One-seller assignment markets with multi-unit demands: core and competitive equilibrium

Francisco Robles*1 and Marina Núñez1

1Departament de Matemàtica Econòmica, Financera i Actuarial
Universitat de Barcelona, Av. Diagonal, 690, 08034 Barcelona, Spain

January 23, 2015

Abstract

We consider an assignment market with one seller who owns several indivisible heterogeneous goods and many buyers each willing to buy up to a given capacity. Our aim is to study the relationship between the core of the game and the set of competitive equilibria. The core is non-empty and it has a lattice structure which contains the allocation in which every buyer gets his marginal contribution to the grand coalition. The set of competitive equilibrium price vectors also has a lattice structure and we determine the minimum and maximum competitive equilibrium prices. Necessary and sufficient conditions under which the buyers-optimal and the seller-optimal core allocations come from a competitive equilibrium are provided. In addition, we characterize in terms of the valuation matrix the coincidence between the core and the set of competitive equilibrium payoff vectors. As a consequence, we obtain that this coincidence always holds if the capacities of all buyers are large enough.

Keywords: Many-to-many assignment markets, core, pairwise-stability, competitive equilibrium.

1 Introduction

We study markets with several buyers and only one seller. The seller owns many indivisible and potentially different objects on sale. Being heterogeneous, the objects are of the same type: for instance different houses or different tasks. On the other side of the market, each buyer has a nonnegative valuation for each object and a desire to acquire...
a certain number of objects. This number is known as the capacity of the buyer. We assume buyers value packages of objects additively. Utility is identified with money and side-payments are allowed.

This market is a particular case of the one considered in Jaume et al. (2012) and Massó and Neme (2014), where there are several sellers, each with a set of heterogeneous objects on sale. It is also a particular case of the package auction of Ausubel and Milgrom (2002), where there is only one seller, but buyers may not value packages additively. A related situation, also with only one seller owning many objects on sale but buyers with unitary demands, was analyzed in Camiña (2006).

Two-sided markets with transferable utility are first considered from the viewpoint of coalitional games in the assignment game (Shapley and Shubik, 1972). In this model, there are two disjoint finite sets of agents: a set of sellers, each supplying one unit of an indivisible good, and a set of buyers, each of them demanding one unit of the good. There is a potential gain of each buyer-seller partnership that is collected in a valuation matrix. The worth of a coalition of buyers and sellers is the maximum profit that can be achieved by optimally matching buyers and sellers inside the coalition.

In the assignment game, the core is non-empty and coincides with those efficient payoff vectors that satisfy pairwise-stability, that is to say, no pair of a buyer and a seller would be better off by breaking their optimal partnership in any optimal matching and being matched together (Shapley and Shubik, 1972). Moreover, the set of competitive equilibria payoff vectors coincides with the set of solutions of the dual linear assignment problem (Gale, 1960), and hence with the core of the assignment game. Even more, the core of the assignment game is a complete lattice with two particular core elements, one of them optimal for all buyers and the other one optimal for all sellers.

When the assumptions of the classical assignment model are relaxed, the coincidence between the core, the set of pairwise-stable outcomes and the set of competitive equilibria payoff vectors does not hold in general. The same happens with the lattice structure of some of these sets.

The first generalization of the classical assignment game considers that each seller owns several units of different goods and can be matched to as many buyers as allowed by the seller’s capacity. On the other hand, buyers’ demand is still unitary (Kaneko, 1976). In this many-to-one assignment model (many agents on the buyers’ side can be matched to a same agent on the sellers’ side), the set of payoff vectors associated to the competitive equilibria is included in the core, which guarantees the non-emptiness of the core. Nevertheless, this inclusion may be strict.

Other generalizations of the assignment game are known as many-to-many markets since both buyers and sellers may have capacities greater than one: the capacity of a buyer is the number of objects he desires to buy and the capacity of a seller is the number of identical objects she offers on sale (Thompson, 1980). A matching describes a set of partnerships between buyers and sellers within their capacities, and allows for multi-unit trade within a same buyer-seller pair. This two-sided market can be considered to represent a job market between heterogeneous firms and workers, when the objects on sale are units of labour as in Sotomayor (2002). Optimal matchings for the many-to-many assignment market are obtained by solving a linear transportation problem. The core of this game is always non-empty but it remains an open problem whether in this
setting an optimal core element for each side of the market does exist, although it is known that a worst core element for each side of the market needs not exist.

A more encompassing many-to-many assignment model is the one with heterogeneous goods and multi-unit demands of Jaume et al. (2012) and Massó and Neme (2014), where there are several sellers each of them with several units of potentially different objects. In these markets, as in the many-to-one markets of Kaneko (1976), the set of competitive equilibrium payoff vectors is non-empty and is strictly included in the core. However, let us point out that the definition of competitive equilibrium in Jaume et al. (2012) and Massó and Neme (2014) assumes that buyers demand as many copies of their preferred object as their capacities allow, being the prices given. Compared to that, in a many-to-many assignment game in which the goods owned by a seller are homogeneous, Sotomayor (2013) defines competitive equilibria by means of a demand in which buyers maximize the utility of the packages they can buy given prices and their capacities.¹

In the present paper, where we have only one seller with heterogeneous goods and multi-unit demands, we focus on the relationship between the core of the game and the set of competitive equilibria of the market. In particular, the coincidence between the core and the competitive equilibria is addressed. We first prove that the game is buyers-submodular. Then, as a consequence of Ausubel and Milgrom (2002), we deduce: a) the core is the non-empty set of efficient payoff vectors where each buyer gets a non-negative payoff bounded by his marginal contribution to the whole market; b) the core is endowed with a lattice structure by the partial order defined from the point of view of buyers, and c) there exists one optimal core element for each side of the market. Moreover, as in the assignment game, in the buyers-optimal core allocation each buyer is paid his marginal contribution, the Vickrey payoff (Vickrey, 1961).

The set of (discriminatory) competitive equilibria of the one-seller assignment market with multi-unit demands does not in general coincide with the core. When we focus on the competitive equilibria, the relevant elements are the buyers and the prices (of objects) no matter who owns the objects. Hence, the set of competitive equilibria does not change if we assume that each object belongs to a different seller. For this related many-to-one assignment market, it can be deduced from Sotomayor (2002) that the set of competitive equilibrium price vectors is non-empty and it is a lattice. Our first result is a characterization of the maximum and the minimum competitive equilibrium price vectors.

The characterization of the minimum competitive equilibrium price vector is inspired by a similar result of Beviá et al. (1999). In their model, there is a finite set of heterogeneous indivisible goods and the agents are buyers that can buy a subset of goods, with no capacity limitation. Utilities are quasi-linear in money so that the preferences of a buyer are represented by a valuation on each possible subset of objects, these valuation functions satisfying submodularity and some additional requirement. Compare to that, our model assumes additivity for the valuation functions of the buyers but, on the other

¹Sotomayor (2013) differentiates between the two definitions of competitive equilibrium. In the one in Jaume et al. (2012) and Massó and Neme (2014) demands are non-discriminatory, since each buyer gets the same utility from all objects in his demanded sets. The demand in Sotomayor (2007, 2013) is discriminatory since in a demanded package the buyer may obtain different utilities from the different objects that form the package. Recently, Arribillaga et al. (2014) consider discriminatory competitive equilibria in assignment markets where sellers own heterogeneous objects.
hand, allows for capacity constraints.

Further, we give conditions for the buyers-optimal core allocation and the seller-optimal core allocation of the one-seller assignment market to be payoff vectors associated to some competitive equilibrium. Finally, we provide conditions on the valuation matrix under which the set of competitive equilibrium payoff vectors coincides with the core.

The paper is organized as follows. In the next section, preliminaries are addressed. In Section 3, we prove the buyers-submodularity property of the coalitional function of the game. In Section 4, we characterize the maximum and the minimum competitive equilibrium price vectors. Section 5 is devoted to study under which conditions the buyers-optimal and the seller-optimal core allocations come from a competitive equilibrium. We also characterize the coincidence between the set of competitive equilibrium payoff vectors and the core.

2 The model and some preliminaries

The one-seller assignment market with multi-unit demands consists of \((M, \{0\}, Q, A, r)\). The finite set of buyers is \(M = \{1, \ldots, m\}\) and the unique seller is denoted by 0. The seller owns a finite set \(Q\) of objects. These objects are indivisible and heterogeneous, but of a similar type, let us say different houses or different part-time jobs.

Each buyer-object pair \((i, j)\) \(\in\) \(M \times Q\) has a potential gain \(a_{ij} \in \mathbb{R}_+\), interpreted as the valuation of object \(j\) by buyer \(i\), where \(\mathbb{R}_+\) stands for the set of non-negative real numbers. The valuation matrix, denoted by \(A = (a_{ij})_{(i, j) \in M \times Q}\), captures each potential gain among all buyer-object pairs. Each buyer \(i \in M\) can acquire \(r_i \in \mathbb{N}\) objects. The vector \(r = (r_i)_{i \in M} \in \mathbb{N}^M\) indicates the buyers’ capacities. We assume that the seller owns some copies of a dummy object, as many as the sum of all buyers’ capacities, \(\sum_{i \in M} r_i\). With some abuse of notation, each copy of this dummy object is denoted by \(j_0\) and each buyer values it at zero. We denote by \(2^Q_{r_i} = \{R \subseteq Q; |R| = r_i\}\) the set of allowable packages of objects for a buyer \(i \in M\) where \(|R|\) denotes the cardinality of \(R\). We assume that buyers valuate packages of objects additively. That is, buyer \(i\) values \(R \in 2^Q_{r_i}\) at \(\sum_{j \in R} a_{ij}\).

A matching \(\mu\) between \(S \subseteq M\) and \(Q\) in the market \((M, \{0\}, Q, A, r)\), is a subset of \(S \times Q\) such that each \(j \in Q\) belongs to at most one pair and each \(i \in S\) belongs to exactly \(r_i\) pairs. Notice that it is possible to match any buyer with dummy objects to complete his capacity. We denote by \(\mathcal{M}(S, Q)\) the set of matchings between \(S \subseteq M\) and \(Q\), and \(\mu(S)\) is the set of objects matched by \(\mu\) to some buyer in \(S\), and when \(S = \{i\}\) we simply write \(\mu(i)\). We denote by \(\mu^{-1}(j)\) the buyer matched to object \(j \in Q\) by matching \(\mu\).

Let \((M, \{0\}, Q, A, r)\) be a market. Given \(S \subseteq M\), a matching \(\mu \in \mathcal{M}(S, Q)\) is optimal for \(S \cup \{0\}\) if

\[
\sum_{(i,j) \in \mu} a_{ij} \geq \sum_{(i,j) \in \mu'} a_{ij} \quad \text{for all } \mu' \in \mathcal{M}(S, Q).
\]

We denote by \(\mathcal{M}_A(S, Q)\) the set of optimal matchings between \(S\) and \(Q\) in this market.
Let us introduce a coalitional game with transferable utility (a game)\(^2\). Consider a one-seller assignment market \((M, \{0\}, Q, A, r)\). The one-seller assignment game related to \((M, \{0\}, Q, A, r)\) is denoted by \((M \cup \{0\}, v_A)\). The worth of each coalition \(T \subseteq M \cup \{0\}\) is given by

\[
v_A(T) = \begin{cases} 
\max_{\mu \in \mathcal{M}(T \setminus \{0\}, Q)} \left\{ \sum_{(i,j) \in \mu} a_{ij} \right\} & \text{if } \{0\} \not\subset T, \\
0 & \text{otherwise}.
\end{cases}
\]

Now, we define competitive equilibrium for one-seller assignment markets with multi-unit demands. First, let us introduce some notions. Given \((M, \{0\}, Q, A, r)\), a price vector \(p = (p_j)_{j \in Q} \in \mathbb{R}_{+}^Q\) consists of one price for each object, with a price of zero for each dummy object. For each \(p \in \mathbb{R}_{+}^Q\), we denote by \(D_i(p) \subseteq 2^{Q \setminus \mu(M)}\) the demand set of buyer \(i\) at level prices \(p\), that is

\[
D_i(p) = \left\{ R \in 2^{Q \setminus \mu(M)} \mid \sum_{j \in R} (a_{ij} - p_j) \geq \sum_{j \in R'} (a_{ij} - p_j) \text{ for all } R' \in 2^{Q \setminus \mu(M)} \right\}.
\]

The demand set of any buyer is never empty, since at sufficiently high prices, the demand set can be formed only by dummy objects.

**Definition 2.1.** A competitive equilibrium for a one-seller assignment market \((M, \{0\}, Q, A, r)\) is a pair \((p, \mu) \in \mathbb{R}_{+}^Q \times \mathcal{M}(M, Q)\), such that the following two conditions hold:

**C.1** For all \(i \in M\), \(\mu(i) \in D_i(p)\),

**C.2** For all \(j \in Q \setminus \mu(M)\), \(p_j = 0\).

If a pair \((p, \mu)\) is a competitive equilibrium, we say that \(p\) is a competitive equilibrium price vector. In a competitive equilibrium, every buyer maximizes his utility given the prices for the objects.

**Definition 2.2.** Let \((M, \{0\}, Q, A, r)\) be a one-seller assignment market and \((p, \mu) \in \mathbb{R}_{+}^Q \times \mathcal{M}(M, Q)\). The payoff vector associated to \((p, \mu)\) is \((U(p, \mu), V(p, \mu)) \in \mathbb{R}^M \times \mathbb{R}\) defined by

\[
U_i(p, \mu) = \sum_{j \in \mu(i)} (a_{ij} - p_j) \quad \text{for each } i \in M, \text{ and}
\]

\[
V(p, \mu) = \sum_{j \in Q} p_j \quad \text{for the seller}.
\]

The following consequences regarding the set of competitive equilibria follow easily for one-seller assignment markets with multi-unit demands. We omit the proofs, they can be derived as a particular case of the model with several sellers in Arribillaga et al. (2014).

**R.1** If \((p, \mu)\) is a competitive equilibrium, then \(\mu\) is optimal and, for all optimal matching \(\mu'\), \((p, \mu')\) is also a competitive equilibrium.

---

\(^2\) A game \((N, v)\) is a pair formed by a finite set of players \(N\) and a characteristic function \(v\) that assigns a real number \(v(S)\) to each coalition \(S \subseteq N\), with \(v(\emptyset) = 0\). The core of a game \((N, v)\) is \(C(v) = \{ x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subseteq N \}\).
3 One-seller assignment game: the core and pairwise-stability

The aim of this section is to analyze the core of the one-seller assignment game. To this end, we use a result in Ausubel and Milgrom (2002) for a similar market where valuations of buyers are more general. They introduce the notion of buyers-submodularity, which means that the marginal contribution of a buyer to a coalition containing the seller decreases as the coalition grows larger. A game \((M \cup \{0\}, v)\) is buyers-submodular if

\[
v((T \cup \{i\}) \cup \{0\}) - v(T \cup \{0\}) \geq v((S \cup \{i\}) \cup \{0\}) - v(S \cup \{0\}),
\]

for all \(T \subseteq S \subseteq M \setminus \{i\}\) and all \(i \in M\). In order to describe the core of the one-seller assignment game, we first show that the coalitional function satisfies the buyers-submodularity condition.

**Theorem 3.1.** Let \((M, \{0\}, Q, A, r)\) be a one-seller assignment market and \((M \cup \{0\}, v_A)\) be its related one-seller assignment game. Then \((M \cup \{0\}, v_A)\) is buyers-submodular.

**Proof.** First, consider \(r_i = 1\) for all \(i \in M\). Let \((M \cup \{0\}, v_A)\) be the one-seller assignment game. We deduce from Theorem 1 in Shapley (1962), that for all \(i, i' \in M\) and all \(S \subseteq M \setminus \{i, i'\}\)

\[
v_A((S \cup \{0\}) \cup \{i\}) - v_A(S \cup \{0\}) \geq v_A((S \cup \{0\}) \cup \{i, i'\}) - v_A((S \cup \{0\}) \cup \{i'\})
\]

and, by repeatedly applying this, we obtain that \((M \cup \{0\}, v_A)\) satisfies (1).

Now, consider \(r_i \geq 1\) for all \(i \in M\). We prove that \((M \cup \{0\}, v_A)\) satisfies (1). Define a related market in which each buyer \(i \in M\) is replicated \(r_i\) times. Denote by \(i(s)\) the \(s\)-th copy of \(i\) and by \(\tilde{M}\) the new set of buyers formed by replicating all buyers in \(M\). Notice that now each buyer has capacity one. Define the valuation matrix \(A = (a_{ij})_{(i(s),j) \in \tilde{M} \times Q}\) by \(a_{ij} = a_{ij}\) for all \((i, j) \in M \times Q\) and all \(s \in \{1, \ldots, r_i\}\). In this way, we obtain \((\tilde{M}, \{0\}, Q, \tilde{A}, \tilde{r})\) with \(\tilde{r}_{i(s)} = 1\) for all \(i(s) \in \tilde{M}\). Notice that \((\tilde{M} \cup \{0\}, v_A)\) and \((\tilde{M} \cup \{0\}, v_{\tilde{A}})\) are related:

\[
v_A(S \cup \{0\}) = v_{\tilde{A}}(\tilde{S} \cup \{0\}) \text{ for all } S \subseteq M,
\]

Let \((M \cup \{0\}, v)\) be a game. Then expression (1) is equivalent to

\[
v(S_1 \cup \{0\}) + v(S_2 \cup \{0\}) \geq v((S_1 \cup S_2) \cup \{0\}) + v((S_1 \cap S_2) \cup \{0\}),
\]

for all \(S_1, S_2 \subseteq M\).
where \( \tilde{S} \) is formed by the replica of all buyers in \( S \). Then, inequality (1) for the game \((M \cup \{0\}, v_A)\) is equivalent to

\[
v_A((\tilde{T} \cup \{i(1), \ldots, i(r_i)\}) \cup \{0\}) - v_A(\tilde{T} \cup \{0\}) \geq v_A((\tilde{S} \cup \{i(1), \ldots, i(r_i)\}) \cup \{0\}) - v_A(\tilde{S} \cup \{0\}),
\]

where \( \tilde{T}, \tilde{S} \) and \( \{i(1), \ldots, i(r_i)\} \) are obtained by replicating all buyers in \( T \), \( S \) and \( \{i\} \), respectively. Define \( S_1 = \tilde{T} \cup \{i(1), \ldots, i(r_i)\}, S_2 = \tilde{S} \) and notice that \( S_1 \cup S_2 = \tilde{S} \cup \{i(1), \ldots, i(r_i)\} \) and \( S_1 \cap S_2 = \tilde{T} \). Since demands of agents in \( \tilde{M} \) are unitary, the game \((\tilde{M} \cup \{0\}, v_A)\) satisfies (1), which is equivalent to (2). Then,

\[
v_A(S_1 \cup \{0\}) + v_A(S_2 \cup \{0\}) \geq v_A((S_1 \cup S_2) \cup \{0\}) + v_A((S_1 \cap S_2) \cup \{0\}),
\]

and by reordering terms we obtain (4). Therefore (1) holds for the game \((M \cup \{0\}, v_A)\).

From Theorem 3.1, the one-seller assignment game is buyers-submodular. Then, by Ausubel and Milgrom (2002), the core is non-empty and can be described as

\[
C(v_A) = \left\{ (U, V) \in \mathbb{R}^M \times \mathbb{R} \mid \sum_{i \in M} U_i + V = v_A(M \cup \{0\}), \ 0 \leq U_i \leq M_i^{v_A} \text{ for all } i \in M \right\},
\]

where \( M_i^{v_A} = v_A(M \cup \{0\}) - v_A((M \setminus \{i\}) \cup \{0\}) \) denotes the marginal contribution of buyer \( i \in M \) to the grand coalition. Furthermore, the core is a lattice with respect to the usual order defined on buyers’ payoffs. Then, we can guarantee the existence of one optimal core allocation for each side of the market. In the buyers-optimal core allocation \((U_i, V) \in \mathbb{R}^M \times \mathbb{R}\), each buyer gets his marginal contribution, that is, \( U_i = M_i^{v_A} \) for all \( i \in M \) and \( V = v_A(M \cup \{0\}) - \sum_{i \in M} M_i^{v_A} \). On the other hand, in the seller-optimal core allocation \((U, V) \in \mathbb{R}^M \times \mathbb{R}\), each buyer \( i \in M \) gets \( U_i = 0 \) and \( V = v_A(M \cup \{0\}) \). Thus, the core of the one-seller assignment game has an optimal core allocation for each market sector as it happens in the classical assignment game. This is not known to be true for other many-to-many assignment models (see e.g. Sotomayor, 2002).

Moreover, as a consequence of \( U_i = M_i^{v_A} \) for all \( i \in M \), in the rule that assigns to each such market \((M, \{0\}, Q, A, r)\) the buyers-optimal core allocation, no buyer has incentives to misrepresent his true valuations.

One of the objectives of this paper is to study in which circumstances this outstanding core element, which is the buyers-optimal core allocation, is supported by competitive equilibrium prices.

### 3.1 Pairwise-stability

With the aim of better analyze competitive equilibria, let us introduce the notion of pairwise-stability which has been widely studied in other many-to-many assignment markets (Sotomayor, 1992, 2002, 2007). It also appears in Sotomayor (2013) with the name of strong stability. This notion of pairwise-stability focuses on the payoff an agent receives due to each single trade she/he takes part in.

Let us define configuration of payoffs and feasible outcomes.
Definition 3.2. Let \((M, \{\emptyset\}, Q, A, r)\) be a one-seller assignment market. Given \(\mu \in \mathcal{M}(M, Q)\), a configuration of payoffs compatible with \(\mu\) is \((u, v) = ((u_{ij})_{(i,j) \in \mu}, (v_j)_{j \in Q}) \in \mathbb{R}^b \times \mathbb{R}^Q\), where \(b = \sum_{i \in M} r_i\), such that:

1. \(u_{ij} + v_j = a_{ij}, u_{ij} \geq 0, v_j \geq 0\) for all \((i, j) \in \mu\),
2. \(v_j = 0\) if \(j \in Q \setminus \mu(M)\).

We can interpret \(u_{ij}\) as the utility of buyer \(i\) acquiring object \(j\). Similarly, \(v_j\) is the utility for the seller because of the sale of object \(j\). If an object is not sold, then the seller receives a profit of zero from this object. Notice that \(u_{ij}\) is not defined for \((i, j) \in (M \times Q) \setminus \mu\).

Definition 3.3. Let \((M, \{\emptyset\}, Q, A, r)\) be a one-seller assignment market. A feasible outcome \((u, v; \mu)\), consists of a matching \(\mu \in \mathcal{M}(M, Q)\) and a configuration of payoffs \((u, v)\) compatible with \(\mu\).

In the following, we define the pairwise-stable outcomes.

Definition 3.4. Let \((M, \{\emptyset\}, Q, A, r)\) be a one-seller assignment market. A feasible outcome \((u, v; \mu)\) is a pairwise-stable outcome if for all \(i \in M\), \(u_{ij} + v_k \geq a_{ik}\) for all \(j \in \mu(i)\) and all \(k \in Q \setminus \mu(i)\).

A feasible outcome is pairwise-stable if there is no pair formed by a buyer \(i \in M\) and an object \(k \in Q\) that are not matched together by \(\mu\) but, if they were (maybe after breaking a previous partnership), then the buyer would be better off and the seller would receive a higher payoff from this object \(k\). A pairwise-stable outcome is stable in strong sense, since the agents do not regret any single partnership. Moreover, if we assume for a moment that each object were owned by a different seller, Definition 3.4 means that the feasible outcome is not blocked by any buyer-seller pair. Our next proposition asserts that, as it happens in the classical assignment market, each pairwise-stable outcome is associated with a competitive equilibrium and vice versa.

Proposition 3.5. Let \((M, \{\emptyset\}, Q, A, r)\) be a one-seller assignment market. The outcome \((u, v; \mu)\) is a pairwise-stable outcome if and only if \((v, \mu)\) is a competitive equilibrium.

Proof. We first prove the “if” part. Let \((v, \mu) \in \mathbb{R}^Q \times \mathcal{M}(M, Q)\) be a competitive equilibrium. Define \(u \in \mathbb{R}^b\) by \(u_{ij} = a_{ij} - v_j\) if \(j \in \mu(i)\), for all \(i \in M\). We see that \((u, v; \mu)\) is a pairwise-stable outcome. Assume on the contrary that \((u, v; \mu)\) is not pairwise-stable, then there is some \(j' \in \mu(i)\) such that, \(a_{ij'} - v_j' < a_{ik} - v_k\) for some \(k \in Q \setminus \mu(i)\). Define \(T = (\mu(i) \setminus \{j'\}) \cup \{k\}\). Notice that \(T \in 2_Q\), then

\[
\sum_{j \in \mu(i)} (a_{ij} - v_j) < \sum_{j \in T} (a_{ij} - v_j),
\]

which implies \(\mu(i) \notin D_i(v)\) and contradicts the fact that \((v, \mu)\) is a competitive equilibrium. Hence \((u, v; \mu)\) is a pairwise-stable outcome.

Now, we prove the “only if” part. Given a pairwise-stable outcome \((u, v; \mu)\), let us see that \((v, \mu)\) is a competitive equilibrium. Stability implies \(u_{ij} = a_{ij} - v_j \geq a_{ik} - v_k\) for all \(j \in \mu(i)\) and all \(k \in Q \setminus \mu(i)\). As a consequence, for any \(T \in 2_Q\), \(\sum_{j \in \mu(i)} (a_{ij} - v_j) \geq \sum_{j \in T} (a_{ij} - v_j)\) and then \(\mu(i) \in D_i(v)\) for each \(i \in M\). Besides, by feasibility of \((u, v; \mu)\), \(v_j = 0\) if \(j \in Q \setminus \mu(M)\). Hence \((v, \mu)\) is a competitive equilibrium. \(\square\)
As a consequence of the coincidence stated in Proposition 3.5, we can easily describe
the set of competitive equilibrium outcomes by means of a finite set of linear equalities
and inequalities, as shown in Example 3.6. Nevertheless, this coincidence does not gen-
erally hold for many-to-many assignment markets (see, e.g. Sotomayor, 2007). However,
as it is also the case for more general markets, the set of competitive equilibrium payoff
vectors of the one-seller assignment game may not coincide with the core. This is shown
in the next example.

Example 3.6. Consider a market with unitary capacities \((M, \{0\}, Q, A, r)\) given by
\(M = \{1, 2\}, Q = \{1', 2'\}\) and \(r = (1, 1)\). For the purposes of this example, we show no
dummy objects. The valuation matrix \(A\) is the following
\[
\begin{pmatrix}
1' & 2' \\
1 & 5 \\
2 & 4
\end{pmatrix}
\]
Consider the one-seller assignment game \((M \cup \{0\}, v_A)\). By (5), the core can be described
by the set of payoff vectors \((U, V) \in \mathbb{R}_+^2 \times \mathbb{R}_+\) such that \(U_1 + U_2 + V = 8, U_2 + V \geq 4\)
which implies that buyer one gets \(U_1 \leq 4\), and \(U_1 + V \geq 5\) which implies that buyer 2
gets \(U_2 \leq 3\).

Take the unique optimal matching \(\mu = \{(1, 2'), (2, 1')\}\). By Proposition 3.5, we know
that any competitive equilibrium \((v, \mu)\), and its related pairwise-stable outcome \((u, v; \mu)\),
satisfies (by feasibility):
\[
[i] \quad u_{12} + v_2 = 4 \quad [ii] \quad u_{21} + v_1 = 4,
\]
and (by pairwise-stability):
\[
[iii] \quad u_{12} + v_1 \geq 5 \quad [iv] \quad u_{21} + v_2 \geq 2,
\]

Making use of \([iii]\) and \([ii]\), we get \(u_{12} - u_{21} \geq 1\). Moreover, from \([iv]\) and \([i]\), we
get \(u_{21} - u_{12} \geq -2\). Then, \(U_1 - U_2 \geq 1\) and \(U_2 - U_1 \geq -2\), where the payoff for buyer
1 is $U_1 = u_{12}$ because of his unitary capacity, and similarly for buyer 2. We denote by $PS(A)$ the set of payoff vectors $(U_1, U_2; V)$ associated to the pairwise-stable outcomes. By Proposition 3.5, we know $PS(A)$ coincides with the set of competitive equilibrium payoff vectors, $CE(A)$. Notice that $C(v_A) \supseteq CE(A) = PS(A)$. In Figure 1, we depict them as subsets of the core of the game.

4 Maximum and minimum competitive equilibrium price vectors

In this section, we focus on the set of competitive equilibrium price vectors of the market. We prove that: the maximum competitive equilibrium price of an object is its marginal contribution to the market; the minimum competitive equilibrium price of an object is its marginal contribution to the market enlarged with a replica of this object under the condition that the two copies of the object cannot be assigned to the same buyer. To show this, we consider a related many-to-one assignment market in which each object is owned by a different seller.

Given a one-seller assignment market $(M, \{0\}, Q, A, r)$, let $(M, Q, A, r)$ represent a many-to-one assignment market where each buyer $i \in M$ values object $j \in Q$ at $a_{ij} \geq 0$ and can establish $r_i$ partnerships. Each object is owned by a different seller, we identify the set of sellers with the set of objects $Q$, and hence each seller can only be matched to at most one buyer. This many-to-one assignment market is studied in the final section of Sotomayor (2002). It can be deduced from there that the set of competitive equilibrium price vectors is a complete lattice with the usual order: $p \geq p'$ if $p_j \geq p'_j$ for all $j \in Q$. This means that if $p, p' \in \mathbb{R}_+^Q$ are competitive equilibrium price vectors then $p \vee p'$ and $p \wedge p'$ are also competitive equilibrium price vectors, where $(p \vee p')_j = \max\{p_j, p'_j\}$ and $(p \wedge p')_j = \min\{p_j, p'_j\}$ for all $j \in Q$. The lattice structure of the set of competitive equilibrium price vectors guarantees the existence of the maximum and the minimum competitive equilibrium prices. In the following, we characterize these two price vectors.

The reader will easily realize that the set of competitive equilibria in the many-to-one assignment market $(M, Q, A, r)$ and in the one seller assignment market $(M, \{0\}, Q, A, r)$ is the same, since in the definition of competitive equilibrium the owners of the objects do not play any role.

Let $(M, Q, A, r)$ be the above many-to-one assignment market and $(M \cup Q, w)$ its associated game. The set of agents is $M \cup Q$ and for each $T \subseteq M \cup Q$, we have

$$w(T) = \begin{cases} \max_{\mu \in \mathcal{M}(T \cap M, T \cap Q)} \left\{ \sum_{(i,j) \in \mu} a_{ij} \right\} & \text{if } T \cap M \neq \emptyset \text{ and } T \cap Q \neq \emptyset, \\ 0 & \text{otherwise}. \end{cases}$$

In the sequel, we first characterize the maximum competitive equilibrium price vector and later on the minimum one.

Now, to obtain the maximum competitive equilibrium prices, let us consider a related one-to-one assignment market. Given a market $(M, Q, A, r)$, we define a one-to-one
assignment market in which each buyer \( i \in M \) is replicated \( r_i \) times. Denote by \( i(s) \) the \( s \)-th copy of \( i \) (each copy has capacity one) and by \( \hat{M} \) the new set of buyers formed by replicating all buyers in \( M \). Define the valuation matrix \( \hat{A} = (\hat{a}_{i(s),j})_{(i(s),j) \in \hat{M} \times Q} \) by \( \hat{a}_{i(s),j} = a_{ij} \) for all \( (i, j) \in M \times Q \) and all \( s \in \{1, ..., r_i\} \). In this way, we obtain the market \((\hat{M}, Q, \hat{A})\). The associated game is \((\hat{M} \cup Q, w_{\hat{A}})\). The set of agents is \( \hat{M} \cup Q \) and for each \( T \subseteq \hat{M} \cup Q \), we have

\[
w_{\hat{A}}(T) = \begin{cases} 
\max_{\mu \in \mathcal{M}(\hat{M} \cup Q)} \left\{ \sum_{(i(s),j) \in \mu} \hat{a}_{i(s),j} \right\} & \text{if } T \cap \hat{M} \neq \emptyset \text{ and } T \cap Q \neq \emptyset, \\
0 & \text{otherwise.}
\end{cases}
\]

Notice that \((M \cup Q, w_A)\) and \((\hat{M} \cup Q, w_{\hat{A}})\) are related as follows:

\[
w_A(M \cup Q') = w_{\hat{A}}(\hat{M} \cup Q'), \quad \text{for all } Q' \subseteq Q. \quad (6)
\]

The game \((\hat{M} \cup Q, w_{\hat{A}})\) is a classical Shapley and Shubik (1972) assignment game. Because of the coincidence of the competitive equilibrium payoff vectors and the core in this model, it is known that the maximum competitive equilibrium payoff vector for the market \((\hat{M}, Q, \hat{A})\) is \( \mathbf{p} \in \mathbb{R}_+^Q \), where

\[
\mathbf{p}_\beta = w_{\hat{A}}(\hat{M} \cup Q) - w_{\hat{A}}(\hat{M} \cup (Q \setminus \{\beta\})) = w_A(M \cup Q) - w_A(M \cup (Q \setminus \{\beta\})) \quad (7)
\]

for each \( \beta \in Q \). The second equality comes from (6).

**Proposition 4.1.** Let \((M, \{0\}, Q, A, r)\) be a one-seller assignment market, \((M \cup Q, w_A)\) the related many-to-one assignment game and \( \mathbf{p} = (\mathbf{p}_\beta)_{\beta \in Q} \) be defined as

\[
\mathbf{p}_\beta = w_A(M \cup Q) - w_A(M \cup (Q \setminus \{\beta\})) \quad \text{for each } \beta \in Q.
\]

Then \( \mathbf{p} \) is the maximum competitive equilibrium price vector in both markets.

**Proof.** Consider the related markets \((M, Q, A, r)\) and \((\hat{M}, Q, \hat{A})\) introduced before. For the one-to-one assignment market \((\hat{M}, Q, \hat{A})\), the core and the set of competitive equilibrium payoff vectors are known to coincide. Moreover, \( \mathbf{p}_\beta \) as defined above is known to be the maximum core payoff of the seller \( \beta \) in the market \((\hat{M}, Q, \hat{A})\). It is straightforward, and hence left to the reader, to see that if \( p \in \mathbb{R}_+^Q \) is a competitive equilibrium price vector for \((\hat{M}, Q, \hat{A})\), it is a competitive equilibrium price vector for \((M, Q, A, r)\). Then \( (\mathbf{p}, \mu) \in \mathbb{R}_+^Q \times \mathcal{M}(M, Q) \) is a competitive equilibrium for \((M, Q, A, r)\). Let \((u, \mathbf{p}) \in \mathbb{R}^M \times \mathbb{R}_+^Q \) be such that \( u_i = \sum_{j \in \mu(i)} (a_{ij} - \mathbf{p}_j) \) for each \( i \in M \). Therefore \((u, \mathbf{p})\) belongs to the core of \((M \cup Q, w_A)\). Since in the core, each agent receives a payoff not exceeding his/her marginal contribution, then \( \mathbf{p}_j \) is the maximum core payoff for any seller \( j \in Q \) in the game \((M \cup Q, w_A)\). It is known from Sotomayor (2002), and can also be deduced from Ma (1998), that for the many-to-one market \((M, Q, A, r)\), the core and the competitive equilibrium payoff vectors coincide. Then \( \mathbf{p} \) is the maximum competitive equilibrium price vector for \((M, Q, A, r)\). Since \((p, \mu)\) is a competitive equilibrium for \((M, Q, A, r)\) if and only if it is a competitive equilibrium for \((M, \{0\}, Q, A, r)\), we get that \( \mathbf{p} \) is the maximum competitive equilibrium price vector for \((M, \{0\}, Q, A, r)\). □
Let \( M \times \mu \) where \( \beta \) not assign both objects \( \beta \) object \( R \) defined at the beginning of this section. Let us prove that \((p,\mu)\) is a competitive equilibrium price vector.

Proof. Given \((M,0,Q,A,r)\) recall the associated many-to-one market \((M,Q,A,\rho)\) defined at the beginning of this section. Let us prove that \((p,\mu)\) is a competitive equilibrium for \((M,Q,A,r)\).

Take \( \mu \in M_{\alpha}(M,Q) \). Assume that there is some buyer \( i^* \) such that \( \mu(i^*) \notin D_{\alpha}(\mu) \) and there exists \( R \in D_{\alpha}(p) \) with \( R \in 2^Q \setminus \{ \mu(i^*) \} \). We will prove that any object \( \alpha \in R \setminus \mu(i^*) \) can be replaced with any \( \beta \in \mu(i^*) \setminus R \) to obtain \((R \setminus \{\alpha}\} \cup \{\beta\} \in D_{\alpha}(p)\).
By way of contradiction, assume that there exists \( \alpha \in R \setminus \mu(i^*) \) and \( \beta \in \mu(i^*) \setminus R \) such that \( (R \setminus \{\alpha\}) \cup \{\beta\} \notin D_{r,p} \). Then

\[
\sum_{j \in R} (a_{i^*j} - p_j) > \sum_{j \in (R \setminus \{\alpha\}) \cup \{\beta\}} (a_{i^*j} - p_j)
\]

which implies \( p_{\beta} - p_{\alpha} > a_{i^*\beta} - a_{i^*\alpha} \).

and from (10) we obtain,

\[
\tilde{w}_{A^\beta}(M \cup Q \cup \{\tilde{\beta}\}) - \tilde{w}_{A^\alpha}(M \cup Q \cup \{\tilde{\alpha}\}) > a_{i^*\beta} - a_{i^*\alpha}.
\]

Take any \( \mu_1 \in \tilde{\mathcal{M}}_{A^\beta}(M, Q \cup \{\tilde{\beta}\}) \) such that \( \beta \in \mu_1(i^*) \) and if \( k \in Q \setminus \mu(M) \) then \( k \notin \mu_1(M) \). Notice that such a matching does exist because of Lemma A.2 in Appendix.

We consider the following cases.

Case 1. \( \alpha \notin \mu_1(i^*) \). Define \( \mu' = (\mu_1 \setminus \{(i^*, \beta)\}) \cup \{(i^*, \tilde{\alpha})\} \) where \( \tilde{\alpha} \) is the replica of object \( \alpha \in Q \). Then \( \mu' \in \tilde{\mathcal{M}}(M, Q \cup \{\tilde{\alpha}\}) \) and

\[
\tilde{w}_{A^\alpha}(M \cup Q \cup \{\tilde{\alpha}\}) \geq \tilde{w}_{A^\beta}(M \cup Q \cup \{\tilde{\beta}\}) - \sum_{j \in \mu_1(i^*)} a_{i^*j} + \sum_{j \in \mu'(i^*)} a_{i^*j}
\]

\[
= \tilde{w}_{A^\beta}(M \cup Q \cup \{\tilde{\beta}\}) - a_{i^*\beta} + a_{i^*\tilde{\alpha}},
\]

and since \( a_{i^*\tilde{\alpha}} = a_{i^*\alpha} \), this contradicts (11).

Case 2. \( \alpha \in \mu_1(i^*) \). Since \( \alpha \in \mu_1(M) \), because of the properties of \( \mu_1 \), then \( \alpha \in \mu(M) \).

Let \( i' \in M \) be such that \( \alpha \in \mu(i') \). Then since \( \alpha \in R \setminus \mu(i^*) \), we deduce \( i' \neq i^* \).

Case 2.1. \( \tilde{\beta} \notin \mu_1(M) \). Define \( \overline{\mu} = \{(i, j) \in M \times (Q \cup \{\tilde{\beta}\})(i, j) \in \mu\} \) and notice that \( \overline{\mu} \in \tilde{\mathcal{M}}_{A^\beta}(M, Q \cup \{\tilde{\beta}\}) \). Define \( \mu' = (\overline{\mu} \setminus \{(i^*, \beta)\}) \cup \{(i^*, \tilde{\alpha})\} \). Notice that \( \mu' \in \tilde{\mathcal{M}}(M, Q \cup \{\tilde{\alpha}\}) \) Then we have

\[
\tilde{w}_{A^\alpha}(M \cup Q \cup \{\tilde{\alpha}\}) - \tilde{w}_{A^\beta}(M \cup Q \cup \{\tilde{\beta}\}) \leq \sum_{i \in M} \left( \sum_{j \in \mu(i)} a_{ij} - \sum_{j \in \mu'(i)} a_{ij} \right) = a_{i^*\beta} - a_{i^*\tilde{\alpha}},
\]

which contradicts (11).

Case 2.2. \( \beta \in \mu_1(M) \). Let \( i'' \) be such that \( \tilde{\beta} \in \mu_1(i'') \).

Case 2.2.1. \( i' = i'' \). Define \( \mu' = (\mu_1 \setminus \{(i', \tilde{\beta})\}) \cup \{(i', \tilde{\alpha})\} \). Then, \( \mu' \in \tilde{\mathcal{M}}(M, Q \cup \{\tilde{\alpha}\}) \),

\[
\tilde{w}_{A^\alpha}(M \cup Q \cup \{\tilde{\alpha}\}) \geq \tilde{w}_{A^\beta}(M \cup Q \cup \{\tilde{\beta}\}) - \sum_{j \in \mu_1(i')} a_{i'j} + \sum_{j \in \mu'(i')} a_{i'j}
\]

\[
= \tilde{w}_{A^\alpha}(M \cup Q \cup \{\tilde{\alpha}\}) - a_{i'\beta} + a_{i'\tilde{\alpha}}.
\]

By the optimality of \( \mu \), we have \( a_{i'\beta} + a_{i'\alpha} \geq a_{i^*\beta} + a_{i^*\alpha} \). As a consequence,

\[
a_{i^*\beta} - a_{i^*\alpha} \geq a_{i'\beta} - a_{i'\alpha} \geq \tilde{w}_{A^\beta}(M \cup Q \cup \{\tilde{\beta}\}) - \tilde{w}_{A^\alpha}(M \cup Q \cup \{\tilde{\alpha}\}),
\]

which contradicts (11).

Case 2.2.2. \( i' \neq i'' \). Since \( \alpha \in \mu(i') \setminus \mu_1(i') \), there is an object \( \beta_0 \in \mu_1(i') \setminus \mu(i') \).
Case 2.2.2.a. \( \beta_0 \in \mu(i^*) \). Define \( \mu' = (\mu_1 \setminus \{(i', \beta_0), (i^*, \beta)\}) \cup \{(i', \tilde{\alpha}), (i^*, \beta_0)\} \). Let us continue denoting by \( \mu' \) the matching that results by interchanging the roles of \( \beta \) and \( \beta' \) above. Then trivially \( \mu' \in \mathcal{M}(M, Q \cup \{\tilde{\alpha}\}) \) and we get

\[
\tilde{w}_{A^\beta}(M \cup Q \cup \{\tilde{\beta}\}) - \tilde{w}_{A^{\mu'}}(M \cup Q \cup \{\tilde{\alpha}\}) \leq \sum_{i \in M} \left( \sum_{j \in \mu_1(i)} a_{ij} - \sum_{j \in \mu'(i)} a_{ij} \right) = (a_{i'\beta_0} - a_{i'\tilde{\alpha}}) + (a_{i^*\beta} - a_{i^*\beta_0}). \tag{12}
\]

By optimality of \( \mu' \), we have \( a_{i'\beta_0} + a_{i^*\beta_0} \leq a_{i^*\beta_0} + a_{i'\alpha} \). Then, \( (a_{i'\beta_0} - a_{i'\alpha}) + (a_{i^*\beta} - a_{i^*\beta_0}) \leq a_{i^*\beta} - a_{i^*\alpha} \) together with (12) contradicts (11).

Case 2.2.2.b. \( \beta_0 \in \mu(i') \). Define \( \mu' = (\mu_1 \setminus \{(i', \beta_0), (i'', \tilde{\beta})\}) \cup \{(i', \tilde{\alpha}), (i'', \beta_0)\} \). Then

\[
\tilde{w}_{A^\beta}(M \cup Q \cup \{\tilde{\beta}\}) - \tilde{w}_{A^{\mu'}}(M \cup Q \cup \{\tilde{\alpha}\}) \leq \sum_{i \in M} \left( \sum_{j \in \mu_1(i)} a_{ij} - \sum_{j \in \mu'(i)} a_{ij} \right) = (a_{i'\beta_0} - a_{i'\tilde{\alpha}}) + (a_{i''\tilde{\beta}} - a_{i''\beta_0}) = (a_{i'\beta_0} - a_{i'\tilde{\alpha}}) + (a_{i''\beta} - a_{i''\beta_0}) = (a_{i^*\alpha} - a_{i^*\beta}) + (a_{i^*\beta} - a_{i^*\alpha}). \tag{13}
\]

By optimality of \( \mu' \), we have \( a_{i'\beta_0} + a_{i''\beta} \leq a_{i^*\beta} + a_{i''\beta_0} + a_{i''\beta} \). Then \( (a_{i'\beta_0} - a_{i'\alpha}) + (a_{i''\beta} - a_{i''\beta_0}) + (a_{i^*\alpha} - a_{i^*\beta}) + (a_{i^*\beta} - a_{i^*\alpha}) \leq a_{i^*\beta} - a_{i^*\alpha} \) together with (13) contradicts (11).

Case 2.2.2.c. \( \beta_0 \notin \mu(i^*) \) and \( \beta_0 \notin \mu(i'') \). Recall that \( \beta_0 \in \mu_1(i') \) and by the assumptions on \( \mu \) and the properties of \( \mu_1 \), there is some \( i_1 \) such that \( \beta_0 \notin \mu(i_1) \). Since \( \beta_0 \notin \mu_1(i_1) \) there is some \( \beta_1 \in \mu_1(i_1) \setminus \mu(i_1) \). If \( \beta_1 \in \mu(i^*) \) or \( \beta_1 \in \mu(i'') \) we finish by an argument similar to cases 2.2.2.a and 2.2.2.b, respectively. Otherwise, we continue. Denote \( i_0 = i' \) and assume we reach a sequence \( \beta_0, \beta_1, ..., \beta_t \) such that \( \beta_t \in \mu_1(i_t) \setminus \mu(i_t) \), \( \beta_t \in \mu(i_{t+1}) \setminus \mu(i_{t+1}) \) for \( t \in \{0, 1, ..., l - 1\} \). The same argument as in the proof of Lemma A.2 allows us to choose all \( \beta_0, \beta_1, ..., \beta_l \) to be different. Hence the procedure ends at some step \( l \geq 1 \) such that either \( \beta_l \in \mu_1(i^*) \) or \( \beta_l \in \mu_1(i'') \). Assume \( \beta_l \in \mu(i^*) \) and define \( \mu' = (\mu_1 \setminus \{(i', \beta_0), (i^*, \beta), (i_1, \beta_1), ..., (i_l, \beta_l)\}) \cup \{(i', \tilde{\alpha}), (i_1, \beta_0), ..., (i_1, \beta_l), (i^*, \beta_l)\} \) then

\[
\tilde{w}_{A^\beta}(M \cup Q \cup \{\tilde{\beta}\}) - \tilde{w}_{A^{\mu'}}(M \cup Q \cup \{\tilde{\alpha}\}) \leq \sum_{i \in M} \left( \sum_{j \in \mu_1(i)} a_{ij} - \sum_{j \in \mu'(i)} a_{ij} \right) = (a_{i'\beta_0} - a_{i'\tilde{\alpha}}) + (a_{i^*\beta} - a_{i^*\beta_l}) + \sum_{k=1}^{l} (a_{i_k\beta_k} - a_{i_k\beta_{k-1}}). \tag{14}
\]

By optimality of \( \mu' \), we have \( a_{i'\beta_0} + a_{i^*\beta} + \sum_{k=1}^{l} a_{i_k\beta_k} \leq a_{i'\alpha} + a_{i^*\beta} + \sum_{k=1}^{l} a_{i_k\beta_{k-1}} \). Then,

\[
(a_{i'\beta_0} - a_{i'\alpha}) + (a_{i^*\beta} - a_{i^*\beta_l}) + \sum_{k=1}^{l} (a_{i_k\beta_k} - a_{i_k\beta_{k-1}}) \leq a_{i^*\beta} - a_{i^*\alpha},
\]

together with (14) contradicts (11).

For \( \beta_l \in \mu(i'') \), take the matching \( \mu' = (\mu_1 \setminus \{(i', \beta_0), (i'', \tilde{\beta}), (i_1, \beta_1), ..., (i_l, \beta_l)\}) \cup \{(i', \tilde{\alpha}), (i_1, \beta_0), ..., (i_1, \beta_{l-1}), (i'', \beta_l)\} \) for $M \cup Q \cup \{\tilde{\alpha}\}$ and proceed analogously.
Therefore, we have proved that \((R \setminus \{\alpha\}) \cup \{\beta\} \in D_t(p)\). By repeatedly applying this procedure that replaces an element of \(R \setminus \mu(i^*)\) with another of \(\mu(i^*) \setminus R\), we obtain a sequence of sets of cardinality \(r_t\) such that \(R_t \in D_t(p)\) for all \(t \in \{1, \ldots, l\}\) and \(R_1 = \mu(i^*)\), which proves that \(\mu(i^*) \in D_t(p)\). Notice finally that, from (10), if \(\beta \in Q \setminus \mu(M)\) then \(p_{\beta} = 0\). This concludes that \((p, \mu)\) forms a competitive equilibrium for the market \((M, Q, A, r)\).

Now, we see that indeed \(\overline{p} = (p_{\alpha})_{\beta \in Q}\) defined in (10) is the minimum competitive equilibrium price vector. Let \((p, \mu)\) be any competitive equilibrium. Then we have that for each optimal matching and for each \(i \in M\), \(\sum_{j \in \mu(i)}(a_{ij} - p_j) \geq \sum_{j \in R}(a_{ij} - p_j)\) for all \(R \in 2^Q\). Consider any \(\beta \in Q\) and its replica \(\hat{\beta}\). Take an optimal matching \(\mu \in M_A(M, Q)\) and \(\mu' \in \widetilde{M}_{A^\alpha}(M, Q \cup \{\hat{\beta}\})\) such that if \(\beta \in \mu(i)\) for some \(i \in M\) and \(k \in Q \setminus \mu(M)\), then \(\beta \in \mu'(i)\) and \(k \notin \mu'(M)\). The existence of \(\mu' \in \widetilde{M}_{A^\alpha}(M, Q \cup \{\hat{\beta}\})\) is guaranteed by Lemma A.2. Notice that if \(\beta, \hat{\beta} \notin \mu'(M)\); or \(\beta \in \mu'(M)\) and \(\hat{\beta} \notin \mu'(M)\); or \(\beta \notin \mu'(M)\) and \(\hat{\beta} \notin \mu'(M)\), then \(w_{A^\beta}(M \cup Q \cup \{\hat{\beta}\}) = w_A(M \cup Q)\) which implies that \(p_{\beta} = 0\) and, since \(p_{\beta} \geq 0\) for all \(\beta \in Q\), we have \(p_{\beta} \leq p_\beta\).

Otherwise, if \(\beta, \hat{\beta} \in \mu'(M)\), then there exists some \(\alpha \in \mu(M) \setminus \mu'(M)\). Let \(i' \in M\) be such that \(\hat{\beta} \in \mu'(i')\). Define \(R_t = \mu'(i)\) for all \(i \in M \setminus \{i'\}\) and \(R_r = (\mu'(i') \setminus \{\hat{\beta}\}) \cup \{\beta\}\). Since \((p, \mu)\) is a competitive equilibrium and because of the properties of \(\mu'\), we obtain

\[
\sum_{i \in M} \left( \sum_{j \in \mu(i)} a_{ij} - p_j \right) \geq \sum_{i \in M} \left( \sum_{j \in R_t} a_{ij} - p_j \right) = \sum_{(i,j) \in \mu'} a_{ij} - \left( \sum_{j \in \mu'(M) \setminus \{\beta\}} p_j + p_{\beta} \right).
\]

Then

\[
w_A(M \cup Q) - p_\alpha \geq w_{A^\beta}(M \cup Q \cup \{\hat{\beta}\}) - p_{\beta},
\]

which implies

\[
p_{\beta} \geq w_{A^\beta}(M \cup Q \cup \{\hat{\beta}\}) - w_A(M \cup Q) + p_\alpha
\]

\[
\geq w_{A^\beta}(M \cup Q \cup \{\hat{\beta}\}) - w_A(M \cup Q) = p_{\beta},
\]

the last inequality holds because any competitive equilibrium price is non-negative, \(p_\alpha \geq 0\) for all \(\alpha \in Q\).

The above Proposition 4.1 and Theorem 4.2 characterize, respectively, the maximum and the minimum equilibrium price vectors of both the one-seller assignment market \((M, \{0\}, Q, A, r)\) and the many-to-one assignment market \((M, Q, A, r)\). It can be deduced from page 11 in Sotomayor (2002) that the core of the many-to-one assignment game \((M \cup Q, w_A)\) coincides with the set of competitive equilibrium payoff vectors. Hence, our above results imply the characterization of the maximum and the minimum core payoff of each seller in the many-to-one assignment market of Sotomayor (2002). In particular, this implies that in such markets, each seller attains her marginal contribution as a core payoff.\footnote{See Pérez-Castrillo and Sotomayor (2014) for a more general market in which a seller with unitary capacity cannot manipulate, by misrepresenting her reservation value, any rule that assigns her maximum equilibrium price to this seller.}
5 Core and competitive equilibria in the one-seller assignment market

In this section, we return to the one-seller assignment market to address the relationship between the core and the set of competitive equilibria. For several generalizations (Kaneko, 1976; Camiña, 2006; Sotomayor, 2013) of the classical assignment game, those outcomes coming from a competitive equilibrium are core elements. However, the coincidence of the set of competitive equilibrium payoff vectors with the core is not preserved in general. The aim of this section is to analyze under which conditions the core of one-seller assignment game coincides with the set of competitive equilibrium payoff vectors.

First, we characterize the case in which the buyers-optimal core allocation comes from a competitive equilibrium. Notice that in Example 3.6, the buyers-optimal core allocation is a competitive equilibrium payoff vector. Notwithstanding, it is easy to find instances in which this is not the case. See the following example.

Example 5.1. Consider a market \((M, \{0\}, Q, A, r)\) given by \(M = \{1, 2\}\), \(Q = \{1', 2', 3', j_0\}\) and \(r = (2, 2)\). For the purposes of this example, we show only one dummy object. The valuation matrix \(A\) is the following

\[
\begin{pmatrix}
1' & 2' & 3' & j_0 \\
10 & 8 & 3 & 6 & 0 \\
2 & 1 & 2 & 0
\end{pmatrix}
\]

The buyers-optimal core allocation is \((7, 5; 7)\). Assume that \((p, µ)\) is a competitive equilibrium such that \((U(p, µ); V(p, µ)) = (7, 5; 7)\). Notice that there is a unique \(µ ∈ M_A(M, Q)\), which is circled in the above market. Therefore, we have that \(U_1(p, µ) = \sum_{j ∈ µ(1)}(a_{1j} - p_j) = 7\), and then \(p_2 + p_3 = 2\). Notice also that \(a_{22} + a_{23} - p_2 - p_3 = 1\) and since \(2 ≥ p_j ≥ 0\) for \(j ∈ \{2, 3\}\), then \(a_{2j} - p_j > 0\) for some \(j ∈ \{2, 3\}\). This contradicts that \((p, µ)\) is a competitive equilibrium because \(j_0 ∈ µ(2)\) and \(a_{20} - p_0 = 0\).

As an application of Theorem 4.2 in the previous section, we can calculate the best competitive equilibrium payoff vector for the buyers. To this end, we obtain the minimum competitive equilibrium prices:

\[
\begin{align*}
\underline{p}_{1'} &= \tilde{w}_{A'}(M ∪ Q ∪ \{1'\}) - w_A(M ∪ Q) = 25 - 19 = 6, \\
\underline{p}_{2'} &= \tilde{w}_{A'}(M ∪ Q ∪ \{2'\}) - w_A(M ∪ Q) = 20 - 19 = 1, \\
\underline{p}_{3'} &= \tilde{w}_{A'}(M ∪ Q ∪ \{3'\}) - w_A(M ∪ Q) = 21 - 19 = 2, \text{ and } \underline{p}_{j_0} = 0.
\end{align*}
\]

Hence the minimum competitive equilibrium price vector is \(\underline{p} = (6, 1, 2, 0)\) and the corresponding payoff vector is \((U(\underline{p}, µ); V(\underline{p}, µ)) = (6, 4; 9)\).

The following proposition characterizes when the buyers-optimal core allocation comes from a competitive equilibrium. To this end, we use the result of the minimum equilibrium price vector stated in Theorem 4.2.
Proposition 5.2. Let \((M, \{0\}, Q, A, r)\) be a one-seller assignment market. The buyers-optimal core allocation is a competitive equilibrium payoff vector if and only if there is an optimal matching \(\mu \in \mathcal{M}_A(M, Q)\) such that for each \(i \in M\) it holds that

\[
M_i^{v^A} = \sum_{j \in \mu(i)} \left( a_{ij} - \left( \bar{w}_A(M \cup Q \cup \{\tilde{j}\}) - w_A(M \cup Q) \right) \right),
\]

where \(\bar{w}_A(M \cup Q \cup \{\tilde{j}\})\) is defined in (9).

Proof. We first prove the ‘if’ part. Given \((M, \{0\}, Q, A, r)\) recall the associated many-to-one assignment market \((M, Q, A, r)\) introduced in the previous section. Recall also that \((p, \mu) \in \mathbb{R}_+^Q \times \mathcal{M}_A(M, Q)\) is a competitive equilibrium for \((M, Q, A, r)\) if and only if it is a competitive equilibrium for \((M, \{0\}, Q, A, r)\). Therefore \((p, \mu) \in \mathbb{R}_+^Q \times \mathcal{M}_A(M, Q)\) where \(p\) is defined in (10) is a competitive equilibrium for \((M, \{0\}, Q, A, r)\) and since (15) is assumed, the corresponding core allocation is buyers-optimal since each buyer gets his marginal contribution.

Now, we prove the ‘only if’ part. Take any competitive equilibrium \((p, \mu)\) for \((M, \{0\}, Q, A, r)\). Since competitive equilibrium payoff vectors are core allocations, \(U_i(p, \mu) \leq M_i^{v^A}\) for each \(i \in M\) with \(U_i(p, \mu)\) from Definition 2.2. Then, if the buyers-optimal core allocation comes from a competitive equilibrium, this is the minimum competitive equilibrium price \(p_j = \bar{w}_A(M \cup Q \cup \{\tilde{j}\}) - w_A(M \cup Q)\) for all \(j \in Q\). Therefore, expression (15) holds.

The above proposition allows us to also draw some consequences on the related many-to-one assignment market \((M, Q, A, r)\) of Sotomayor (2002), where \(Q\) is the set of sellers. Recall that for this market the core and the set of competitive equilibrium payoff vectors coincide and moreover the set of competitive equilibrium price vectors coincides with the set of competitive equilibrium price vectors of the one-seller assignment market \((M, \{0\}, Q, A, r)\). Example 7 in Sotomayor (2002) shows that a buyer may not reach his marginal contribution in the core of the game \((M \cup Q, w_A)\). Since the marginal contribution of a buyer in both markets is the same, Proposition 5.2 determines when a buyer achieves his marginal contribution in the core of the many-to-one assignment market \((M, Q, A, r)\).

Another consequence is that if equality (15) is satisfied for a buyer \(i \in M\), then this buyer has no incentives to misrepresent his true valuations in the rule that selects for each market \((M, Q, A, r)\) the minimum competitive equilibrium price vector.

We now focus on the sellers-optimal core allocation. As we can see in Example 3.6, the seller-optimal core allocation may not be a competitive equilibrium payoff vector. In order to study when this core allocation comes from a competitive equilibrium, let us first define the set of desirable objects, \(Q_A^*\). We say that an object is desirable if at least one buyer valuates it positively

\[
Q_A^* = \{j \in Q \mid a_{ij} > 0 \text{ for some } i \in M\}.
\]

The conditions for the seller-optimal core allocation to be a competitive equilibrium payoff vector require that each sold object is acquired by the buyer who valuates it the most and that all desirable objects are sold.
**Proposition 5.3.** Let \( (M, \{0\}, Q, A, r) \) be a one-seller assignment market. The seller-optimal core allocation is a competitive equilibrium payoff vector if and only if there is an optimal matching \( \mu \in \mathcal{M}_{A}(M, Q) \) and the following two conditions are satisfied:

(a) for all \( j \in \mu(M) \) and all \( i \in M \setminus \{\mu^{-1}(j)\} \), \( a_{ij} \leq a_{\mu^{-1}(j)j} \),
(b) \( Q_{A} \subseteq \mu(M) \).

**Proof.** We first prove the ‘if’ part. Assume that \( \mu \in \mathcal{M}_{A}(M, Q) \) satisfies conditions (a) and (b). Define \( p_{j} = a_{\mu^{-1}(j)j} \) for all \( j \in \mu(M) \) and \( p_{j} = 0 \) for all \( j \in Q \setminus \mu(M) \). We show that \( \mu(i) \in D_{i}(p) \) for all \( i \in M \). Take any \( i \in M \) and consider any \( R \in 2^{Q} \). Since \( \mu \) satisfies (a) and (b), and by definition of \( p \), we have

\[
\sum_{j \in R}(a_{ij} - p_{j}) = \sum_{j \in R \cap \mu(M)}(a_{ij} - a_{\mu^{-1}(j)j}) + \sum_{j \in R \setminus \mu(M)}(a_{ij} - 0) \leq 0 = \sum_{j \in \mu(i)}(a_{ij} - p_{j}),
\]

and thus \( \mu(i) \in D_{i}(p) \) for all \( i \in M \). Besides, by definition of \( p \), we get \( p_{j} = 0 \) for each \( j \in Q \setminus \mu(M) \). Notice that \( (U(p, \mu), V(p, \mu)) \) is the seller-optimal core allocation.

Now, we prove the ‘only if’ part. Assume that \( (p, \mu) \) is a competitive equilibrium and \( (U(p, \mu), V(p, \mu)) \) is the seller-optimal core allocation. By property R.1 in page 5, we have that \( \mu \in \mathcal{M}_{A}(M, Q) \). Moreover, in the seller-optimal core allocation the seller’s payoff is equal to \( v_{A}(M \cup \{0\}) \).

We claim that

\[ p_{j} = a_{\mu^{-1}(j)j} \text{ for all } j \in \mu(M). \quad (16) \]

If \( p_{j} > a_{\mu^{-1}(j)j} \) for some \( j \in \mu(M) \), then for all \( R \in D_{\mu^{-1}(j)j}(p) \) we have \( j \notin R \), and as a consequence \( (p, \mu) \) is not a competitive equilibrium. On the other hand, if \( p_{j} < a_{\mu^{-1}(j)j} \) for some \( j \in \mu(M) \) then \( \sum_{j \in Q}p_{j} < v_{A}(M \cup \{0\}) \) and the seller-optimal core allocation is not the payoff vector of \( (p, \mu) \).

Now taking (16) into account, we shall prove that \( \mu \) satisfies condition (a) of the statement. Assume on the contrary that there is some \( i \in M \) such that \( a_{ij} > a_{\mu^{-1}(j)j} \) for some \( j \in Q \) with \( i \in M \setminus \{\mu^{-1}(j)\} \). Let \( R \in \mathcal{Q}^{Q} \) be the package formed by object \( j \) and copies of the dummy object, i.e., \( R = \{j, j_{0}^{1}, j_{0}^{2}, ..., j_{0}^{r-1}\} \). Since \( \sum_{j \in R}(a_{ij} - p_{j}) > 0 = \sum_{j \in \mu(i)}(a_{ij} - p_{j}) \), we obtain that \( \mu(i) \notin D_{i}(p) \) in contradiction with \( (p, \mu) \) being a competitive equilibrium. Then \( \mu \) satisfies (a). In order to show (b), assume on the contrary that, there is some \( j \in Q_{A}^{*} \setminus \mu(M) \). By definition of competitive equilibrium, the price of this object is \( p_{j} = 0 \). Since \( j \in Q_{A}^{*} \), there is some \( i \in M \) such that \( a_{ij} > 0 \). This implies that \( \mu(i) \notin D_{i}(p) \) because \( \sum_{j \in R}(a_{ij} - p_{j}) > \sum_{j \in \mu(i)}(a_{ij} - p_{j}) \) where \( R = \{j, j_{0}^{1}, j_{0}^{2}, ..., j_{0}^{r-1}\} \). This contradicts \( (p, \mu) \) being a competitive equilibrium. Hence, \( \mu \) satisfies (b). \( \square \)

Notice that condition (a) in Proposition 5.3 is not satisfied in Example 3.6, while in Example 5.1, both (a) and (b) are satisfied.

The following theorem is the main result of this section. We characterize the coincidence between the set of competitive equilibrium payoff vectors and the core of the one-seller assignment game.

**Theorem 5.4.** Let \( (M, \{0\}, Q, A, r) \) be a one-seller assignment market and \( (M \cup \{0\}, v_{A}) \) be its associated one-seller assignment game. Then the core of \( (M \cup \{0\}, v_{A}) \) coincides
with the set of competitive equilibrium payoff vectors if and only if there is an optimal matching \( \mu \in \mathcal{M}_A(M, Q) \) that satisfies the following three conditions:

(a) for all \( j \in \mu(M) \) and all \( i \in M \setminus \{ \mu^{-1}(j) \} \), \( a_{ij} \leq a_{\mu^{-1}(j)j} \);

(b) \( Q_A \subseteq \mu(M) \);

(c) \( M_i^{v_A} \leq \sum_{j \in \mu(i)} \left( a_{ij} - \max_{t \in M \setminus \{i\}} \{ a_{ij} \} \right) \) for all \( i \in M \).

Proof. We first prove the ‘if’ part. Assume that some \( \mu \in \mathcal{M}_A(M, Q) \) satisfies (a), (b) and (c). We show that any \((U, V) \in C(v_A)\) is the payoff vector of some competitive equilibrium. By conditions (a) and (c), for each \( i \in M \), we can find some \((\alpha_{ij})_{j \in \mu(i)} \in \mathbb{R}^n\) such that \( a_{ij} \geq \alpha_{ij} \geq \max_{t \in M \setminus \{i\}} \{ a_{ij} \} \) for all \( j \in \mu(i) \) and \( M_i^{v_A} = \sum_{j \in \mu(i)} (a_{ij} - \alpha_{ij}) \).

Take any \((U, V) \in C(v_A)\) and define \( b_i = M_i^{v_A} - U_i \) for all \( i \in M \). Since for all \( i \in M \) we have \( M_i^{v_A} \geq U_i \geq 0 \), then \( M_i^{v_A} \geq b_i \geq 0 \).

Let us define \( p \in \mathbb{R}_+^Q \) by

\[
p_j = \begin{cases} 
\alpha_{\mu^{-1}(j)j} + \frac{a_{\mu^{-1}(j)j} - \alpha_{\mu^{-1}(j)j}}{M_i^{v_A}} b_{\mu^{-1}(j)} & \text{for } j \in \mu(M) \text{ and } M_i^{v_A} > 0, \\
a_{\mu^{-1}(j)j} & \text{for } j \in \mu(M) \text{ and } M_i^{v_A} = 0, \\
0 & \text{for } j \in Q \setminus \mu(M).
\end{cases}
\]

Notice that \( p \in \mathbb{R}_+^Q \). We show that \( \mu(i) \in D_i(p) \) for all \( i \in M \). It is sufficient to see that \( a_{ij} - p_j \geq a_{ik} - p_k \) for all \( j \in \mu(i) \) and all \( k \in Q \setminus \mu(i) \). To this end, let us see that for all \( i \in M \) and all \( j \in \mu(i) \) it holds \( a_{ij} - p_j \geq 0 \) while \( a_{ik} - p_k \leq 0 \) for all \( k \in Q \setminus \mu(i) \). On one hand, take \( i \in M \) such that \( M_i^{v_A} > 0 \). Then \( a_{ij} - p_j = a_{ij} - \alpha_{ij} - \frac{a_{ij} - \alpha_{ij}}{M_i^{v_A}} b_i = (a_{ij} - \alpha_{ij})(1 - \frac{b_i}{M_i^{v_A}}) \geq 0 \) for all \( j \in \mu(i) \). Take \( i \in M \) such that \( M_i^{v_A} = 0 \). Then \( a_{ij} - p_j = a_{ij} - a_{ij} = 0 \) for all \( j \in \mu(i) \). On the other hand, take \( k \in \mu(M) \) such that \( M_k^{v_A} - M_i^{v_A} > 0 \). Then for any \( i \in M \setminus \{ \mu^{-1}(k) \} \), we have

\[
a_{ik} - p_k = a_{ik} - \alpha_{\mu^{-1}(k)k} - \frac{a_{\mu^{-1}(k)k} - \alpha_{\mu^{-1}(k)k}}{M_i^{v_A}} b_{\mu^{-1}(k)} \leq 0 \text{ because } a_{\mu^{-1}(k)k} - \alpha_{\mu^{-1}(k)k} \geq a_{ik}.
\]

Take \( k \in \mu(M) \) such that \( M_k^{v_A} - M_i^{v_A} = 0 \). Then for any \( i \in M \setminus \{ \mu^{-1}(k) \} \), we have \( a_{ik} - p_k = a_{ik} - a_{\mu^{-1}(k)k} \leq 0 \) because of assumption (a). Finally, consider \( k \in Q \setminus \mu(M) \). Then for any \( i \in M \), \( a_{ik} - p_k = 0 \) because of (b). Thus \( \mu(i) \in D_i(p) \) for all \( i \in M \) and \( p \in \mathbb{R}_+^Q \). Hence, \((p, \mu)\) is a competitive equilibrium. Then, the payoffs in this competitive equilibrium are

\[
U_i(p, \mu) = \sum_{j \in \mu(i)} (a_{ij} - p_j) = \sum_{j \in \mu(i)} (a_{ij} - \alpha_{ij} - \frac{a_{ij} - \alpha_{ij}}{M_i^{v_A}} b_i) = \sum_{j \in \mu(i)} (a_{ij} - \alpha_{ij}) \left(1 - \frac{b_i}{M_i^{v_A}}\right) = M_i^{v_A} - b_i = U_i,
\]

for all \( i \in M \) such that \( M_i^{v_A} > 0 \), where the last equality comes from \( \sum_{j \in \mu(i)} (a_{ij} - \alpha_{ij}) = M_i^{v_A} \). Take now any \( i \in M \) such that \( M_i^{v_A} = 0 \). From the definition of \( p_j \) in (17), we have \( U_i(p, \mu) = \sum_{j \in \mu(i)} (a_{ij} - p_j) = \sum_{j \in \mu(i)} (a_{ij} - a_{ij}) = 0 = U_i \). Since \((U(p, \mu), V(p, \mu)) \in C(v_A)\) for any competitive equilibrium \((p, \mu)\), by efficiency the seller’s payoff is \( V(p, \mu) = v_A(M \cup \{0\}) - \sum_{i \in M} U_i(p, \mu) = v_A(M \cup \{0\}) - \sum_{i \in M} U_i = V \). This completes the proof of the ‘if’ part.
Now, we prove the ‘only if’ part. Assume that the core and the set of payoff vectors associated with the competitive equilibria coincide. By Proposition 5.3, conditions (a) and (b) hold for some optimal matching \( \mu \in \mathcal{M}_A(M,Q) \). Then, we only have to prove (c). Assume on the contrary that for this \( \mu \), there is some buyer \( i' \in M \) such that \( M^\mu_{i'} > \sum_{j \in \mu(i')} (a_{ij} - \max_{t \in M \setminus \{i'\}} \{a_{tj}\}) \). Recall the description of the core in (5) and consider \((U,V) \in C(v_A)\) with \( U_i' = M^\mu_{i'}\) for the buyer \( i'\) and \( U_i = 0 \) for all \( i \in M \setminus \{i'\}\). By assumption, there is a competitive equilibrium \((p,\mu')\) such that \((U,V)\) is its payoff vector. Take this competitive equilibrium price vector \( p \) and the matching \( \mu \in \mathcal{M}_A(M,Q) \) such that \( M^\mu_{i'} > \sum_{j \in \mu(i')} (a_{ij} - \max_{t \in M \setminus \{i'\}} \{a_{tj}\}) \). Then \((p,\mu)\) is a competitive equilibrium (recall R.1 in page 5). Therefore \( p_j = a_{i'-1(j)j} \) for all \( j \in \mu(M \setminus \{i'\}) \) and \( M^\mu_i = \sum_{j \in \mu(i')} (a_{ij} - p_j) \). We obtain \( \sum_{j \in \mu(i')} (a_{ij} - p_j) = M^\mu_{i'} > \sum_{j \in \mu(i')} (a_{ij} - \max_{t \in M \setminus \{i'\}} \{a_{tj}\}) \). As a consequence, \( \sum_{j \in \mu(i')} \max_{t \in M \setminus \{i'\}} \{a_{ij}\} > \sum_{j \in \mu(i')} p_j \) which implies that there is some \( i \in M \setminus \{i'\} \) such that \( a_{ij} > p_j \) for some \( j \in \mu(i') \). We have that \( \mu(i) \notin D_i(p) \) because \( a_{ik} - p_k = 0 < a_{ij} - p_j \) for all \( k \in \mu(i) \) and the above \( j \notin \mu(i) \). This contradicts that \((p,\mu)\) is a competitive equilibrium. Hence, condition (c) holds.

The above theorem gives a characterization of the coincidence between the core and the set of competitive equilibrium payoff vectors in one-seller assignment markets with multi-unit demands. As a consequence of this result, when the buyers have a sufficiently large capacity, the core coincides with the set of competitive equilibrium payoff vectors.

Indeed, when there are no capacity constraints (or each buyer has a capacity greater than the number of non-dummy objects), an optimal matching assigns each object to one of the buyers who value it the most. Hence, conditions (a) and (b) are satisfied. Moreover, when a buyer leaves the market, his objects are assigned to the buyers with the second best valuation. This implies that for all \( i \in M \), condition (c) holds with an equality.

6 Concluding Remarks

Two-sided markets with only one agent on one of the sides have been considered in the literature, see for instance Stuart (2007) for a market with a monopolist seller or Tauman et al. (1997) for price competition in a multiproduct oligopoly market where each firm produces only one product and there is only one consumer.

In particular, markets with only one seller and several buyers are the setting of an auction, even in frameworks where valuations of buyers over packages of objects are more general than ours. For instance, Ausubel and Milgrom (2002) show that if the coalitional function, that depends on buyers’ valuations, is buyers-submodular then the Vickrey outcome is in the core. Moreover, in that case, an ascending package auction they design leads to the Vickrey outcome.

Under the assumption that buyers valuate packages of objects additively, we prove under which conditions the Vickrey outcome, which is a core allocation, comes from a competitive allocation. Even more, we characterize under which conditions the whole core can be priced by means of competitive prices.
A Appendix

We prove in this Appendix a lemma that is needed in the proof of Theorem 4.2. First we make a remark that will be useful in the sequel.

Remark A.1. For all \( \mu \in \tilde{\mathcal{M}}_{A^\beta}(M, Q \cup \{\tilde{\beta}\}) \) such that \( \tilde{\beta} \in \mu(i_1) \) and \( \beta \in \mu(i_2) \), the matching \( \mu = (\mu \setminus \{(i_1, \tilde{\beta}), (i_2, \beta)\}) \cup \{(i_1, \beta), (i_2, \tilde{\beta})\} \) also belongs to \( \tilde{\mathcal{M}}_{A^\beta}(M, Q \cup \{\tilde{\beta}\}) \). That is, we can interchange \( \beta \) with \( \tilde{\beta} \).

The next lemma relates the original market \((M, Q, A, r)\) and the market with a replica \((M, Q \cup \{\tilde{\beta}\}, A^\beta, r)\). The object \( \beta \in Q \) was optimally assigned in the original market. We see that in the market \((M, Q \cup \{\tilde{\beta}\}, A^\beta, r)\), there exists a restricted optimal matching such that those objects that were unassigned remain unassigned and the object \( \beta \) remain assigned to the same buyer.

Lemma A.2. Let \((M, Q, A, r)\) be a many-to-one assignment market, \( \mu \in \mathcal{M}_A(M, Q) \) and \( \beta \in \mu(M) \). Then there exists \( \mu' \in \tilde{\mathcal{M}}_{A^\beta}(M, Q \cup \{\tilde{\beta}\}) \) such that

(a) if \( \gamma \in Q \setminus \mu(M) \) then \( \gamma \in Q \setminus \mu'(M) \) and,

(b) if \( \beta \in \mu(i) \) then \( \beta \in \mu'(i) \).

Proof. First, we prove condition (a) of the statement. Consider any \( \mu_1 \in \tilde{\mathcal{M}}_{A^\beta}(M, Q \cup \{\tilde{\beta}\}) \) and if \( \beta \notin \mu_1(M) \) recall Remark A.1 and interchange \( \beta \) and \( \tilde{\beta} \) in \( \mu_1 \).

So, we can assume without loss of generality that \( \beta \in \mu_1(M) \). If \( Q \setminus \mu(M) = Q \setminus \mu_1(M) \) then trivially \( \mu' = \mu_1 \) satisfies condition (a).

Otherwise, if there is some \( \gamma \in \mu_1(M) \setminus \mu(M) \), then there is a buyer \( i_1 \) such that \( \gamma \in \mu_1(i_1) \setminus \mu(i_1) \). Because of the capacity constraint of \( i_1 \), there is some object \( \alpha_1 \in \mu(i_1) \setminus \mu_1(i_1) \). Notice that \( \alpha_1 \neq \beta \) because \( \beta \notin \mu(M) \). If \( \alpha_1 \notin \mu_1(M) \), define \( \mu' = (\mu_1 \setminus \{(i_1, \gamma)\}) \cup \{(i_1, \alpha_1)\} \). Notice that \( \mu' \in \tilde{\mathcal{M}}(M, Q \cup \{\tilde{\beta}\}) \). Then

\[
0 \leq a_{i_1i_1} - a_{i_1\gamma} = \sum_{j \in \mu'(i_1)} a_{i_1j} - \sum_{j \notin \mu_1(i_1)} a_{i_1j} \Rightarrow \sum_{j \in \mu'(i_1)} a_{i_1j} \geq \sum_{j \notin \mu_1(i_1)} a_{i_1j},
\]

where the first inequality holds by the optimality of \( \mu \). Then \( \mu' \in \tilde{\mathcal{M}}_{A^\beta}(M, Q \cup \{\tilde{\beta}\}) \) and \( k \notin \mu'(M) \).

If otherwise \( \alpha_1 \in \mu_1(M) \), then there is a buyer \( i_2 \neq i_1 \) such that \( \alpha_1 \in \mu_1(i_2) \). Since \( \alpha_1 \in \mu_1(i_2) \setminus \mu_1(i_2) \) there is some \( \alpha_2 \in \mu_1(i_2) \) and \( \alpha_1 \neq \alpha_2 \). If \( \alpha_2 \notin \mu_1(M) \), we finish as above by taking \( \mu' = (\mu_1 \setminus \{(i_1, \gamma), (i_2, \alpha_1)\}) \cup \{(i_1, \alpha_1), (i_2, \alpha_2)\} \).

Otherwise, if \( \alpha_2 \in \mu_1(M) \), there is some \( i_3 \) such that \( \alpha_2 \in \mu_1(i_3) \setminus \mu(i_3) \) and some \( \alpha_3 \in \mu(i_3) \setminus \mu_1(i_3) \). We continue and obtain a sequence \( \gamma, \alpha_1, ..., \alpha_t \) such that \( \gamma \in \mu_1(i_1) \setminus \mu(i_1) \), \( \alpha_t \in \mu(i_t) \setminus \mu_1(i_t) \). 

Although buyers in the above sequence can be repeated, the objects \( \alpha_1, ..., \alpha_t \) can be taken to be all different. Indeed, assume as induction hypothesis that for some \( 2 \leq s < l \), \( \alpha_1, ..., \alpha_s \) are all different (we already know that \( \alpha_1 \neq \alpha_2 \)). Assume that \( i_s = i_k \) for some \( k \in \{0, ..., s-1\} \), that is both \( \alpha_k, \alpha_s \in \mu_1(i_k) \setminus \mu(i_k) \). By assumption on \( \alpha_1, ..., \alpha_s \), we have \( \alpha_k \neq \alpha_s \). Hence there exists \( \alpha_{s+1} \in \mu(i_k) \setminus \mu_1(i_k) \) different from \( \alpha_{k+1} \). The fact that \( \alpha_1, ..., \alpha_l \) can be taken to be all different guarantees that the sequence finishes with some \( l \geq 1 \) such that \( \alpha_l \notin \mu_1(M) \).
Take then \( \mu' = (\mu_1 \setminus \{(i_1,k),(i_2,\alpha_1),\ldots,(i_l,\alpha_{l-1})\}) \cup \{(i_1,\alpha_1),(i_2,\alpha_2),\ldots,(i_l,\alpha_l)\} \). Moreover, we can assume without loss of generality that \( \mu' \) satisfies that if \( \beta \in \mu'(i) \) and \( \tilde{\beta} \in \mu'(i') \) then \( i \neq i' \). Indeed, \( \tilde{\beta} \neq \alpha_t \) for all \( t \in \{1,\ldots,l\} \), we have that \( \beta, \tilde{\beta} \in \mu'(i) \) can only happen if some \( t \in \{1,\ldots,l\} \), \( \beta = \alpha_t \) and \( \tilde{\beta} \in \mu_1(i_t) \). But by definition of \( \alpha_t \), this means that \( \alpha_t \in \mu_1(i_1) \) and then by Remark A.1, we can interchange \( \beta \) and \( \tilde{\beta} \) in \( \mu_1 \) in such a way that \( \beta \in \mu_1(i_t) \). This means, because of \( \beta \in \mu(i_t) \), that \( \beta \) will be different from any \( \alpha_t \) for \( t \in \{1,\ldots,l\} \).

It only remains to prove that \( \mu' \in \tilde{\mathcal{M}}_{\mathcal{A}^p}(M,Q \cup \{\tilde{\beta}\}) \). To see this, denote \( \alpha_0 = \gamma \). Then, by optimality of \( \mu \), we have

\[
\sum_{t=1}^{l} \sum_{j \in \mu(i_t)} a_{ij} \geq \sum_{t=1}^{l} \sum_{j \in (\mu(i_t) \setminus \{\alpha_t\}) \cup \{\alpha_{t-1}\}} a_{ij},
\]

which leads to

\[
0 \leq \sum_{t=1}^{l} \left( a_{\alpha_t \alpha_t} - a_{\alpha_t \alpha_{t-1}} \right) = \sum_{t=1}^{l} \left( \sum_{j \in \mu'(i_t)} a_{ij} - \sum_{j \in \mu_1(i_t)} a_{ij} \right).
\]

As a consequence \( \mu' \in \tilde{\mathcal{M}}_{\mathcal{A}^p}(M,Q \cup \{\tilde{\beta}\}) \) and \( \gamma \notin \mu'(M) \).

If there is more than one object \( \gamma \) such that \( \gamma \in \mu_1(M) \setminus \mu(M) \), repeat the above procedure starting now from \( \mu' \) to construct \( \mu'' \) and so on, in order to get a matching under the desired requirements.

Now, we prove condition (b) of the statement. Take \( \mu \in \mathcal{M}_A(M,Q) \) and let \( i_1 \in M \) be such that \( \beta \in \mu(i_1) \). Let \( \mu_1 \in \tilde{\mathcal{M}}_{\mathcal{A}^p}(M,Q \cup \{\tilde{\beta}\}) \) be a matching that satisfies the requirements of condition (a). Trivially, if \( \beta \notin \mu_1(i_1) \) we are done. Otherwise, we have \( \beta 
 \in \mu_1(i_1) \). If \( \tilde{\beta} \notin \mu_1(M) \) notice that \( \mu' = \{(i,j) \in M \times (Q \cup \{\tilde{\beta}\}) \mid (i,j) \in \mu\} \) belongs to \( \tilde{\mathcal{M}}_{\mathcal{A}^p}(M,Q \cup \{\tilde{\beta}\}) \), \( \beta \in \mu'(i_1) \) and condition (a) is also satisfied. If \( \tilde{\beta} \in \mu_1(i_1) \) interchange \( \beta \) and \( \tilde{\beta} \) in \( \mu_1 \) and we are done. Finally, consider that \( \tilde{\beta} \in \mu_1(M) \setminus \{\mu_1(i_1)\} \). Since \( \beta \in \mu(i_1) \setminus \mu_1(i_1) \) there is some object \( \alpha_1 \in \mu(i_1) \setminus \mu_1(i_1) \). Notice that \( \alpha_1 \neq \beta \) and \( \alpha_1 \neq \tilde{\beta} \). Since \( \mu_1 \) satisfies the requirements of condition (a) and \( \alpha_1 \in \mu_1(M) \), we have that \( \alpha_1 \in \mu(M) \). Then there is some buyer \( i_2 \neq i_1 \) such that \( \alpha_1 \in \mu(i_2) \).

If \( \beta \in \mu_1(i_2) \), define \( \mu' = (\mu_1 \setminus \{(i_1,\alpha_1),(i_2,\beta)\}) \cup \{(i_1,\beta),(i_2,\alpha_1)\} \). Then

\[
\sum_{k=1}^{2} \left( \sum_{j \in \mu'(i_k)} a_{ikj} - \sum_{j \in \mu(i_k)} a_{ikj} \right) = a_{i_1\beta} + a_{i_2\alpha_1} - a_{i_1\alpha_1} - a_{i_2\beta}
\]

\[
\sum_{k=1}^{2} \sum_{j \in \mu(i_k)} a_{ikj} - \sum_{j \in (\mu(i_1) \setminus \{\beta\}) \cup \{\alpha_1\}} a_{ij} - \sum_{j \in (\mu(i_2) \setminus \{\alpha_1\}) \cup \{\beta\}} a_{ij} \geq 0.
\]

where the inequality comes from the optimality of \( \mu \). Therefore,

\[
\sum_{k=1}^{2} \sum_{j \in \mu'(i_k)} a_{ikj} \geq \sum_{k=1}^{2} \sum_{j \in \mu(i_k)} a_{ikj},
\]

implies that \( \mu' \in \tilde{\mathcal{M}}_{\mathcal{A}^p}(M,Q \cup \{\tilde{\beta}\}) \) and it satisfies all the requirements. The case where \( \tilde{\beta} \in \mu_1(i_2) \) is analogous.
If $\beta \notin \mu_1(i_2)$ and $\tilde{\beta} \notin \mu_1(i_2)$, since $\alpha_1 \in \mu(i_2) \setminus \mu_1(i_2)$, there is $\alpha_2 \in \mu(i_3)$. We continue with this procedure and we obtain a sequence of objects $\beta, \alpha_1, \ldots, \alpha_{l-1}$ with $l > 1$, each one different from $\tilde{\beta}$, such that $\beta \in \mu(i_1) \setminus \mu_1(i_1)$, $\alpha_t \in \mu_1(i_t) \setminus \mu(i_t)$, $\alpha_t \in \mu_1(i_t) \setminus \mu(i_{t+1})$ with $t \in \{1, \ldots, l-1\}$ and $\beta \in \mu_1(i_t)$ or $\tilde{\beta} \in \mu_1(i_t)$. This can be guaranteed since, by an argument similar to the one used in the proof of part (a), the elements $\alpha_1, \ldots, \alpha_{l-1}$ can be chosen to be all different. Assume $\beta \in \mu_1(i_t)$ (similarly for $\tilde{\beta} \in \mu_1(i_t)$), then define $\mu' = (\mu_1 \setminus \{(i_1, \alpha_1), (i_2, \alpha_2), \ldots, (i_t, \beta)\}) \cup \{(i_1, \beta), (i_2, \alpha_1), \ldots, (i_t, \alpha_{t-1})\}$ and by applying the same argument of (18) and (19) we obtain that $\mu' \in \widetilde{M_\beta}(M, Q \cup \{\tilde{\beta}\})$ and agents unassigned by $\mu$ remain unassigned by $\mu'$ and moreover $\beta \in \mu'(i_1)$.

Reference


