A Beauty Contest with Flexible Information Acquisition

Alexandros Rigos∗
University of Leicester
April 15, 2015

Preliminary and incomplete. Please do not distribute or cite.

Abstract

This paper studies a beauty-contest coordination game. A continuum of players get payoffs based on the squared distance of their action from an unobserved fundamental state of the world and the average action among all players. Each player receives a signal whose probability distribution conditional on the value of the fundamental is part of their strategy. This flexible information acquisition technology allows players to choose not only how precise but also what kind of information they want to get about the fundamental. Information is costly, in particular cost is linear in Shannon’s mutual information measure between the prior of the fundamental and the player’s chosen conditional distribution. When unit costs are high enough, there is a unique equilibrium where players do not obtain information. For lower information unit costs, players restrict their attention around the expected value of the fundamental while paying little attention to fundamental values away from it. As costs get lower, players follow the value of the fundamental more closely. A stronger coordination motive or a more concentrated distribution of the fundamental have the same effect as a higher information cost. When information costs exceed a certain threshold, players do not acquire any information and play the ex-ante expected value of the fundamental with probability 1. The case of a normally distributed fundamental is examined in more detail. Only in this case there exist equilibria whereby the average action of the population is an affine function of the realized value of the fundamental. In most parameter combinations, there exists a unique equilibrium within the classes of affine equilibria and equilibria without information acquisition. Interestingly, when the coordination motive is high and for relatively high information costs, there is multiplicity of equilibria within the classes considered.

Keywords: Flexible information acquisition, Coordination games, Beauty contest, Rational attention

JEL classification: C72, D83

∗Email: alexandros.rigos@gmail.com.
1 Introduction

This paper studies a beauty-contest coordination game under a “flexible information acquisition” technology. In a beauty contest game agents have two motives: a “fundamental” motive in the sense that they want to take actions “close” to the value of some unobserved but “real” random variable and a “coordination” motive in the sense that they want their action to be “close” to the average action of the population. In such environments, information about the value of the unobserved random variable (henceforth the fundamental) can be valuable to the players as it can help them both to be close to the realized value of the fundamental and also serve as a coordination device. If acquiring such information is costly, then players face a trade-off between the benefit of the extra piece of information weighed against its cost. Most researchers that have explored beauty contest games with information acquisition usually assume particular functional forms of the distribution (typically taken to be normal) of the fundamental as well as of the signals that the players observe.

This paper addresses the way in which players in this type of environments acquire information when they are allowed to do so in a “flexible” manner. This flexibility of information acquisition stems from the ability of the players to freely choose what kind of signals they want to receive as well as the functional form of the distribution over the signals they will receive conditional on the value of the fundamental. This technology gives players the option to determine the accuracy of the information they will receive about the fundamental as well as on which events this attention should be focused. Thus, the questions that are being answered here are: “To which events do players pay more attention?” and “how do changes in information costs, the coordination motive and the distribution of the fundamental affect the way players acquire information?”

Beauty contest games have been extensively studied in the literature. They capture situations where “[…] players wish to do the right thing […] and do it together” (Myatt and Wallace, 2012). Such examples of interactions come from industrial organization where players are assumed to be firms under Bertrand competition (see Myatt and Wallace, 2012), financial markets where players are traders and try to forecast the value of the fundamental while competing with each other (as in the parable of Keynes, 1936, which gave the game its name) or investment games (as in Angeletos and Pavan, 2004). This form of games has also been used to model non-economic interactions such as the policy choices of political party members (Dewan and Myatt, 2008).

In beauty contest games, the structure from which players obtain information is crucial. Since the seminal work of Morris and Shin (2002), there has been a growing strand of literature that studies the value of information in these coordination games. In particular, the social value of information has been in focus. This literature considers whether more “public” or more “private” information will be socially optimal. Most authors consider exogenous information structures where players cannot affect the information they get and only make decisions based on that. Exceptions to the rule include Myatt and Wallace (2012), Dewan and Myatt (2008) and Hellwig and Veldkamp (2009). In these models, the way players obtain information is endogenized as they can affect the information they get by purchasing more signals from different sources or bearing a cost in order to increase the precision of their signals.

Recently, Yang (forthcoming) studied a model where two players are playing a coordination game with two strategies (invest or not invest) that has a global-game structure: Players want to always invest if the value of the fundamental is high, to never invest when the fundamental is low and invest only if the other player invests as well for intermediate fundamental values. He introduced the notion of flexible information acquisition based on the work of Sims (1998, 2003) on rational inattention. In his model, each player observes one signal and makes the decision of whether or not to invest based on that. Crucially, the players decide the distribution that their signal will take conditional on the value
of the fundamental. This comes to a cost that is linear in Shannon’s measure of mutual information between the distribution of the fundamental and the player’s chosen distribution of signals.\footnote{Mutual information as Shannon (1948) defines it has to do with reduction of entropy. It measures the reduction of the “randomness” of one random variable once the value of another random variable is known.} Yang’s findings suggest that only if the distribution of the fundamental is concentrated enough between the two “threshold” values described before and unit information costs are low enough, then players want to obtain information about the fundamental. When costs are relatively high, he identifies a unique equilibrium of the game whereas for low information costs there is multiplicity of equilibria.

Similar information acquisition technologies have been used by Woodford (2008, 2009) in state-dependent pricing settings where producers can revise their prices in continuous time and they do so depending on market conditions. Both Woodford’s and Yang’s work consider situations where information is important in order to make binary decisions (invest/not invest, revise/not revise the price). It is natural to ask whether this flexible information technology can be applied to a model of continuous choice. The present paper answers this question in a positive manner.

The model presented here shares things with Yang’s in the sense that players still play a coordination game but with a continuous action set. In this way one can also explore the exact way the coordination motive affects strategic outcomes. In this model it is socially optimal for all players to choose the realized value of the fundamental in the same sense that in Yang (forthcoming) it is socially optimal for the 2 players to choose the invest action when the fundamental exceeds a certain threshold value. As mentioned before, in Yang’s setting – which resembles a stag hunt – multiple equilibria are obtained when the cost of information is low which is natural for this kind of game. In the model presented here, it turns out that when information cost is low there exist unique equilibria (among the classes considered) when the fundamental follows a normal prior. Multiplicity appears when coordination motive is high and costs are relatively high but below a certain threshold.

The layout of the paper is as follows: Section 2 sets up the model while in Section 3 best responses that follow a smoothness condition are being calculated. Necessary conditions for “well-behaved” equilibria and equilibria where no information acquisition takes place are derived in Section 4. Section 5 applies the theory developed in the previous sections to beauty contests with a normally distributed fundamental. Section 6 concludes.

\section{Model Setup}

This section introduces the beauty contest game as is set up by Myatt and Wallace (2012) (based on Morris and Shin, 2002) and the information acquisition technology of Yang (forthcoming).

A continuum of identical expected utility-maximizing players (indexed by \( i \in [0, 1] \)) are playing a coordination game. Each of them obtains payoff given by:

\[
 u_i = \bar{a} - (1 - \gamma)(a_i - \theta)^2 - \gamma(a_i - \bar{a})^2 - C(S_i, q_i)
\]

In the above payoff function, \( a_i \) is player \( i \)’s chosen action whereas \( \bar{a} \) represents the agents’ average action.\footnote{It is assumed that agents act in a way such that the value of \( \bar{a} \) is well-defined. For a discussion on this point see Myatt and Wallace (2012, footnotes 3 and 6).} The variable \( \theta \) (the fundamental motive) is a random variable that follows a continuous probability distribution with probability density function \( p \) over \( \Theta = \mathbb{R} \). The distribution \( p \) is assumed to be common knowledge (or that it is a common prior of the players on the fundamental). Players are incentivized in two different ways: a) they want to coordinate and b) they want to get close to the realized value of \( \theta \). The parameter \( \gamma \in (0, 1) \) determines how strong the coordination motive is. More than that,
agents can obtain information on the value of \( \theta \). They do so by choosing an information structure that consists of a set of signals \( S_i \) and a selection of probability distributions over \( S_i \) conditional on the value of \( \theta \), i.e., \( q_i : \Theta \to \Delta(S_i) \).\(^3\) The signal that player \( i \) receives is drawn independently of the signals of the rest of the players and will follow the conditional distribution \( q_i(\theta) \). The function \( C((S_i, q_i)) \) represents the cost of obtaining this information.

The cost of information acquisition is linear in Shannon’s mutual information measure between \( q_i \) and \( p \). That is, the more informative the information acquisition strategy \( (S_i, q_i) \) is, the higher the cost of information. More explicitly, the cost of information is given by

\[
C((S_i, q_i)) = \mu \cdot I((S_i, q_i)) = \mu \left( \int_{\Theta} \int_{S_i} q_i(s_i|\theta) \log q_i(s_i|\theta) ds_i p(\theta) d\theta - \int_{S_i} \int_{\Theta} q_i(s_i|\theta) p(\theta) d\theta \log \left( \int_{\Theta} q_i(s_i|\theta) p(\theta) d\theta \right) ds_i \right).
\]

Here, \( \mu \) is the cost per unit of information.\(^4\)

**Timing:** The timing is as follows:

1. All players decide upon their information acquisition strategy.
2. The value of \( \theta \) is realized.
3. Players receive a signal according to their information acquisition strategy.
4. Players take their actions \( a \in \Theta \) contingent on the signal they received.

Player \( i \) has to decide upon a strategy that consists of three parts: (a) a signal space \( S_i \in \mathbb{R} \); (b) a \( p \)-integrable mapping \( q_i \) that gives a probability distribution on \( S_i \) conditional on the value of \( \theta \);\(^5\) and (c) a mapping \( \sigma_i : S_i \to \Delta(A_i) \) from the signal space to the space of probability distributions over the action space \( A_i = \mathbb{R} \). The probability density with which player \( i \) follows action \( a_i \) conditional on receiving signal \( s_i \) will be denoted by \( \sigma_i(a_i|s_i) \).

Let player \( i \)’s strategy be denoted by \( m_i = (S_i, q_i, \sigma_i) \). The whole population’s strategy profile will be denoted by \( \mathbf{m} \) and the strategy profile of the population excluding player \( i \) by \( \mathbf{m}_{-i} \).

### 3 Best Responses

In this section the best response correspondence for a player \( i \) is being calculated. The steps taken for this calculation are the following: It is firstly shown that an optimal strategy for a player \( i \) should use (a) a signal space that has the same cardinality as the action space and (b) a probability distribution over actions conditional on the signal received that is degenerate (i.e., assigns all probability mass to a single action).\(^6\) Given the result from the first step, player \( i \)’s strategy best response can be summarized by

---

\(^3\)Throughout the paper, \( \Delta(X) \) will represent the space of probability distributions over \( X \).

\(^4\)Throughout the paper, \( \log \) denotes the natural logarithm and so the unit of measurement of information is the nat. If the logarithms were taken with a base 2, the unit of measurement of information would be the bit. As Yang (forthcoming) points out, the choice of the unit of measurement does not change the results as 1 bit equals \( \log 2 \) nats.

\(^5\)Formally, \( q_i : \Theta \to \Delta(S_i) \) and \( q_i \in L^1(\Theta, p) \). We will also write \( q_i(s_i|\theta) \) to denote the distribution on \( S_i \) conditional on the value of \( \theta \).

\(^6\)This result is well known for cases where the number of choices is finite (e.g., Woodford, 2008; Yang, forthcoming).
a function \( r_i : \Theta \to \Delta(A_i) \) that gives a probability distribution over the action space conditional on the value of \( \theta \). As a second step, necessary conditions that \( r_i \) has to satisfy if it is to be a best response are derived. This is done by considering (local) variations of \( r_i \) and demanding that an optimal \( r_i \) should do at least as well as any of these variations. By taking first order conditions, a unique form of \( r_i \) when \( \bar{a}(\theta) \) is strictly increasing is identified.

### 3.1 Determining the signal space and the signal-to-action function

In order to determine the optimal signal space that a player will use let \( m_{-i} \) be any strategy profile that player \( i \)'s opponents are using. Let also \( s_i \) be the signal that player \( i \) obtained by following the information acquisition strategy \((S_i, q_i)\). Given these two components, player \( i \) will form a belief on the value of \( \bar{a} \).

**Lemma 1.** For any strategy profile of player \( i \)'s opponents \( m_{-i} \), any information acquisition strategy \((S_i, q_i)\) of player \( i \), and any signal \( s_i \) that player \( i \) may have received, player \( i \) has a unique optimal action.

**Proof.**

Let \( p_i(\theta|s_i; m_i) \) denote the distribution of \( \theta \) conditional on the event of player \( i \) receiving signal \( s_i \) while using strategy \( m_i = (S_i, q_i, \sigma_i) \). This distribution has to satisfy Bayes's law:

\[
p_i(\theta|s_i; m_i) = \frac{q_i(s_i|\theta) p(\theta)}{\int_\Theta q_i(s_i|\theta) p(\theta) d\theta}
\]

(2)

If player \( i \) knows player \( j \)'s strategy, he can also infer the distribution of player \( j \)'s signal (conditional on \( i \) receiving signal \( s_i \)) by using Bayes's rule. That would be:

\[
q_j(s_j|s_i; m_i, m_j) = \int_\Theta q_j(s_j|\theta) p_i(\theta|s_i; m_i) d\theta
\]

(3)

So, player \( j \)'s action distribution from player \( i \)'s viewpoint will be:

\[
\nu_j(a_j|s_i; m_i, m_j) = \int_{S_j} \sigma_j(a_j|s_j) q_j(s_j|s_i; m_i, m_j) ds_j
\]

(4)

and player \( j \)'s expected action from \( i \)'s viewpoint will be:

\[
a_j(s_i; m_i, m_j) = \int_{A_j} a_j \nu_j(a_j|s_i; m_i, m_j) da_j
\]

(5)

So, from player \( i \)'s viewpoint, the expected average action of his opponents (which is equal to the expected average action over the whole population as player \( i \)'s action cannot affect the mean action in a continuum population) will be

\[
\bar{a}_i(s_i; m_i, m_{-i}) = \bar{a}_{-i}(s_i; m_i, m_{-i}) = \int_0^1 \int_{A_j} a_j \nu_j(a_j|s_i; m_i, m_j) da_j dj
\]

(6)

\(^7\)Formally it should be \( \bar{a}_{-i} \) but as the contribution of a single player to the mean action of a continuum of players is zero, \( \bar{a}_{-i} = \bar{a} \).
From $i$’s viewpoint, given that he received signal $s_i$, $i$’s expected utility is:

$$E_i(u_i|s_i) = \int_{\Theta} \left( \bar{u} - (1-\gamma)(a_i - \theta)^2 - \gamma(a_i - \bar{a})^2 - C(m_i) \right) p_i(\theta|s_i; m_i) \, d\theta$$

(7)

Now assume that player $i$ has already fixed his information acquisition strategy and all other players have decided upon their strategy. Player $i$ then receives his signal $s_i$. Given that he is maximizing expected utility, his action has to satisfy the following first-order condition:

$$a_i^*(m_{-i}|s_i; (S_i, q_i)) = (1-\gamma) \int_{\Theta} \theta p_i(\theta|s_i; m_i) \, d\theta + \gamma \int_{\Theta} \bar{a} p_i(\theta|s_i; m_i) \, d\theta$$

$$= (1-\gamma) \int_{\Theta} \theta p_i(\theta|s_i; m_i) \, d\theta + \gamma \bar{a}(s_i|m_i, m_{-i})$$

(8)

From the equation above it is seen that – as long as $\int_{\Theta} \theta p_i(\theta|s_i; m_i) \, d\theta$ and $\bar{a}(s_i|m_i, m_{-i})$ are well-defined – there is a unique value of $a_i^*$ that satisfies the above condition.

In light of the above result, in an optimal strategy (best response) of player $i$ each signal should map to a unique action $a_i \in A_i$ rather than a distribution over actions. Therefore, the signal space should have the same cardinality as the action space.\(^8\) So, player $i$’s signal space can be reduced to be the whole of $A_i = \Theta = \mathbb{R}$ for all $i$ but the action space will be explicitly mentioned in order to avoid confusion.

### 3.2 Determining the form of the information acquisition strategy

The remaining question has to do with how players decide upon their information acquisition strategy.

Keeping in mind the result of the previous paragraph, player $i$’s strategy can be summarized by a function $r_i: \Theta \rightarrow \Delta(A_i)$ which gives a probability distribution on the action space of player $i$ conditional on the value of $\theta$. So, by writing that the information acquisition strategy of player $i$ is $r_i(a_i|\theta)$, it is meant that player $i$ is using strategy $m_i = (\mathbb{R}, \delta(a_i - s_i), q_i(a_i|s_i))$ with $q_i(a_i|\theta) = r_i(a_i|\theta)$.

Now, observe that from player $i$’s point of view, the only way that the other players are affecting his payoff is through the effect of their strategies to the mean action $\bar{a}$. Thus, player $i$ is not being affected by the way that the particular $\bar{a}(\theta)$ comes about. This means that the object to which he is best-responding is the function $\bar{a}(\theta)$ which summarizes all of his opponent's strategies.

So, the decision problem of player $i$ is to maximize the following:

$$V(r_i, r_{-i}) = U(r_i, r_{-i}) - \mu I(r_i).$$

(9)

Where

$$U(r_i, r_{-i}) = \bar{u} - (1-\gamma) \int_{\Theta} \int_{A_i} (a_i - \theta)^2 r_i(a_i|\theta) p(\theta) \, da_i \, d\theta - \gamma \int_{\Theta} \int_{A_i} (a_i - \bar{a}(\theta))^2 r_i(a_i|\theta) \, da_i \, d\theta$$

(10)

\(^8\)This should happen even if some of these signals are never used. Of course, if any of the signals is not to be used, this would immediately mean that the corresponding action would never be used by player $i$. 

6
with \( \bar{a}(\theta) \) given by

\[
\bar{a}(\theta) = \int_0^1 \int_{A_j} a_j r_j(a_j|\theta) da_j \, dj.  \tag{11}
\]

The mutual information between \( r_i \) and \( p \) is given by

\[
I(r_i) = \int_{\Theta} \int_{A_i} \log \left( \frac{r_i(a_i|\theta)p(\theta)}{p(\theta) \int_{\Theta} r_i(a_i|\theta)p(\theta) d\theta} \right) r_i(a_i|\theta)p(\theta) da_i \, d\theta = \int_{\Theta} \int_{A_i} \log(r_i(a_i|\theta)) r_i(a_i|\theta) p(\theta) \, da_i \, d\theta - \int_{A_i} \log(R_i(a_i)) R_i(a_i) \, da_i
\]

where \( R_i(a_i) = \int_{\Theta} r_i(a_i|\theta)p(\theta) \, d\theta \) is the marginal of action \( a_i \).

As a first result, it is easy to show that if information is costless, player \( i \) has a unique best response to any \( \bar{a}(\theta) \).

**Proposition 2.** If \( \mu = 0 \), then for any \( \bar{a}(\theta) \), player \( i \) has a unique best response that gives a probability mass of 1 to the action \( a^*(\theta) = (1 - \gamma)\theta + \gamma \bar{a}(\theta) \).

**Proof.** As \( \mu = 0 \), player \( i \) can obtain perfect information on the value of \( \theta \) without paying any costs. So, his optimization problem becomes

\[
\max_{a^*(\theta)} -(1 - \gamma)(a^*(\theta) - \theta)^2 - \gamma(a^*(\theta) - \bar{a}(\theta))^2
\]

Taking a first order condition, one obtains that

\[
a^*(\theta) = (1 - \gamma)\theta + \gamma \bar{a}(\theta).
\]

So, given any \( \bar{a}(\theta) \), player \( i \) has a unique best action \( a^*(\theta) \). Thus, his best response would be to give a probability mass of 1 to that action (conditional on \( \theta \)). That is, his best response would be to use \( r_i(a_i|\theta) = \delta(a_i - a^*(\theta)) \) with \( \delta \) being Dirac’s delta function.

The analysis from now on will focus on strategy profiles that satisfy some continuity and smoothness conditions. In particular the following are defined:

**Definition 1 (Monotone full-support profile).** A strategy profile \( r \) will be called a monotone, full-support profile if it satisfies the following conditions:

1. the average action \( \bar{a}(\theta) = \int_0^1 \int_{A_j} a_j r_j(a_j|\theta) da_j \, dj \) is well-defined for all \( \theta \in \Theta \) and is twice continuously differentiable (\( a^* \in C^2 \))
2. the “best action” function \( a^* : \Theta \to A \) defined as \( a^*(\theta) = (1 - \gamma)\theta + \gamma \bar{a}(\theta) \) is bijective with \( a^*(\theta) > 0 \) for all \( \theta \in \Theta \).

The inverse of \( a^* \) will be denoted by \( \phi(x) \), i.e. \( a^*(\theta) = x \Longleftrightarrow \phi(x) = \theta \).

**Definition 2 (Smooth strategy).** A strategy \( r_i \) of player \( i \) will be called smooth if \( r_i(a_i|\theta) \) is continuous with respect to the Lebesgue measure (contains no atoms) for all \( \theta \in \Theta \).

The following result holds.
**Proposition 3.** Consider a beauty contest with flexible information acquisition with coordination motive $\gamma \in (0, 1)$. Information cost $\mu > 0$ and fundamental $\theta$ that follows a continuous common prior $p \in \Delta(\mathbb{R})$ with full support and well-defined mean.

For any monotone, full-support strategy profile $r_i$ of player $i$’s opponents for which $F^{-1}_r(\exp(\mu \pi^2 \xi^2)) \cdot F(x[p(x)]\phi'(x))[(\xi)](x)$ exists, player $i$ has a unique smooth best response.

**Proof.** See Appendix.

In the above, $F_x$ and $F^{-1}_x$ denote the Fourier and inverse Fourier transforms defined as

$$F_x[f(x)](\xi) = \int_{-\infty}^{+\infty} f(x) \exp(-2\pi i x \xi) dx$$

and $F^{-1}_x[F(\xi)](x) = \int_{-\infty}^{+\infty} F(\xi) \exp(2\pi i x \xi) d\xi$

respectively ($i$ is the imaginary unit). Also notice that $F_x[p(x)]$ is closely related to the characteristic function of the prior $p$.

Some examples are provided to demonstrate what solutions to the decision problem look like.

**Example 1.** Let $p(\theta) = \exp(-\pi \theta^2)$ and $\bar{a}(\theta) = \theta$. Then $a^*(\theta) = \theta$ and $\phi(x) = x$. Thus $\phi'(x) = 1$. So,

$$R_i(x) = F^{-1}_r(\exp(\mu \pi^2 \xi^2)) \cdot F_x[\exp(-\pi x^2)](\xi)(x) = \frac{1}{\sqrt{1 - \pi \mu}} \exp \left(-\frac{\pi x^2}{1 - \pi \mu}\right)$$

For the above to be integrable, it has to be that $\mu < 1/\pi$. Notice that $\int_{-\infty}^{+\infty} R_i(x) dx = 1$ which is a condition that is required. So

$$r(a_i | \theta) = \frac{1}{\sqrt{1 - \pi \mu}} \exp \left(-\frac{\pi a_i^2}{1 - \pi \mu}\right) \exp(\pi \theta^2) \frac{1}{\sqrt{\pi \mu}} \exp \left(-\frac{(a_i - \theta)^2}{\mu}\right)$$

The expected action of player $i$ conditional on $\theta$ can be calculated to be:

$$\mathbb{E}a_i(\theta) = \int_{-\infty}^{+\infty} a_i r(a_i | \theta) da_i = (1 - \pi \mu) \theta$$

**Example 2.** Let $p(\theta) = (\sigma \sqrt{2\pi})^{-1} \cdot \exp(-(\theta - \theta_0)^2/2\sigma^2)$ and $\bar{a}(\theta) = \theta + \lambda \theta$. Then, $\phi(x) = (x - \gamma a_0)/(1 + \gamma(\lambda - 1))$ and $\phi'(x) = (1 + \gamma(\lambda - 1))^{-1}$. We can calculate $R_i(\cdot)$ to be:

$$R_i(a) = \frac{1}{\sqrt{2\pi(1 + \gamma(\lambda - 1))^2}\sigma^2 - \pi \mu} \exp \left(-\frac{(a_i - (\gamma a_0 + (1 - \gamma)\theta_0 + \gamma \lambda \theta_0))^2}{2(1 + \gamma(\lambda - 1))^2 \sigma^2 - \mu}\right)$$

It is required that $\mu < 2(\sigma(1 + \gamma(\lambda - 1))^2)$. Otherwise, there is no smooth strategy that solves the decision making problem of player $i$. The full solution is therefore

$$r(a_i | \theta) = R_i(a_i) \frac{1 + \gamma(\lambda - 1)}{\sqrt{\pi \mu}} \sqrt{2\pi \sigma} \exp \left(\frac{(\theta - \theta_0)^2}{2\sigma^2}\right) \exp \left(-\frac{(a_i - (a_0 + \lambda \theta))^2}{\mu}\right).$$

---

9In particular, the characteristic function of $p$ is $\psi(\omega) = \int_{-\infty}^{+\infty} \exp(i \omega x)p(x) dx$ whereas $F_x[p(x)](\xi) = \int_{-\infty}^{+\infty} \exp(-2\pi i x \xi)p(x) dx$. 

8
4 Equilibrium

4.1 Smooth, monotone, full-support equilibria

Having established conditions for the existence of a smooth best response to monotone, full-support strategy profiles, attention is now shifted towards equilibria. In particular, the focus will be on the class of (Nash) equilibria whereby individual strategies are smooth and the whole strategy profile of the population is monotone and full-support. Such equilibria are defined formally in the following definition.

**Definition 3** (Smooth, monotone, full-support Equilibrium). *An action profile \( r \) will be called a smooth, monotone, full-support equilibrium (SMFE) if*

1. \( r \) is a monotone, full-support profile,
2. \( r_i \) is smooth for all \( i \in [0, 1] \) and
3. \( r_i \) is a best response to \( \bar{a}(\theta) = \int_0^1 \int_{A_j} a_j r_j(a_j|\theta) da_j \, dj \) for all \( i \in [0, 1] \).

From the above, it is easy to prove the following result.

**Corollary 4.** Consider a beauty contest with flexible information acquisition. Then all SMFE are symmetric i.e. in equilibrium all players use strategies that are equal the same strategy \( r \) almost surely.

**Proof.** As any single player \( i \) cannot influence the average action taken by the population for any value of \( \theta \), then all players face the same decision problem. Now, as the solution any player’s decision problem is unique (up to deviations of measure zero) then, in equilibrium, the strategies that the players are using should be equal to the same strategy \( r \) almost surely. \( \square \)

It is now possible to characterize the SMFE of the game.

**Proposition 5.** Consider a beauty contest with flexible information acquisition. A strategy profile \( r \) is an SMFE if and only if all players use the same strategy \( r \) that satisfies

\[
\int_{-\infty}^{+\infty} a_i r(a_i|\theta) da_i = \bar{a}(\theta)
\]

Where

\[
r(a_i|\theta) = R_i(a_i) \frac{1 + \gamma (\bar{a}'(\theta) - 1)}{p(\theta) \sqrt{\pi \mu}} \exp \left( - \frac{(a_i - (1 - \gamma) \theta - \gamma \bar{a}(\theta))^2}{\mu} \right)
\]

and

\[
R_i(x) = \mathcal{F}^{-1}[\exp(\mu x^2) \cdot \mathcal{F}_x[p(\phi(x))\phi'(x)|(\xi)](x)\].

**Proof.** The result is straightforward from Corollary 4 and Proposition 3. \( \square \)

**Proposition 6.** Consider a beauty contest with flexible information acquisition. Then \( r \) is the strategy that all players use in an SMFE if and only if it satisfies the following conditions:

\[
r(a_i|\theta) = R_i(a_i) \frac{1 + \gamma (\bar{a}'(\theta) - 1)}{p(\theta) \sqrt{\pi \mu}} \exp \left( - \frac{(a_i - (1 - \gamma) \theta - \gamma \bar{a}(\theta))^2}{\mu} \right)
\]

(13)

\[
\bar{a}(\theta) = \mathcal{G}[r(a_i|\theta)](\theta)
\]

(14)

\[
\bar{a}'(\theta) \geq 0
\]

(15)
with
\[ R_i(x) = \mathcal{F}_x^{-1}[\exp(\mu \pi^2 \xi^2) \cdot \mathcal{F}_x[p(\phi(x))\phi'(x)](\xi)](x). \]
and the operator \( \mathcal{G} \) is defined as
\[ \mathcal{G}[r_i(a_i|\theta)](\theta) = -\frac{1}{2\pi i} (\mathcal{F}_a_i[r_i(a_i|\theta)])'(0) \]
and \( \phi : \mathbb{R} \to \mathbb{R} \) is the inverse of \( a^* \) defined by \( a^*(\theta) = (1 - \gamma)\theta + \gamma \bar{a}(\theta). \)

**Proof.** The proof is straightforward from the result of Proposition 5 and the fact that the mean of a distribution \( f(x) \) is equal to \((-2\pi i)^{-1}(\mathcal{F}_x[f(x)])'(0). \)

In the following proposition a necessary condition for SMFE is established.

**Proposition 7.** Consider a beauty contest with flexible information acquisition. Then, any equilibrium in which information acquisition is taking place should satisfy the following condition:
\[ \bar{a}(\theta) = \theta + \frac{\mu}{2(1 - \gamma)(1 + \gamma(a'(\theta) - 1))} \left( \frac{p'(\theta)}{p(\theta)} - \frac{\gamma \bar{a}''(\theta)}{1 + \gamma(a'(\theta) - 1)} \right) \]
(16)

or equivalently:
\[ a^*(\theta) = \theta + \frac{\mu \gamma}{2(1 - \gamma)} \frac{1}{a''(\theta)} \int \log \left( \frac{p(\theta)}{a''(\theta)} \right) d\theta \]
(17)

**Proof.** See Appendix.

4.2 Equilibria without information acquisition

It might be possible for equilibria where no information acquisition takes place to exist. Such an equilibrium would be one where players’ information acquisition strategy does not vary conditional on \( \theta. \) In this case, the cost of information acquisition is zero and from player \( i \)'s point of view, the average action does not vary with \( \theta \) \( (\bar{a}(\theta) = \bar{a}). \) A necessary condition for such an equilibrium to exist is derived in what follows.

**Lemma 8.** In a symmetric equilibrium where no information acquisition takes place, all players play \( \bar{\theta} = \int_\Theta \theta p(\theta) d\theta \) with probability 1 independently of the value of \( \theta. \)

**Proof.** Assuming that the rest of the population is using strategies without information acquisition, player \( i \) is facing a population where \( \bar{a}(\theta) = \bar{a}. \) Player \( i \)'s expected payoff from not acquiring information is
\[ \tilde{V}(a_i) = \bar{a} + \int_\Theta \left( -\gamma(a_i - \bar{a})^2 - \gamma(a_i - \bar{a})^2 \right) p(\theta) d\theta \]
and so, if he maximizes, he has to choose
\[ a^*_i = \gamma \bar{a} + (1 - \gamma)\bar{\theta}. \]

Now for a strategy like that to be an equilibrium strategy, it has to be that \( a^*_i = \bar{a} \) and thus, \( \bar{a} = \bar{\theta}. \)

Notice that for the above result the assumption that \( \gamma < 1 \) is crucial. This is because if the players have no fundamental motive \( (\gamma = 1) \), then the condition \( a^*_i = \bar{a} \) is not enough to pin down the equilibrium as the usual coordination problems arise.
Proposition 9. Consider a beauty contest with flexible information acquisition. Then, there exists a symmetric Nash equilibrium without information acquisition only if

\[ F^{-1}_\xi \left[ \exp \left( \mu \pi^2 \xi^2 - 2\pi i \frac{\gamma \Theta}{1 - \gamma} \xi \right) \cdot F_x[p(x)]((1 - \gamma)\xi) \right] \]

does not exist.

Proof. From Lemma 8, if an equilibrium without information acquisition were to exist, then all players would follow the strategy \( r_i(a_i|\theta) = \delta(a_i - \bar{\theta}) \). Now notice that the expression

\[ R_i(a_i) = F^{-1}_\xi \left[ \exp \left( \mu \pi^2 \xi^2 - 2\pi i \frac{\gamma \Theta}{1 - \gamma} \xi \right) \cdot F_x[p(x)]((1 - \gamma)\xi) \right] (a_i) \]

gives the marginal of action \( a_i \) in a best response to a strategy profile where (almost) all the players are using \( r_i(a_i|\theta) = \delta(a_i - \bar{\theta}) \).

Now, if the above expression is well-defined (i.e. the inverse Fourier transform of the expression in the square brackets exists), then the (unique) best response to the strategy profile described above will be a strategy where information acquisition occurs (as described in Proposition 3).

From the equilibrium characterization (Proposition 5), it is required that

\[ \bar{a}(\theta) = \bar{\theta} = \int_{-\infty}^{+\infty} a_i r_i(a_i|\theta) da_i \quad \text{for all } \theta \in \Theta. \]

Thus, \( r(a_i|\theta) = \delta(a_i - \bar{\theta}) \) is not a best response as all \( a_i \in \mathbb{R} \) receive positive probability density. Therefore, the consistency condition of Proposition 5 fails to hold and the strategy profile is not a Nash equilibrium.

5 Application: Normally Distributed Fundamental

In this section the Nash equilibria of beauty contests with flexible information acquisition with a normally distributed fundamental are studied. Comparative statics results are also provided.

Assume that the fundamental motive of a beauty contest game follows a normal distribution with mean \( \theta_0 \) and variance \( \sigma^2 \) i.e. \( p(\theta) = (\sqrt{2\pi}\sigma)^{-1} \exp(-((\theta - \theta_0)^2)/(2\sigma^2)) \). Using the result of Proposition 6, the SMFE be calculated. First the mean action is calculated. It turns out that the average action of the population conditional on the value of \( \theta \) has to satisfy the following differential equation:

\[ \bar{a}(\theta) = \theta + \frac{\mu(\theta - \theta_0)}{2\sigma^2(1 - \gamma)(1 + \gamma(\bar{a}'(\theta) - 1))} + \frac{\mu \gamma}{2(1 - \gamma)(1 + \gamma(\bar{a}'(\theta) - 1))^2} \bar{a}''(\theta) \]

or equivalently, the best action \( a^* \) has to satisfy

\[ a^*(\theta) = \theta + \frac{\mu \gamma}{2(1 - \gamma) a''(\theta)} \left( -\frac{\theta - \theta_0}{\sigma^2} - \frac{d}{d\theta} \left( \log a''(\theta) \right) \right). \quad (18) \]

As it is difficult to identify closed-form solutions to the above equation in the general case, attention is focused to SMFE where the best action \( a^* \) function takes an affine form i.e. \( a^*(\theta) = \kappa \theta + b \) for some constants \( \kappa > 0 \) and \( b \in \mathbb{R} \). Such SMFE will be called affine.

\[ ^{10} \text{One can show that such SMFE exist only if } p \text{ is a normal distribution.} \]
Using $a^*(\theta) = \kappa \theta + b$ into equation (18), one obtains two solutions for the value of $\kappa$ in an affine SMFE:

$$
\kappa_+ = \frac{1}{2} + \frac{\sqrt{\sigma^2 - \frac{2\mu \gamma}{\gamma - 1}}}{2\sigma} \quad \kappa_- = \frac{1}{2} - \frac{\sqrt{\sigma^2 - \frac{2\mu \gamma}{\gamma - 1}}}{2\sigma}
$$

whereas the corresponding values for $b$ are

$$
b_+ = \frac{\mu \gamma}{2(1-\gamma)\sigma^2 \kappa_+} \theta_0 \quad b_- = \frac{\mu \gamma}{2(1-\gamma)\sigma^2 \kappa_-} \theta_0.
$$

Of course, for these solutions to exist, it is required that $\sigma^2 - \frac{2\mu \gamma}{\gamma - 1} \geq 0$ which is equivalent to $\mu \leq \frac{(1-\gamma) \sigma^2}{2\gamma}$ for $\theta \in (0,1)$. It is easy to confirm that $a^*(\theta) = \theta \iff \theta = \theta_0$.

Now, recall that for an SMFE, it is required that $\kappa \in (0,1)$. So, the first derivative of $a^*(\theta) = \kappa \theta$ is constant with $a^\gamma = \kappa_+$. It is easy to confirm that

$$(1-\gamma)\sigma^2, 2(1-\gamma)^2 \sigma^2)
$$

1. For an affine SMFE with $a^\gamma(\theta) = \kappa_+$ to exist, it has to be that

   either $\mu < (1-\gamma)\sigma^2$ or $\gamma < \frac{1}{2}$ and $\mu \in (1-\gamma)\sigma^2, 2(1-\gamma)^2 \sigma^2)$

2. For an affine SMFE with $a^\gamma(\theta) = \kappa_-$ to exist, it has to be that

   $$
   \mu > 2(1-\gamma)^2 \sigma^2
   $$

From the above conditions, together with the restriction that $\mu \leq \frac{(1-\gamma) \sigma^2}{2\gamma}$, one obtains that:

1. For an affine SMFE with $a^\gamma(\theta) = \kappa_+$ to exist, it has to be that

   $$
   \mu \leq \frac{(1-\gamma) \sigma^2}{2\gamma}
   $$

2. For an affine SMFE with $a^\gamma(\theta) = \kappa_-$ to exist, it has to be that

   $$
   \gamma \geq \frac{1}{2} \quad \text{and} \quad \mu \in \left[2(1-\gamma)\sigma^2, \frac{(1-\gamma)}{2\gamma} \sigma^2\right]
   $$

Finally, the equilibrium where no information acquisition is taking place is taken into consideration. In this equilibrium (as shown in Proposition 9) the inverse Fourier transform (with respect to $\xi$) of

$$
\exp\left(\mu \pi^2 \xi^2 - 2\pi \gamma \frac{\theta}{1-\gamma} \xi\right) \cdot \mathcal{F}_\pi [p(x)]((1-\gamma)\xi)
$$

should not exist. In the case of a normally distributed fundamental, this condition implies that

$$
\mu \geq 2(1-\gamma)^2 \sigma^2
$$

Notice that in such equilibria $a^\gamma(\theta) = 1-\gamma$ for all values of $\theta$. So, the first derivative of $a^*$ is constant even though this is not an SMFE as the strategy that the players use is not smooth.

Interestingly, if $\gamma \geq \frac{1}{2}$ and $\mu \in \left(2(1-\gamma)^2 \sigma^2, \frac{(1-\gamma)}{2\gamma} \sigma^2\right)$ there is a multiplicity of equilibria i.e. there exist three equilibria of the classes that are being considered: two SMFE (one with $a^\gamma = \kappa_+$ and one with $a^\gamma = \kappa_-$) and an equilibrium with out information acquisition ($a^\gamma = 1-\gamma$). In all other cases, there is a unique equilibrium within the classes under consideration. These results are summarized in Figure 5.
6 Conclusion

A beauty contest game with under flexible information acquisition was studied. Necessary conditions for Smooth Monotone, Full-support Equilibria and equilibria where information acquisition does not take place were derived. Higher information costs, higher coordination motives and a more concentrated fundamental lead to the players paying less attention to the fundamental. In the case of a normally-distributed fundamental, most parameter combinations lead to a unique equilibrium within the classes considered. When information costs are relatively high and the coordination motive is high, there is multiplicity of equilibria.

Appendix

A Omitted Proofs

Proof of Proposition 3

Consider variations of the information acquisition strategy of player $i$. These variations will be of the type $\tilde{r} = r + \epsilon \eta$ for some $\epsilon > 0$. These variations should still be feasible. That is, for all $\theta$, it is required that $r(\theta) + \epsilon \eta(\theta)$ is a probability distribution over $A_i$. It is required, thus, that for all $\theta$, $\int_{A_i} r(a_i|\theta) + \epsilon \eta(a_i|\theta) \, da_i = 1$ which leads to the condition that for all $\theta$, $\int_{A_i} \eta(a_i|\theta) \, da_i = 0$. It also has to be that $r(a_i|\theta) + \epsilon \eta(a_i|\theta) \geq 0$ and so $\eta(a_i|\theta) \geq -r(a_i|\theta)/\epsilon$ for all $a_i$ and $\theta$. From the above equations, the
following is calculated:\textsuperscript{11}
\begin{align*}
U(r_i + \varepsilon \eta, r_{-i}) &= \bar{u} - (1 - \gamma) \int_{\Theta} \int_{A_i} (a_i - \theta)^2(r_i(a_i|\theta) + \varepsilon \eta(a_i|\theta))p(\theta)da_i d\theta - \\
&\quad - \gamma \int_{\Theta} \int_{A_i} (a_i - \bar{a}(\theta))^2(r_i(a_i|\theta) + \varepsilon \eta(a_i|\theta))p(\theta)da_i d\theta.
\end{align*}
(19)

And the derivatives:
\begin{align*}
\frac{dU(r + \varepsilon \eta, r_{-i})}{d\varepsilon} \bigg|_{\varepsilon=0} &= - (1 - \gamma) \int_{\Theta} \int_{A_i} (a_i - \theta)^2 \eta(a_i|\theta)p(\theta)da_i d\theta - \\
&\quad - \gamma \int_{\Theta} \int_{A_i} (a_i - \bar{a}(\theta))^2 \eta(a_i|\theta)p(\theta)da_i d\theta \tag{20}
\end{align*}
\begin{align*}
\frac{dI(r + \varepsilon \eta)}{d\varepsilon} \bigg|_{\varepsilon=0} &= \int_{\Theta} \int_{A_i} \log(r_i(a_i|\theta))\eta(a_i|\theta)p(\theta)da_i d\theta - \\
&\quad - \int_{A_i} \log(R_i(a_i))H(a_i)da_i \tag{21}
\end{align*}

with $H(a_i) = \int_{\Theta} \eta(a_i|\theta)p(\theta) d\theta$.

Since the perturbations considered have to be feasible, player $i$ has to solve the following constrained optimization problem:

$$
\max_{r_i \in L^1(\Theta, \rho)} U(r_i, r_{-i}) - \mu I(r_i)
$$

s.t. $\int_{A_i} r_i(a_i|\theta)da_i = 1$ for all $\theta \in \Theta$.

So, the Lagrangian for player $i$'s decision problem will be

$$
\mathcal{L}(r_i, k(\theta)) = V(r_i, r_{-i}) - \int_{\Theta} k(\theta) \left( \int_{A_i} r_i(a_i|\theta)da_i - 1 \right) p(\theta)d\theta.
$$

This means that for any given $\theta \in \Theta$ and all possible perturbations $\eta$, an optimal strategy $r$ should satisfy the following first order conditions:

$$
\frac{d\mathcal{L}(r_i + \varepsilon \eta, k(\theta))}{d\varepsilon} \bigg|_{\varepsilon=0} = 0 \Rightarrow
$$
\begin{align*}
\int_{\Theta} \int_{A_i} &\left[ -(1 - \gamma)(a_i - \theta)^2 - \gamma(a_i - \bar{a}(\theta))^2 - \mu \left( \log(r_i(a_i|\theta)) - \log(R_i(a_i)) \right) - k(\theta) \right] \eta(a_i|\theta)p(\theta)da_i d\theta = 0 \tag{22}
\end{align*}

and $\int_{A_i} r_i(a_i|\theta)da_i = 1$ for all $\theta \in \Theta$. \tag{23}

Since condition (22) has to be satisfied for all $\eta$, it has to be the case that

$$
-(1 - \gamma)(a_i - \theta)^2 - \gamma(a_i - \bar{a}(\theta))^2 - \mu \left[ \log(r_i(a_i|\theta)) - \log(R_i(a_i)) \right] = k(\theta) \text{ for all } \theta \in \Theta.
$$

\textsuperscript{11}The effect of the other players' strategies is incorporated in $\bar{a}(\theta)$. 

14
So \( r(a_i|\theta) \) has to be:

\[
 r(a_i|\theta) = R_i(a_i) \exp \left( -\frac{k(\theta)}{\mu} \right) \exp \left( \frac{u_i(a_i, \theta)}{\mu} \right). \tag{24}
\]

Where \( u_i(a_i, \theta) = -(1 - \gamma)(a_i - \theta)^2 - \gamma(a_i - \bar{a}(\theta))^2 \). So, (24) can be rewritten as

\[
 r(a_i|\theta) = R_i(a_i)K(\theta) \exp \left( \frac{u_i(a_i, \theta)}{\mu} \right). \tag{25}
\]

where \( K(\theta) = \exp \left( -\frac{k(\theta)}{\mu} \right) \). All that remains to be done is to determine the functions \( K(\cdot) \) and \( R_i(\cdot) \).

Now, using the fact that player \( i \)'s opponents are using a monotone full-support strategy profile and from the definition of \( R_i(a_i) = \int_{\Theta} r(a_i|\theta)p(\theta)\,d\theta \Rightarrow \int_{\Theta} \frac{r(a_i|\theta)}{R_i(a_i)}\,d\theta = 1 \) one gets:

\[
 \int_{-\infty}^{+\infty} K(\theta) \exp \left( \frac{(a_i - a^*(\theta))^2}{\mu} \right) \exp \left( -\frac{\gamma(1 - \gamma)(\theta - \bar{a}(\theta))^2}{\mu} \right) p(\theta)\,d\theta = 1 \tag{26}
\]

In the above, \( a^*(\theta) = (1 - \gamma)\theta + \gamma\bar{a}(\theta) \). By assumption (monotone full-support strategy profile), \( a^* \) is invertible with \( \phi \) being the inverse of \( a^* \). With a change of the variable of integration from \( \theta \) to \( x = a^*(\theta) \), taking into account assumption 2, and by defining \( g(\cdot) \) as

\[
 g(x) = \frac{K(\phi(x)) \exp \left( -\frac{\gamma(1 - \gamma)(\phi(x) - \bar{a}(\phi(x)))^2}{\mu} \right) p(\phi(x))}{(1 - \gamma) + \gamma \bar{a}'(\phi(x))} \tag{27}
\]

condition (26) can be rewritten as

\[
 \int_{-\infty}^{+\infty} g(x) \exp \left( -\frac{1}{\mu}(a_i - x)^2 \right) \,dx = 1. \tag{28}
\]

Notice that the above condition has to hold for all \( a_i \). This can only happen if \( g(x) = 1 / \sqrt{\pi \mu} \).\(^{12}\) So now \( K(\theta) \) can be calculated.

\[
 K(\theta) = \frac{1 + \gamma(\bar{a}'(\theta) - 1)}{p(\theta)\sqrt{\pi \mu}} \exp \left( \frac{\gamma(1 - \gamma)(\theta - \bar{a}(\theta))^2}{\mu} \right) \tag{29}
\]

Using (29) in (25) yields

\[
 r(a_i|\theta) = R_i(a_i) \frac{1 + \gamma(\bar{a}'(\theta) - 1)}{p(\theta)\sqrt{\pi \mu}} \exp \left( -\frac{(a_i - a^*(\theta))^2}{\mu} \right). \tag{30}
\]

\(^{12}\)Proof: Notice that the left-hand side of equation (28) is the convolution of \( g \) and \( f : f(x) = \exp(-\mu^{-1}x^2) \). Now, take the Fourier transform on both sides and use the convolution theorem:

\[
 \mathcal{F}_x [(g * f)(x)](\xi) = \mathcal{F}_x [g(x)](\xi) \cdot \mathcal{F}_x [f(x)](\xi) = \delta(\xi) \Rightarrow \mathcal{F}_x [g(x)](\xi) = \frac{\delta(x)}{\sqrt{\pi \mu} \exp(-\mu^{-1}\xi^2)} = \frac{1}{\sqrt{\pi \mu}} \delta(\xi).
\]

Where \( \delta(\cdot) \) is Dirac's delta function. By taking the inverse Fourier transform on both sides, the statement is proven:

\[
 g(x) = \mathcal{F}_x^{-1} \left[ \frac{1}{\sqrt{\pi \mu}} \delta(\xi) \right](x) = \frac{1}{\sqrt{\pi \mu}} 
\]
The solution has to also satisfy \( \int_{-\infty}^{+\infty} r(a_i|\theta) \, da_i = 1 \) for all \( \theta \). Again, changing the variable from \( \theta \) to \( x = a^\ast(\theta) \), this condition yields

\[
\int_{-\infty}^{+\infty} R_i(a_i) \exp \left( -\frac{(x - a_i)^2}{\mu} \right) \, da_i = \sqrt{\pi \mu} p(\phi(x)) \phi'(x).
\] (31)

Notice that the left-hand side of equation (31) is the convolution of \( R_i \) and \( f \). Now, take the Fourier transform on both sides and use the convolution theorem

\[
\mathcal{F}_x[R_i(x)](\xi) \cdot \mathcal{F}_x[f(x)](\xi) = \sqrt{\pi \mu} \cdot \mathcal{F}_x[p(\phi(x)) \phi'(x)](\xi) \Rightarrow \\
\mathcal{F}_x[R_i(x)](\xi) = \mathcal{F}_x^{-1}[\exp(\mu \pi^2 \xi^2) \cdot \mathcal{F}_x[p(\phi(x)) \phi'(x)](\xi)](x) = \\
R_i(x) = \mathcal{F}_x^{-1}[\exp(\mu \pi^2 \xi^2) \cdot \mathcal{F}_x[p(\phi(x)) \phi'(x)](\xi)](x)
\]

Notice that as by assumption the above expression is well-defined, this solution is unique. \( \square \)

**Proof of Proposition 7**

From the result of Proposition 6, an equilibrium strategy needs to satisfy

\[
\frac{1}{-2\pi i} (\mathcal{F}_{a_i}[r_i(a_i|\theta)])' (0) = \tilde{a}(\theta).
\]

So the following calculations are obtained:

\[
r(a_i|\theta) = \mathcal{F}_x^{-1}[\exp(\mu \pi^2 \xi^2) \cdot \mathcal{F}_x[p(\phi(x)) \phi'(x)](\xi)](a_i) \times \\
\times \frac{1 + \gamma(\tilde{a}'(\theta) - 1)}{p(\theta) \sqrt{\mu}} \exp \left( -\frac{(a_i - (1 - \gamma) \theta - \gamma \tilde{a}(\theta))^2}{\mu} \right)
\] (32)

Using the convolution theorem and the definitions of the operation of convolution and the Fourier transform:

\[
\mathcal{F}_{a_i}[r(a_i|\theta)](a) = \\
= \frac{1 + \gamma(\tilde{a}'(\theta) - 1)}{p(\theta) \sqrt{\mu}} \left( \exp(\mu \pi^2 \xi^2) \mathcal{F}_x[p(\phi(x)) \phi'(x)](\xi) \right) * \left( \mathcal{F}_{a_i} \left[ \exp \left( -\frac{(a_i - (1 - \gamma) \theta - \gamma \tilde{a}(\theta))^2}{\mu} \right) \right] (a) \right) = \\
= \frac{1 + \gamma(\tilde{a}'(\theta) - 1)}{p(\theta) \sqrt{\mu}} \int_{-\infty}^{+\infty} \exp(\mu \pi^2 t^2) \left( \int_{-\infty}^{+\infty} \exp(-2\pi i t x) p(\phi(x)) \phi'(x) \, dx \right) \times \\
\times \left( \int_{-\infty}^{+\infty} \exp(-2\pi i a_i(a - t)) \exp \left( -\frac{(a_i - (1 - \gamma) \theta - \gamma \tilde{a}(\theta))^2}{\mu} \right) \, da_i \right) \, dt
\]

And taking the first derivative with respect to \( a \):

\[
\frac{d}{da} \mathcal{F}_{a_i}[r(a_i|\theta)](a) = \\
= \frac{1 + \gamma(\tilde{a}'(\theta) - 1)}{p(\theta) \sqrt{\mu}} \int_{-\infty}^{+\infty} \exp(\mu \pi^2 t^2) \left( \int_{-\infty}^{+\infty} \exp(-2\pi i t x) p(\phi(x)) \phi'(x) \, dx \right) \times \\
\times \left( \exp(-2\pi i a_i(a - t)) \exp \left( -\frac{(a_i - (1 - \gamma) \theta - \gamma \tilde{a}(\theta))^2}{\mu} \right) \, da_i \right) \, dt
\]
And so

\[
\frac{1}{-2\pi i} \frac{d}{da} \mathcal{F}_a [r(a_i \theta)](a) \bigg|_{a=0} =
\]

\[
= \frac{1 + \gamma(\bar{a}'(\theta) - 1)}{p(\theta) \sqrt{\pi \mu}} \int_{-\infty}^{+\infty} \exp(\mu \pi^2 t^2) \left( \int_{-\infty}^{+\infty} \exp(-2\pi i t x) p(\phi(x)) \phi'(x) \, dx \right) \times
\]

\[
\times \left( \exp(2\pi i t a_i) a_i \exp \left( \frac{-\bar{a}(\theta) - (1 - \gamma) \theta - \gamma \bar{a}(\theta)}{\mu} \right) \right) \, dt
\]

\[
= \frac{1 + \gamma(\bar{a}'(\theta) - 1)}{p(\theta) \sqrt{\pi \mu}} \int_{-\infty}^{+\infty} \exp(\mu \pi^2 t^2) \left( \int_{-\infty}^{+\infty} \exp(-2\pi i t x) p(\phi(x)) \phi'(x) \, dx \right) \times
\]

\[
\times \mathcal{F}_a^{-1} \left( a_i \exp \left( \frac{-\bar{a}(\theta) - (1 - \gamma) \theta - \gamma \bar{a}(\theta)}{\mu} \right) \right) (t) \, dt
\]

\[
= \frac{1 + \gamma(\bar{a}'(\theta) - 1)}{p(\theta) \sqrt{\pi \mu}} \int_{-\infty}^{+\infty} \exp(\mu \pi^2 t^2) \left( \int_{-\infty}^{+\infty} \exp(-2\pi i t x) p(\phi(x)) \phi'(x) \, dx \right) \times
\]

\[
\times \frac{1}{2\pi i} \frac{d}{dt} \mathcal{F}_a^{-1} \left( \exp(2\pi i t ((1 - \gamma) \theta + \gamma \bar{a}(\theta))) \sqrt{\pi \mu} \exp(-\pi \mu t^2) \right) \, dt
\]

\[
= \frac{1 + \gamma(\bar{a}'(\theta) - 1)}{p(\theta)} \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} \exp(-2\pi i t x (1 - \gamma) \theta - \gamma \bar{a}(\theta)) p(\phi(x)) \phi'(x) \, dx \right) \times
\]

\[
\times \exp(2\pi i t ((1 - \gamma) \theta + \gamma \bar{a}(\theta)) (2\pi i ((1 - \gamma) \theta + \gamma \bar{a}(\theta)) - 2\pi \mu t^2) \, dt
\]

\[
= \frac{1 + \gamma(\bar{a}'(\theta) - 1)}{p(\theta)} \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} \exp(-2\pi i t x (1 - \gamma) \theta - \gamma \bar{a}(\theta)) p(\phi(x)) \phi'(x) \, dx \right) \times
\]

\[
\times ((1 - \gamma) \theta + \gamma \bar{a}(\theta) + \mu \pi t) \, dt
\]

\[
= \frac{1 + \gamma(\bar{a}'(\theta) - 1)}{p(\theta)} \left( (1 - \gamma) \theta + \gamma \bar{a}(\theta) \right) \int_{-\infty}^{+\infty} \exp(2\pi i t ((1 - \gamma) \theta - \gamma \bar{a}(\theta)) x)
\]

\[
\times \left( \int_{-\infty}^{+\infty} \exp(-2\pi i t x) p(\phi(x)) \phi'(x) \, dx \right) \, dt +
\]

\[
+ \mu \pi t \int_{-\infty}^{+\infty} t \exp(2\pi i t ((1 - \gamma) \theta + \gamma \bar{a}(\theta))) \left( \int_{-\infty}^{+\infty} \exp(-2\pi i t x) p(\phi(x)) \phi'(x) \, dx \right) \, dt
\]
\[ x = (1 - \gamma)\phi(x) + \gamma\bar{a}(\phi(x)) \]

\[ \phi'(x) = \frac{1}{1 + \gamma(\bar{a}'(\phi(x)) - 1)} \]

\[ \phi''(x) = -\frac{\gamma\bar{a}''(\phi(x))}{(1 + \gamma(\bar{a}'(\phi(x)) - 1))^3} \]

Thus, as in equilibrium it has to be that \( \bar{a}(\theta) = \frac{1}{-2\pi i} \frac{d}{da} \mathcal{F}_a [r(a)] [\alpha] \bigg|_{\alpha = 0} \), the following equilibrium condition is obtained:

\[ \bar{a}(\theta) = \theta + \frac{\mu}{2(1 - \gamma)(1 + \bar{a}'(\theta) - 1)} \left( \frac{p'(\theta)}{p(\theta)} - \frac{\gamma\bar{a}''(\theta)}{1 + \gamma(\bar{a}'(\theta) - 1)} \right) \].

And as \( a^*(\theta) = (1 - \gamma)\theta + \gamma\bar{a}(\theta) \), this is equivalent to

\[ a^*(\theta) = \theta + \frac{\mu\gamma}{2(1 - \gamma) a'^*(\theta)} \frac{1}{d\theta} \left( \log \left( \frac{p(\theta)}{a'^*(\theta)} \right) \right) \].

References


Keynes, J. M. (1936). *The general theory of interest, employment and money*.


