - \Delta^L(p,r)m_{HL} \geq \tau^{(1,0)}(H) - \tau^{(1,0)}(L) \geq m_{HL} - \Delta^L(p,r)m_{LL}, \]  
(147)

\[ \Delta^H(p,r)m_{HH} - m_{HL} \geq \tau^{(0,1)}(H) - \tau^{(0,1)}(L) \geq \Delta^H(p,r)m_{HL} - m_{LL}. \]  
(148)

Thus, an incentive compatible payment difference exists if and only if the following conditions are satisfied:

\[ \Delta^H(p,r)m_{HH} - \Delta^L(p,r)m_{HL} \geq \Delta^H(p,r)m_{HL} - \Delta^L(p,r)m_{LL}, \]  
(149)

\[ m_{HH} - m_{HL} \geq m_{HL} - m_{LL}, \]  
(150)

\[ m_{HH} - \Delta^L(p,r)m_{HL} \geq m_{HL} - \Delta^L(p,r)m_{LL}, \]  
(151)

\[ \Delta^H(p,r)m_{HH} - m_{HL} \geq \Delta^H(p,r)m_{HL} - m_{LL}. \]  
(152)

Observe from (144) that \( \Delta^\theta(p,r) \leq 1 \). Hence, (152) implies (149) to (151). Similarly, (150) implies (149). (150) follows from supermodularity. To see that (152) holds whenever the Positive Assortative Policy is optimal, we need to compare it explicitly to the boundary of the Positive Assortative Policy. Reformulating (152) gives
\[ m_{HL} \leq \frac{\Delta^H(p,r)}{1 + \Delta^H(p,r)} m_{HH} + \frac{1}{1 + \Delta^H(p,r)} m_{LL}. \]  

(153)

We need to argue that (153) holds whenever the Positive Assortative Policy is optimal, i.e., \( m_{HL} \leq m_{HL}^1 \). To this end, we will show that the multipliers of \( m_{HH} \) and \( m_{LL} \) are (weakly) larger than the corresponding factors in \( m_{HL}^1 \). First, consider the factor attached to \( m_{HH} \). By (144),

\[ \frac{\Delta^H(p,r)}{1 + \Delta^H(p,r)} = \frac{\delta p}{1 - \delta + 2\delta p}, \]

which coincides with the multiplier of \( m_{HH} \) in \( m_{HL}^1 \). Second, for the factor attached to \( m_{LL} \) we obtain

\[ \frac{1}{1 + \Delta^H(p,r)} = \frac{1 - \delta + \delta p}{1 - \delta + 2\delta p}. \]

(154)

(154) is larger than the multiplier of \( m_{LL} \) in \( m_{HL}^1 \) if and only if

\[ \frac{1 - \delta + \delta p}{1 - \delta + 2\delta p} \geq \frac{\delta(1 - p)}{1 + \delta - 2\delta p} \iff \delta \leq 1. \]

Thus, whenever the Positive Assortative Policy is optimal, we can find an incentive compatible payment pair, for every market report \( \Theta_s \).

We construct payments which are positive and individual rational: Set

\[ \tau^{(0,0)}(L) = \tau^{(1,0)}(L) = \Delta^L(p,r)m_{LL} \geq 0, \quad (155) \]
\[ \tau^{(0,1)}(L) = \tau^{(1,1)}(L) = m_{LL} \geq 0. \quad (156) \]

By construction, payments (155) and (156) set the unproductive agent’s expected utility to zero and are therefore individual rational. For every market report, choose, given the unproductive type’s payment, the maximal payment for the productive type that is consistent with (149) - (152), i.e., such that the payment pair is incentive compatible:

\[ \tau^{(0,0)}(H) = \Delta^L(p,r)m_{LL} + \Delta^H(p,r)m_{HH} - \Delta^L(p,r)m_{HL}, \]
\[ \tau^{(1,1)}(H) = m_{LL} - m_{HL} + m_{HH}, \]
\[ \tau^{(1,0)}(H) = \Delta^L(p,r)m_{LL} + m_{HH} - \Delta^L(p,r)m_{HL}, \]
\[ \tau^{(0,1)}(H) = m_{LL} + \Delta^H(p,r)m_{HH} - m_{HL}. \]

Individual rationality of the payments for the unproductive type and incentive compatibility yield individual rationality for the productive type. Given that (149) - (152) are satisfied whenever the Positive Assortative Policy is opti-
mal, we can deduce that
\[
\tau^{(0,0)}(H) \geq \Delta^L(p,r)m_{LL} + \Delta^H(p,r)m_{HL} - \Delta^L(p,r)m_{LL} = \Delta^H(p,r)m_{HL} \geq 0, \\
\tau^{(1,1)}(H) \geq m_{LL} + m_{HL} - m_{LL} = m_{HL} \geq 0, \\
\tau^{(1,0)}(H) \geq \Delta^L(p,r)m_{LL} + m_{HL} - \Delta^L(p,r)m_{LL} = m_{HL} \geq 0, \\
\tau^{(0,1)}(H) \geq m_{LL} + \Delta^H(p,r)m_{HL} - m_{LL} = \Delta^H(p,r)m_{HL} \geq 0.
\]

As all payments are positive, budget balancedness is satisfied. Furthermore, payments support efficient exit because they are charged upon arrival. These observations conclude the proof. ■

**Proof of Proposition 4:**

For incentive compatibility, we need to find a single payment pair \((\tau(H), \tau(L))\) such that the difference \(\tau(H) - \tau(L)\) satisfies Conditions (145) to (148). Observe that (148) yields the lowest upper bound, whereas (151) yields the highest lower bound on the payment difference. Hence, payments that do not vary with the market report are incentive compatible iff

\[
\Delta^H(p,r)m_{HH} - m_{HL} \geq \tau(H) - \tau(L) \geq m_{HL} - \Delta^L(p,r)m_{LL}. \tag{157}
\]

Hence, incentive compatible payments exist iff

\[
m_{HL} \leq \Delta^H(p,r)\frac{m_{HH}}{2} + \Delta^L(p,r)\frac{m_{LL}}{2}. \tag{158}
\]

To see that (158) describes a strict subset of the parameter region in which the Positive Assortative Policy is optimal, we compare it to the boundary of the Positive Assortative Policy \(m_{HL}^1\). We argue that (158) is more restrictive than \(m_{HL} \leq m_{HL}^1\) by separately comparing the factors in front of \(m_{HH}\) and \(m_{LL}\). For the factor attached to \(m_{HH}\) we note that

\[
\frac{1}{2} \frac{\delta p}{(1 - \delta(1 - p))} < \frac{\delta p}{1 - \delta(1 - 2p)} \iff \delta < 1. \tag{159}
\]

Similarly, for the factor in front of \(m_{LL}\) observe that

\[
\frac{1}{2} \frac{\delta(1 - p)}{1 - \delta p} < \frac{\delta(1 - p)}{1 + \delta(1 - 2p)} \iff \delta < 1. \tag{160}
\]

Set \(\tau(L) = \Delta^L(p,r) \geq 0\). For market reports \((1,1)\) and \((0,1)\), an arriving unproductive type’s expected utility from reporting truthfully is
\[ m_{LL} - \Delta^L(p,r) \geq 0. \]

For market reports (1,0) and (0,0), an arriving unproductive type’s expected utility is

\[ \Delta^L(p,r) - \Delta^L(p,r) = 0. \]

Given \( \tau(L) \) we choose

\[ \tau(H) = \Delta^L(p,r)m_{LL} + \Delta^H(p,r)m_{HH} - m_{HL} \]

which is consistent with incentive compatibility by (157). With this payment the expected utility of a productive type from truth-telling for market reports (1,1) and (1,0) is

\[ (1 - \Delta^H(p,r))m_{HH} + m_{HL} - \Delta^L(p,r)m_{LL} \geq 0, \]

and for market reports (0,0) and (0,1) he receives

\[ m_{HL} - \Delta^L(p,r) \geq 0. \]

Thus, the pair \((\tau(H), \tau(L))\) is individual rational. Furthermore, for the parameter region characterized by (158) it holds that

\[ \tau(H) \geq \Delta^L(p,r)m_{LL} + m_{HL} - \Delta^L(p,r)m_{LL} = m_{HL} \geq 0, \]

therefore, the payments induce a balanced budget. Payments are charged only upon arrival and hence support efficient exit.

\[ \blacksquare \]

**Proof of Proposition 5:**

Fix the share \( \alpha \) of the productive agent in the mixed pair. The incentive constraint for the productive and the unproductive type are, for market report (0,0),

\[ \Delta^H(p,r)m_{HH} \frac{m_{HL}}{2} \geq \Delta^L(p,r)\alpha m_{HL}, \]

\[ \Delta^H(p,r)(1 - \alpha)m_{HL} \leq \Delta^L(p,r)\frac{m_{LL}}{2}, \]

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for market report (1,1),

\[
\frac{m_{HH}}{2} \geq \alpha m_{HL}, \\
(1 - \alpha)m_{HL} \leq \frac{m_{LL}}{2},
\]

for market report (1,0),

\[
\frac{m_{HH}}{2} \geq \Delta^L(p, r)\alpha m_{HL}, \\
(1 - \alpha)m_{HL} \leq \Delta^L(p, r)\frac{m_{LL}}{2},
\]

and for market report (0,1),

\[
\Delta^H(p, r)\frac{m_{HH}}{2} \geq \alpha m_{HL}, \\
\Delta^H(p, r)(1 - \alpha)m_{HL} \leq \frac{m_{LL}}{2}.
\]

Observe that, for every market report, the incentive constraint of the productive agent provides an upper bound on \(\alpha\), whereas the incentive constraint of the unproductive agent gives a lower bound on \(\alpha\):

\[
\frac{1}{2} \Delta^H(p, r)m_{HH} \geq \alpha \geq \frac{\Delta^H(p, r)m_{HL} - \Delta^L(p, r)m_{LL}}{m_{HL}},
\]

\[
\frac{1}{2} \frac{m_{HH}}{m_{HL}} \geq \alpha \geq \frac{m_{HL} - \frac{m_{LL}}{2}}{m_{HL}},
\]

\[
\frac{1}{2} \frac{m_{HH}}{m_{HL}} \geq \alpha \geq \frac{m_{HL} - \Delta^L(p, r)m_{LL}}{m_{HL}},
\]

\[
\frac{1}{2} \frac{m_{HH}}{m_{HL}} \geq \alpha \geq \frac{\Delta^H(p, r)m_{HL} - \frac{m_{LL}}{2}}{m_{HL}}.
\]

The incentive constraint of the productive agent given market report (0,1) yields the lowest upper bound, cf. (164), and the incentive constraint of the unproductive agent for market report (1,0) provides the highest lower bound, cf. (163). Thus, any incentive compatible match value split has to satisfy

\[
\frac{1}{2} \frac{\Delta^H(p, r)m_{HH}}{m_{HL}} \geq \alpha \geq \frac{m_{HL} - \Delta^L(p, r)m_{LL}}{m_{HL}}.
\]

Note that

\[
\frac{1}{2} \frac{\Delta^H(p, r)m_{HH}}{m_{HL}} \geq 0 \quad \text{and} \quad 1 \geq \frac{m_{HL} - \Delta^L(p, r)m_{LL}}{m_{HL}} \geq \frac{1}{2},
\]

(165) reveals that an incentive compatible match value split exists iff
Rearranging terms yields
\[ m_{HL} \leq \frac{\Delta^H(p, r)m_{HH}}{2} + \frac{\Delta^L(p, r)m_{LL}}{2}, \tag{166} \]
which coincides with (158).

The Positive Assortative Policy without payments supports efficient exit and provides all agents with (expected) utility of at least zero. Thus, as changes in the sharing rule of the mixed pair only affect off-path behavior, individual rationality is satisfied which concludes the proof.