The optimal group size in microcredit contracts

Najmeh Rezaei

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Utrecht University School of Economics (USE), Kriekenpitplein 21-22, 3884 EC Utrecht, Netherlands

Abstract

We analyze a model of a repeated microcredit lending and study how group size affects the optimal group lending contracts with joint liability. The story is that one benevolent lender gives microcredit to a group of \( n \) borrowers to be invested on \( n \) projects. The outcome of each risky project is not observable by the lender. Therefore in case some of the borrowers default on their loan repayments, the lender is not able to identify strategic default. We characterize the optimal contract and determine the optimal size of the group of borrowers endogenously. Our analysis suggests that Joint liability has positive effects on the repayment rate and borrowers’ welfare, and that this effect can increase in the size of the group. However joint liability contracts are feasible under a smaller set of parameter values than individual liability contract. When projects have lower chance of success, the amount of loan that can be granted to borrowers under Joint liability is higher and it is also increasing in the group size.

*JEL classification: C70, D82, G21*

*Keywords: Microfinance, Group lending, Individual lending, Strategic default*

1 Introduction

Microfinance institutions lend to poor people who are excluded from conventional financial services. They offer small loans without collateral to an individual or a group of borrowers. This type of lending has been seen as one of the significant recent improvements in development and has attracted enormous public and academic attention. Microcredit lending has a long history of practice\(^1\), but it took more attention after the

\(^1\)Credit cooperatives were active in nineteenth-century in Germany (Guinnane, 2001). The major source of funding for these institutions was members’ contributions.
successful experience of Grameen bank in Bangladesh in the mid 1970s conducted by Dr. Muhammad Yunus who received Nobel Peace Prize for his efforts in poverty reduction in 2006. Grameen bank’s success has led to a rapid reproduction of a similar type of lending all over the world\(^2\).

Microcredit market as other financial markets suffers from two main classes of imperfections, informational asymmetry and enforcement problems. Informational asymmetry problems are due to inability of lenders to know about borrower’s riskiness, effort, or their actual realized returns (Stiglitz and Weiss, 1981; Townsend, 1979). Enforcement problems arise of lack of collateral and the fact that many borrowers are unable to finance high return investments (Bulow and Rogoff, 1989). In developing countries, problems become worse due to the lack of sound legal infrastructure and credit scoring mechanisms\(^3\).

Group lending contracts with joint liability (JL) have shown some success in assuaging market imperfections, although not successful everywhere\(^4\). In JL contracts, each member receives loan individually, but the entire group is responsible for the default of every one member on his repayment. Group lending with JL has received much attention from researchers and has been considered an important vehicle in assuaging market imperfections, as instances we could mention Stiglitz (1990) and Ghatak and Guinnane (1999). However not all microcredit institutions lend to a group\(^5\), and the ones that are using group lending with JL differ in size of borrower’s group\(^6\). This paper attempts to explain this trend by exploring the optimal group size for JL contracts.

Consider a benevolent lender (she) who wants to give loans to a group of \(n\) borrowers (he) with JL. The loans will be invested in \(n\) projects and realized outcome of projects are unknown to the lender but known to the group members. How large this \(n\) should be in order to maximize each borrower’s benefit while to leave the lender break even? On the one side, larger group size can have a positive effect on the repayment rate, as having more people liable for repaying defaulted payments assures a higher rate of repayment. On the other side, larger group size can be a treat towards members who repay their loans successfully, since they should pledge repaying for all their defaulting peers, and it may happen that every body else in the group is defaulting on his repayment. In this

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\(^2\)According to the Microcredit Summit, as of 31 December 2009, 3,589 microcredit institutions reported reaching 190,135,080 clients (Reed 2011, p. 5).

\(^3\)See Basu (2006) for evidences from India


\(^5\)Bank Rakyat Indonesia (BRI), for example, never used group lending.

\(^6\)Although microfinance lenders such as FINCA International successfully work with ‘village banks’ of twenty or more jointly liable borrowers most microfinance providers tend to limit group size to between two and seven borrowers, Conning (2005).
paper we reconsider the optimal design of uncollateralized lending contracts with JL, and examine positive and negative effects of larger groups on feasibility of JL contracts and welfare of borrower and lender and propose an algorithm to find the optimal group size that allows JL to provide larger loans to borrowers.

We analyze a model of an infinitely repeated microlending game\textsuperscript{7}. Inspired by Tedeschi’s (2006), we define two phases in our model, lending phase where the game starts and continues until the group defaults on repayment, and then punishment phase is where no new loans are extended to the borrowers. We stay aligned with Bhole and Ogden’s (2010) simple group lending model that is designed for groups of two borrowers, and extend their established results on groups of \(n\) borrowers. In our model, borrowers receive contracts individually that determine the amount of loan and repayment. They invest their loans on \(n\) disjoint projects that are identical in terms of mean return and chance of success. Projects will be either successful with high return or unsuccessful with low return. After the outcomes of projects are realized, each borrower decides to repay his loan or not. Only those who had successful project can repay. We distinguish between two types of defaults, strategic and non-strategic. A borrower defaults strategically when he doesn’t want to repay in spite of having high income and defaults non-strategically when he is not able to repay as a result of a negative economic shock and having low outcome of his project. The lender can not identify strategic default, but members of the group can. If some one defaults non-strategically, his group members will repay his loan. The lender deprives a defaulting group from loans in future in order to decrease incentives for strategic default, however non-strategic defaults resulted from an unanticipated economic shock are unavoidable and repayment is not completely insured. We deviate from Bhole and Ogden’s (2010) where we assume that group members play grim-trigger between themselves. As a result if some one defaults strategically, other members of the group will not repay his loan as well as their own loans, and no further loans are granted to the group members. Individual liability (IL) which is our benchmark can be seen as a special case of JL in which \(n = 1\).

Our results suggest that when chances of success in projects are not very high, then feasibility of JL will be increasing in the group size meaning that larger groups are more reliable to repay the loan when projects are risky. But group size can not grow too large. We suggest an algorithm to calculate the optimal group size given the amount of high and low returns of projects, the discount factor of borrowers for future loans, and the chance of success in projects. Furthermore, we show that JL has a positive effect on

\textsuperscript{7}Ensuring repayment incentives through refinancing is modeled in the context of microfinance by Hulme and Mosley (1996) and Armendariz de Aghion and Morduch (2000).
borrower’s welfare and repayment rate compared to IL, and this effect is increasing in
the group size, although JL contracts are feasible under a smaller set of parameter values
than IL contract. JL lending can also outperform IL lending in terms of maximum loan
that can be offered to borrower if the group size is not too large.

The following section reviews the related literature. Section 3 presents our model.
Sections 4 and 5 characterize the optimal IL and JL contracts, Section 5 also deals with
comparison between IL and JL and discusses conditions under which JL is the optimal
decision (conditions are made on borrowers discount factor).

2 Related literature

Our study is in line with the recent and growing literature in economic theory that
explores conditions, if any, under which JL can enhance lending results and alleviate
market imperfections relative to other forms of lending. As recent instances, De Quidt
et al. (2014) compare JL lending to group lending without JL and to IL lending; Baland
et al. (2013) and Giné and Karlan (2014) compare JL lending with IL lending; and Ahlin
and Waters (2013) compare JL lending to dynamic lending. We argue that JL lending
outperforms IL lending in terms of repayment rate and borrower’s welfare, although it
is feasible under smaller parameter setting.

The existing literature on JL devotes much attention to different attributes of JL
as peer selection, peer monitoring, social sanction, and reduced transaction costs as
responsible factors for its relative success, Varian (1990), Stiglitz (1990) and Besley and
Coate (1995) are among the well-known examples. However little attention has been
paid so far to group size as a feature that may affect the lending outcome. Theoretical
studies mostly analyze lending models of groups of two borrowers. The reason for this
may be avoiding the complexity of calculations for larger groups. In fact experimental
studies suggest the importance of group size (Abbink et al., 2006); and in practice, most
of the time groups have more than two members in JL lending.

Although not in the context of JL lending, Diamond (1984) and Laux (2001) prove
that larger groups can improve efficiency. Conning (2005) and Ahlin (2013) also argue in
favor of larger groups, but they suggest that group size can not grow too large. Conning
(2005) specifies that it becomes more and more costly to contain free-riding as group
size increases. Ahlin (2013) mentions that presence of a social asset – local borrower

8Some of the other often cited articles that consider adverse selection issues in group lending are
Besley (1995); Ghatak and Guinmane (1999); Ghatak (1999); Ghatak (2000); Laffont and NGuessan
(2000); Armendariz de Aghion (1999); Armendariz de Aghion and Gollier (2000); Laffont (2003), and
many more.
information – is necessary for large groups to have any impact.

However Baland et al. (2013), in a setting similar to Diamond (1984) and Laux (2001), characterize loan contracts as a function of borrower’s initial wealth, and prove that a smaller group size in JL lending can raise efficiency. Bourjade and Schindele (2012) argue that if group members have social ties, a rational lender should choose a group of limited size. They explain that there is a trade-off between raising profits through increased group size and providing incentives for borrowers with less social ties.

3 Model

Consider an infinitely repeated principal-agent model, with \( n \) agents (borrowers) and a benevolent principal (lender) playing grim-trigger with each other. Following Tedeschi’s (2006), we consider a two-phase model in which the lender and borrowers start in a “lending phase”, if one loan is successfully repaid by the group, another loan is given. In any period, if borrowers default, the lender and borrowers then engage in a “punishment phase”, where no new loans are extended to the borrowers.

Extending Bhole and Ogden (2010) over \( n \) borrower, each period of our game has three steps:

\[ s = 0 \quad \text{Each member receives a contract} \ (L, R) \ \text{individually, specifying the loan} \ L \ \text{and repayment} \ R. \]

\[ s = 1 \quad \text{Each borrower invests} \ L \ \text{on his project that will be either successful with chance of} \ \alpha \in [0, 1] \ \text{and its return will be high, i.e.} \ P(Y = Y^H) = \alpha, \ \text{or not successful with chance of} \ 1 - \alpha, \ \text{and its return will be low, i.e.} \ P(Y = Y^L) = 1 - \alpha, \ \text{and} \ 0 \leq Y^L < Y^H. \ \text{Project returns are public information to all members of the group of borrowers.} \]

\[ s = 2 \quad \text{Each borrower decides either to repay his share} \ R \ \text{or not, and the lender announces the remaining payment to the members. If some members default and the total repayment is less than} \ nR \ \text{successful members should pay their share of the remaining. Otherwise there will be no loan from the next period on.} \]

These three stages will be repeatedly played until the lender realizes that the group of borrowers are not entitled for financing next period, and each period of not receiving loan, borrowers’ utility will be zero. Note that it is assumed that projects do not differ in their riskiness (i.e. \( \alpha \) is the same for all borrowers); high or low returns of projects
are borrowers private information; and projects that do not change through periods and each borrower always invests on the same project.

Two types of defaults are possible: *strategic default* in which borrower does not repay although he had high outcome $Y^H$, and *nonstrategic default* as a result of obtaining low outcome $Y^L$ resulted of a bad luck or negative economic shock. We assume that the lender is unable to observe whether a borrower’s default is strategic or nonstrategic. However borrowers are able to observe strategic defaults of their peers costlessly.

We assume that the borrowers among themselves play a repeated game in which they play a grim-trigger strategy (this is different from Bhole and Ogden, 2010). They start out cooperating, thus from period 1 to period $T$ of the lending game, they continue to repay if they can, and if all others did repay or some defaulted non-strategically. And they also repay the remaining share of their peers. If at some period $T + 1$ some members did not repay strategically, they stop repaying themselves and stop repaying other players’ shares. They will therefore not get a loan in the next period.

As we assumed that the lender is benevolent (as in Rai and Sjostrom, 2004) it would be good for her to maximize the payoff of each borrower contingent on:

1. each borrower must be willing to accept a loan (repayment amount must be affordable for him),

2. each borrower must have correct incentives to repay for herself and his share for each defaulting peer, when he is able to pay; in the worst case that all other members default, he must be still willing to repay for the entire group, and

3. the lender must break even, meaning that she must maintain a sustainable lending operation over the entire loan portfolio by charging the appropriate repayment.

The exact terms of the maximization problem depends on whether we are examining the individual lending, group lending or flexible group lending and will be discussed in the following sections.

4 Individual liability (IL)

Individual lending can be considered as a special type of group lending with $n = 1$. Timing is stated as:

$s = 0$ borrower receives his loan $L$.

$s = 1$ the loan is invested into a project that can be either successful with probability $\alpha$, or a failure with probability $1 - \alpha$. 

borrower decides to either repay his share $R$ or not, and bank determines if he is entitled to receive further loan or not, contingent upon the accomplishment of his repayment.

These three stages will be repeatedly played until the lender realizes that borrower is not allowed for financing next period, and each period of not receiving loan, borrowers utility will be zero.

To impose the equilibrium that “borrower is repaying when his project is successful”, we need to ensure that the payoff of strategically defaulting is NOT larger than or equal to the payoff of repaying and being refinanced $V^D < V^R$, where $V^R$ is the expected lifetime utility of a borrower from next period onwards in which he gets financing. Put it more mathematically, we must have $Y^H < Y^H - R + \delta V^R$, that can be simplified to $R < \delta V^R$ in which $0 \leq \delta \leq 1$ represents the borrower’s discount factor or his valuation of his future utility of accessing financing.

Today value of the expected lifetime utility of a borrower who plays a repayment strategy at any period onwards in which he gets financing is determined as

$$V^R_{IL} = \mathbb{E}(Y) - \alpha R + \alpha \delta V^R_{IL}$$

that can be rewritten as

$$V^R_{IL} = \frac{\mathbb{E}(Y) - \alpha R}{1 - \alpha \delta}$$ (1)

Now the lender’s maximization problem can be formulated as $\max_{R,L} V^R_{IL}$ subject to:

1. The stipulated repayment amount for a successful borrower must be affordable for him and can not exceed his output,

$$R \leq Y^H$$ (2)

2. Each borrower has to have incentive to repay his loan if his projects successfully has high return (repayment must be better than default),

$$R \leq \delta V^R$$ (3)

3. Bank has to be able to break-even. In another word, the expected repayment amount of the borrower must be at least as large as $L$,

$$R \geq \frac{L}{\alpha}$$ (4)
in which $0 \leq Y^L < Y^H$, $0 \leq \alpha \leq 1$, and $0 \leq \delta \leq 1$.

If there are some $(L, R)$ that satisfy constraints (2), (3) and (4), then individual lending will be feasible, and these constraints will define its feasibility region.

**Proposition 1:** Individual lending is feasible iff $L \leq \alpha \delta \mathbb{E}(Y)$. For any $\alpha \neq 0$, the lender demands optimally the repayment $R = \frac{L}{\alpha}$. The borrower’s expected lifetime utility will be

$$V_{IL}^R = \frac{\mathbb{E}(Y) - L}{1 - \alpha \delta}$$

Proposition 1 says that the repayment amount asked by lender and the expected lifetime utility of each borrower have both reverse relations with the chance of success in project. The larger the $\alpha$ becomes, the lender asks for a smaller repayment amount and borrower is better off. Second suggestion of this Proposition is that the borrower is also better off, the more patient he is and the more value he put on his future financing for investment.

## 5 Joint liability (JL)

Timing in group lending is as stated below:

$s = 0$ each member of the group receives a loan $L$.

$s = 1$ the loans are invested into $n$ projects that can be either successful with probability $\alpha$ or a failure with probability $1 - \alpha$, which is public information to all borrowers.

$s = 2$ each borrower decides to either repay his share $R$ or not, and the lender announces the remaining payment. In this case, if $i$ members default non-strategically on their loan, other partners will be asked to additionally pay an amount $\frac{iR}{n-i}$ to the lender for their defaulting peers. If the total repayment is equal to $nR$ or more, the group receives future financing. Otherwise, the entire group will be excluded from financing next period.

The game will be repeatedly played until lender realizes that the group is not entitled for loaning next period, and each period of not receiving loan, borrowers utility will be zero.

To ensure that “each borrower is repaying when his project is successful” is a subgame perfect equilibrium, we need to ensure that the payoff of strategically defaulting can never be larger than or equal to the payoff of repaying and being refinanced, i.e. $V^D < V^R$. 


where $V^R$ is the expected lifetime utility of a borrower from next period onwards in which he gets financing. Put it more mathematically, we must have $Y^H < Y^H - nR + \delta V^R$, and $0 \leq \delta \leq 1$ is borrower’s discount factor that determines his valuation of tomorrow’s utility of financing and investment. Additionally, the lender needs to induce a successful borrower to pay $nR$ when all his partners are unsuccessful. Therefore it is necessary to have $0 < \delta V^R - nR$ or

$$nR < \delta V^R$$

that also implies $R < \delta V^R$, which guarantees that a successful borrower pays repayment $R$ when all his partners are successful.

The expected payoff in each period $T$ for a borrower who plays the repayment strategy can be stated as follows:

$$\mathbb{E}(Y) - R[1 - (1 - \alpha)^n]$$

(see the appendix for detailed calculations of (5)). Note that he repays, only when he has earned high outcome $Y^H$ of his project. As the game is played for an infinite number of rounds, today value of the expected lifetime utility of a borrower who plays a repayment strategy at any period onwards in which he gets financing, using formula (5), is determined as

$$V^R_{JL} = \mathbb{E}(Y) - R[1 - (1 - \alpha)^n] + [1 - (1 - \alpha)^n] \delta V^R_{JL}$$

that can be rewritten also as

$$V^R_{JL} = \frac{\mathbb{E}(Y) - R[1 - (1 - \alpha)^n]}{1 - \delta [1 - (1 - \alpha)^n]} \delta V^R_{JL}$$

(6)

Now the lender’s optimization problem for any $0 \leq Y^L < Y^H$, $0 \leq \alpha \leq 1$, and $0 \leq \delta \leq 1$ can be stated as $\max_{L,R} V^R_{JL}$ subject to:

1. The stipulated repayment amount for a successful borrower can not exceed his output (even in worse case that everyone else has failed) and it must be affordable,

$$nR \leq Y^H$$

(7)

2. Each borrower must have correct incentive to repay for herself and for defaulting members (he must be willing to pay for the entire group in worst case that everybody else has failed). In another word, repaying must be better than default even
in worst case that is repaying for all peers,

\[ nR \leq \delta V^R \tag{8} \]

3. The lender must be able to sustain the lending game over periods and at least break even. So using the fact that the expected repayment amount of the group have to be at least as large as \( nL \), we must have

\[ R \geq \frac{L}{[1 - (1 - \alpha)^n]} \tag{9} \]

(for detailed calculation of (9) see the appendix).

If there are some \((L, R)\) that satisfy constraints (7), (8) and (9), then individual lending will be feasible, and these constraints will define its feasibility region.

Proposition 2:

1. If \( \delta < \tilde{\delta}(n, \alpha, Y^H, Y^L) \), then JL lending is feasible iff \( L < \tilde{L}(n, \alpha, \delta, Y^H, Y^L) \); otherwise, JL lending is feasible iff \( L < \hat{L}(n, \alpha, Y^H) \), where

\[
\tilde{\delta}(n, \alpha, Y^H, Y^L) = \frac{Y^H}{E(Y) + \frac{n-1}{n} [1 - (1 - \alpha)^n] Y^H} \\
\tilde{L}(n, \alpha, \delta, Y^H, Y^L) = \frac{\delta E(Y) [1 - (1 - \alpha)^n]}{n - \delta (n-1) [1 - (1 - \alpha)^n]} \\
\hat{L}(n, \alpha, Y^H) = \frac{Y^H [1 - (1 - \alpha)^n]}{n}
\]

2. Whenever JL lending is feasible,

(a) For any \( \alpha \neq 0 \), the lender demands the optimal repayment \( R^* = \frac{L}{[1 - (1 - \alpha)^n]} \)

from each borrower.

(b) The expected lifetime utility for each borrower will amount to

\[ V_{JL}^R = \frac{E(Y) - L}{1 - \delta [1 - (1 - \alpha)^n]} \]

Proposition 2 suggest that an optimal repayment \( R^* \) that is decreasing in \( n \). Therefore a larger group can be charged less that in turn increases the borrowers welfare. It also
shows that changes in feasibility of JL are similar to changes of

\[ f(n, \alpha, \delta, Y^H, Y^L) = \begin{cases} 
\hat{L}(n, \alpha, \delta, Y^H, Y^L) & \delta < \hat{\delta}(n, \alpha, Y^H, Y^L) \\
\check{L}(n, \alpha, Y^H) & \delta \geq \hat{\delta}(n, \alpha, Y^H, Y^L) 
\end{cases} \]

that according to the proof of Proposition 2 can be also written as

\[ f(n, \alpha, \delta, Y^H, Y^L) = \min \{ \hat{L}(n, \alpha, \delta, Y^H, Y^L), \check{L}(n, \alpha, Y^H) \} \]

Function \( f(n, \alpha, \delta, Y^H, Y^L) \) will be called feasibility function from now on. Feasibility function defines an upper bound for the maximum loan that can be offered to the borrower. Besides \( \hat{L} \) and \( \check{L} \), feasibility function also depends on whether \( \delta \) belongs to \( (0, \hat{\delta}(n, \alpha, Y^H, Y^L)) \) or \( (\hat{\delta}(n, \alpha, Y^H, Y^L), 1) \).

**Group size and feasibility of JL:** We continue this section by trying to discover how the feasibility function and consequently feasibility of JL lending is affected by the group size and what is the optimal group size that results in maximum feasibility. Later we discuss the necessary circumstances under which JL lending outperforms IL lending.

To be able to predict behavior of feasibility function with respect to changes of \( n \), in Lemma 1 and Lemma 2 we take a closer look at the changes of \( \hat{L}(n, \alpha, \delta, Y^H, Y^L) \), \( \check{L}(n, \alpha, Y^H) \) and \( \hat{\delta}(n, \alpha, Y^H, Y^L) \) with respect to changes of \( n \) when other parameters \( (\alpha, \delta, Y^H, Y^L) \) are given.

**Lemma 1:** Assume \( \hat{L}(n, \alpha, \delta, Y^H, Y^L) \) and \( \check{L}(n, \alpha, Y^H) \) are functions defined in Proposition 2,

1. \( \hat{L}(n, \alpha, \delta, Y^H, Y^L) \) is strictly decreasing in \( n \) if \( 0 < \hat{\delta}(n, \alpha) < \delta < \hat{\delta}(n, \alpha, Y^H, Y^L) \), where

\[ \hat{\delta}(n, \alpha) = \frac{(1-\alpha)^n \ln(1-\alpha)^n + [1 - (1-\alpha)^n]}{[1 - (1-\alpha)^n]^2} \]

2. \( \check{L}(n, \alpha, Y^H) \) is strictly decreasing in \( n \).

As we see in Lemma 1, changes of \( \hat{L}(n, \alpha, \delta, Y^H, Y^L) \) with respect to \( n \) is affected by changes of \( \hat{\delta}(n, \alpha) \) with respect to \( n \). So we also need to have understanding about how \( \hat{\delta}(n, \alpha) \) changes with respect to changes of \( n \).
**Lemma 2:** Assume $\tilde{\delta} (n, \alpha, Y^H, Y^L)$ and $\hat{\delta} (n, \alpha)$ are functions defined in Proposition 2 and Lemma 1,

1. $\hat{\delta} (n, \alpha)$ is strictly increasing in both $n$ and $\alpha$.
2. $\tilde{\delta} (n, \alpha, Y^H, Y^L)$ is strictly decreasing in both $n$ and $\alpha$.
3. For any given $n$ and $\alpha$,
   \[ \frac{1}{2} \leq \hat{\delta} (n, \alpha) \leq 1 \]
4. For any given $n$ and $\alpha$,
   \[ \frac{1}{2} \leq \tilde{\delta} (n, \alpha, Y^H, Y^L) \leq \frac{Y^H}{Y^L} \]

A direct result of Lemma 2 is that interval $\left( \hat{\delta} (n, \alpha), \tilde{\delta} (n, \alpha, Y^H, Y^L) \right)$ becomes tighter by the increase of $n$ or $\alpha$, and it becomes wider by the decrease of $n$ or $\alpha$. Therefore to keep the interval $\left( \hat{\delta} (n, \alpha), \tilde{\delta} (n, \alpha, Y^H, Y^L) \right)$ non-empty, the larger the $\alpha$ is, the smaller the $n$ must be chosen in order to offset the contraction of the interval resulted by large $\alpha$. And in general, $n = 2$ always provides the widest interval for any given $\alpha$. Below in Proposition 3, we prove that regardless of the magnitude of $n$, for small enough $\alpha$, the interval $\left( \hat{\delta} (n, \alpha), \tilde{\delta} (n, \alpha, Y^H, Y^L) \right)$ is never empty. However if $\alpha$ becomes too small, then project may not be considered for financing.

**Proposition 3:** Assume $\tilde{\delta} (n, \alpha, Y^H, Y^L)$ and $\hat{\delta} (n, \alpha)$ are functions defined in Proposition 2 and Lemma 1.

1. If $\alpha < \bar{\alpha}$ and $\delta$ is such that
   \[ \hat{\delta} (n, \alpha) < \delta < \tilde{\delta} (n, \alpha, Y^H, Y^L) \]
   for some $n$, then feasibility of JL will be increasing in $n \in [2, N_{\alpha,\delta}]$, where
   \[ N_{\alpha,\delta} = \min \left\{ \left[ \hat{\delta}^{-1} (\alpha, \delta) \right], \left[ \tilde{\delta}^{-1} (\alpha, \delta, Y^H, Y^L) \right] \right\} \]
2. For $\alpha > \bar{\alpha}$, maximum feasibility happens at $n = 2$.
3. $\bar{\alpha}$ is approximately 0.5.
4. For very large $n$, feasibility of JL is decreasing in $n$. 

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(a) It is assumed that $\alpha = 0.3$.

(b) It is assumed that $\alpha = 0.8$.

(c) It is assumed that $\alpha = 0.3$ and $\delta = 0.85$.

(d) It is assumed that $\alpha = 0.3$ and $\delta = 0.85$.

Figure 1: As shown in (a) and (b), larger $n$ are possible as long as given $\delta$ belongs to the interval $\left(\hat{\delta}(n,\alpha), \tilde{\delta}(n,\alpha,Y^{H},Y^{L})\right)$. And largest possible $n$ locates at $[\hat{\delta}^{-1}(\alpha,\delta)]$ or $[\tilde{\delta}^{-1}(\alpha,\delta,Y^{H},Y^{L})]$. (c) and (d) show that maximum feasible loan under JL is increasing in $n$ as long as given $\delta$ belongs to the interval $\left(\hat{\delta}(n,\alpha), \tilde{\delta}(n,\alpha,Y^{H},Y^{L})\right)$. 

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Proposition 3 is telling us that if the chance of success in the project is low (less than 50%), larger loans could be given to larger groups (see Figure 1, parts (a) and (c)). While for projects with higher chance of success (more than 50%), the maximum loan that can be given to a group of two members is higher than any other group (see Figure 1, parts (b) and (d)). Proposition 3 together with the Lemma 2 also provide a necessary condition for feasibility function to be increasing in $n$, that is the given $\delta$ must be at least $\frac{1}{2}$, as both $\hat{\delta}(n, \alpha)$ and $\tilde{\delta}(n, \alpha, Y^H, Y^L)$ are proved to be larger than $\frac{1}{2}$.

Proposition 3 also suggests that the group size can not grow too large. Intuitively group size has two countervailing effects. On one side, larger group can provide stronger repayment insurance and is able to handle riskier projects and repay successfully. On the other side, a large group can be a threat towards feasibility of group lending. The threat comes from the point that each successful member is in charge of all defaulting peers and if everybody else has failed, he must repay the entire loan of the group. And this becomes very difficult if the group is too large. Proposition 3 suggests an algorithm to find the optimal group size when parameters $\alpha$ and $\delta$ are given.

In Figure 1, we try to add some intuition to Proposition 3. For simplicity it is assumed that $Y^H = 1$ and $Y^L = 0$. In parts (a) and (b), we depict $\hat{\delta}(n, \alpha, Y^H, Y^L)$ and $\tilde{\delta}(n, \alpha)$ with respect to $n$ when chance of success in project is small $\alpha = 0.3$, and when it is large $\alpha = 0.8$ respectively. As shown in part (a), for $\alpha = 0.3$, there are some $n$ for which the given $\delta = 0.85$ belongs to the interval $\left(\hat{\delta}(n, \alpha), \tilde{\delta}(n, \alpha, Y^H, Y^L)\right)$, and the largest of such $n$ lies at $\lfloor \hat{\delta}^{-1}(\alpha, \delta) \rfloor = \lfloor 7.136 \rfloor = 7$. Note that if the given $\delta$ is very close to 1, then the largest $n$ would lies at $\lfloor \tilde{\delta}^{-1}(\alpha, \delta, Y^H, Y^L) \rfloor$. However in part (b), when $\alpha$ is large, for any $n > 1$, the interval $\left(\hat{\delta}(n, \alpha), \tilde{\delta}(n, \alpha, Y^H, Y^L)\right)$ is empty.

Parts (c) and (d) of Figure 1 depict $\hat{L}(n, \alpha, \delta, Y^H, Y^L)$ and $\tilde{L}(n, \alpha, Y^H)$ with respect to $n$ when chance of success in project is small $\alpha = 0.3$, and when it is large $\alpha = 0.8$ respectively. As shown in part (c), when $\alpha$ is small, $\hat{L}(n, \alpha, \delta, Y^H, Y^L)$ defines the boundary for maximum feasible loan, and it reaches its maximum at $n = 7$. Therefore $n = 7$ is the group size that maximizes the feasibility of JL. However in part (d), when $\alpha$ is large, $\hat{L}(n, \alpha, \delta, Y^H, Y^L)$ defines the boundary for maximum feasible loan which is decreasing in $n$. Therefore $n = 2$ is the optimum group size that maximizes the feasibility of JL.

**When JL works better than IL?** Up to now, we discussed that feasibility of JL lending can be increasing in group size when chance of success in project is small. However we don’t know yet that it can perform more efficiently than IL lending does. In Proposition 4, we prove that the lender can charge borrowers less under JL lending.
comparing to IL lending while staying still break-even. The reason is that “no repayment” is something that happens less often under JL than IL. And in turn smaller amount of repayment leads to a higher level of welfare for the borrowers.

**Proposition 4:** When both IL and JL contracts are feasible, then

1. Borrowers repayment amount is lower and his welfare is higher with JL than with IL.

2. If \( \alpha < \alpha \) and \( \delta \) is such that

\[
\frac{\alpha n - [1 - (1 - \alpha)^n]}{\alpha (n - 1)[1 - (1 - \alpha)^n]} < \delta < \frac{Y^H [1 - (1 - \alpha)^n]}{n \alpha E(Y)}
\]

at least for some \( n \), then JL lending is feasible for larger amount of loans than IL lending. Otherwise IL lending will be feasible for larger range of loan \( L \) than JL lending.

3. If \( \alpha > \alpha \), then IL lending is feasible for larger amount of loan \( L \) than JL lending.

4. \( \alpha \) is approximately 0.764.

5. For very large \( n \), IL lending does always better than JL lending.

Proposition 4 shows that JL contract has a positive effect on borrower’s welfare and repayment rate compared to IL contract. It suggests a necessary condition, the relative feasibility condition, under which JL contract can offer larger maximum loan than IL contract, and as discussed before in Proposition 3, the amount of loan can be increasing in the group size. Proposition 4 also discuses the circumstances that are necessary for the relative feasibility condition to be satisfied. It proves that if the chance of success in project is high (higher than 75%), the maximum loan that can be given to a borrower under IL contract is higher than JL contract. While for projects with lower chance of success (lower than 75%), larger loans could be given only under JL contracts. As it was expected, the group size can not grow too large. JL contract with groups of more than 10 members can have better outcome than IL contract only when project has very little chances of success (35%), otherwise IL contract has higher outcome than JL with groups of too many members.

Figure 2 illustrates the situation discussed in Proposition 4. For simplicity it is assumed that \( Y^H = 1 \) and \( Y^L = 0 \). As shown in part (a), when chance of success in project is small enough so that the relative feasibility condition is satisfied (for example
Figure 2: For smaller $\alpha$, the maximum loan that can be offered to borrowers is higher under JL than IL; for larger $\alpha$, the maximum loan that can be offered to borrowers is higher under IL than JL.

$\alpha = 0.4$), for any $0 < \delta < 1$, $\hat{L} \left( n, \alpha, \delta, Y^H, Y^L \right)$ defines the boundary for maximum feasible loan under JL which is always equal to or larger than maximum feasible loan under IL. However when chance of success in project is large (for example $\alpha = 0.9$), maximum feasible loan under IL is strictly higher than maximum feasible loan under JL.

6 Conclusion

The original question that motivates our study is how group size affects the efficiency of microcredit lending under JL contracts and what is the optimal group size. We analyze a model of repeated microcredit lending in which a microcredit is given to a group of $n$ members by a benevolent lender. Borrowers invest the loan on their separate projects and the lender does not observe the borrowers outcome of their risky project. Therefore, in case of some default on borrower’s loan repayments, the lender is not able to identify the strategic default.

Most of the existing theoretical papers in microfinance discuss JL lending in groups of only two members, although experimental studies talk about the importance of group size (Abbink et al., 2006), and in real life group sizes are most of the time larger than two. Our results show that group size can be an influential factor in improving lending efficiency and the assumption $n = 2$ that is largely used in the literature may result in
neglecting a potential important factor.

In this study, we found the optimal group size endogenously while deriving the optimal contract that maximizes the borrower’s welfare subject to: repayment being affordable for each borrower; repayment being better than default for each borrower; and lender breaking even.

We discussed in Proposition 3 that although JL contracts are feasible under a smaller set of parameter values than IL, the feasibility can increase in the group size for riskier projects.

Intuitively there are costs and benefits in being a member of a large group of borrowers. From one side, it enhances the chance of assured repayment for a defaulting member. On the other side, there are also a higher threat of repaying for other defaulting members of the group. We proved it formally that for risky projects, lender should consider financing larger groups while for less risky projects, lending to smaller groups and some times even to individuals can be the optimum decision.

Intuitively, large groups would have an additional attraction when borrowers are loss-averse, since they can assure borrower’s payoff through insurance provided by the group members. Hence large group can also be seen as a way of reducing the risk linked with group lending (Stiglitz, 1990). We leave this issue for further research.

7 Appendix

Calculation of Formula (5):

\[
\begin{align*}
\binom{n-1}{0} & \alpha^n (1-\alpha)^0 (Y^H - R) + \\
\binom{n-1}{1} & \alpha^{n-1} (1-\alpha) \left[ Y^H - \left( R + \frac{R}{n-1} \right) \right] + \\
\binom{n-1}{2} & \alpha^{n-2} (1-\alpha)^2 \left[ Y^H - \left( R + \frac{2R}{n-2} \right) \right] + \\
& \vdots \\
\binom{n-1}{n-1} & \alpha (1-\alpha)^{n-1} \left[ Y^H - (nR) \right] + (1-\alpha) Y^L = \\
\alpha Y^H - R \sum_{i=0}^{n-1} \binom{n-1}{i} & \alpha^{n-i} (1-\alpha)^i \left( \frac{n}{n-i} \right) + (1-\alpha) Y^L =
\end{align*}
\]
\[
\alpha Y^H + (1 - \alpha) Y^L - R [1 - (1 - \alpha)^n] = \\
\mathbb{E}(Y) - R [1 - (1 - \alpha)^n]
\]

Calculation of Formula (9):

\[
\alpha^n \binom{n}{0} nR + \alpha^{n-1} (1 - \alpha) \binom{n}{1} nR + \ldots + \alpha^1 (1 - \alpha)^{n-1} \binom{n}{n-1} nR \geq nL
\]

\[
\Rightarrow nR \sum_{i=0}^{n} \alpha^{n-i} (1 - \alpha)^i \binom{n}{i} - \alpha^0 (1 - \alpha)^n \binom{n}{n} nR \geq nL
\]

\[
\Rightarrow nR \left[ 1 - \alpha^0 (1 - \alpha)^n \binom{n}{n} \right] \geq nL
\]

\[
\Rightarrow R \geq \frac{L}{[1 - (1 - \alpha)^n]}
\]

Note that we are using the fact that the expected repayment amount of the group have to be at least as large as \(nL\).

Proof of Lemma 1:

1. \(\hat{L}(n, \alpha, \delta, Y^H, Y^L) = \frac{\delta \mathbb{E}(Y) [1 - (1 - \alpha)^n]}{[n - \delta (n - 1) [1 - (1 - \alpha)^n]]} \)

\[
\frac{\partial \hat{L}}{\partial n} = \frac{\delta \mathbb{E}(Y) \left[ - (1 - \alpha)^n \ln (1 - \alpha)^n + \delta [1 - (1 - \alpha)^n]^2 - [1 - (1 - \alpha)^n] \right]}{[n - \delta (n - 1) [1 - (1 - \alpha)^n]]^2}
\]

Sign of \(\frac{\partial \hat{L}}{\partial n}\) can be determined by looking only at the sign of

\[G(n, \alpha, \delta) \equiv - (1 - \alpha)^n \ln (1 - \alpha)^n + \delta [1 - (1 - \alpha)^n]^2 - [1 - (1 - \alpha)^n] \]

(a) \(\frac{\partial \hat{L}}{\partial n} < 0\) iff \(G(n, \alpha, \delta) < 0\) that results in

\[\delta < \frac{[1 - (1 - \alpha)^n] + (1 - \alpha)^n \ln (1 - \alpha)^n}{[1 - (1 - \alpha)^n]^2} \equiv \hat{\delta}(n, \alpha)\]

, so \(\hat{L}(n, \alpha, \delta, Y^H, Y^L)\) is strictly decreasing in \(n\) for any \(0 < \delta < \hat{\delta}(n, \alpha)\).
(b) \( \frac{\partial \hat{L}}{\partial n} > 0 \) iff \( G(n, \alpha, \delta) < 0 \), so \( \hat{L}(n, \alpha, \delta, Y^H, Y^L) \) is strictly increasing in \( n \) for any \( \hat{\delta}(n, \alpha) < \delta < \check{\delta}(n, \alpha, Y^H, Y^L) \).

(c) \( \frac{\partial \hat{L}}{\partial n} = 0 \) at \( \delta = \hat{\delta}(n, \alpha) \). \( \hat{L}(n, \alpha, \delta, Y^H, Y^L) \) has a minimum at \( \delta = \hat{\delta}(n, \alpha) \).

2.

\[
\hat{L}(n, \alpha, Y^H) = \frac{Y^H [1 - (1 - \alpha)^n]}{n}
\]

\[
\frac{\partial \hat{L}}{\partial n} = \frac{(1 - \alpha)^n [1 - \ln (1 - \alpha)^n] Y^H - Y^H}{n^2}
\]

Sign of \( \frac{\partial \hat{L}}{\partial n} \) can be determined by knowing the sign of \( (1 - \alpha)^n [1 - \ln (1 - \alpha)^n] - 1 \).

Claim: For any \( \alpha \neq 0 \), \( H(\alpha, n) = (1 - \alpha)^n [1 - \ln (1 - \alpha)^n] < 1 \).

Proof: Assume \( \rho = (1 - \alpha)^n \), then \( H(\rho) = \rho [1 - \ln \rho] \). Since \( \frac{\partial H}{\partial \rho} = -\ln \rho \) is positive for any \( 0 < \rho < 1 \), zero for \( \rho = 1 \), and negative for any \( \rho > 1 \), then \( H(\rho) \) has a maximum at \( \rho = 1 \). Therefore \( H(\alpha, n) \) has a maximum at \( (1 - \alpha)^n = 1 \) or more precisely at \( \alpha = 0 \). Then for any \( \alpha \neq 0 \), \( H(\alpha, n) < 1 \).

So for any \( \alpha \neq 0 \), \( \frac{\partial \hat{L}}{\partial n} < 0 \) and \( \hat{L}(n, \alpha, Y^H) \) is strictly decreasing in \( n \).

QED

Proof of Lemma 2:

\[
\hat{\delta}(n, \alpha) = \frac{(1 - \alpha)^n \ln (1 - \alpha)^n + [1 - (1 - \alpha)^n]}{[1 - (1 - \alpha)^n]^2}
\]

\[
\check{\delta}(n, \alpha, Y^H, Y^L) = \frac{Y^H}{\mathbb{E}(Y) + \frac{n}{n - 1} [1 - (1 - \alpha)^n] Y^H}
\]

1. \( \hat{\delta}(n, \alpha) \) is strictly increasing in both \( n \) and \( \alpha \):

(a)

\[
\frac{\partial \hat{\delta}}{\partial n} = (1 - \alpha)^n \ln (1 - \alpha)^n \times \frac{\ln (1 - \alpha)^n [1 + (1 - \alpha)^n] + 2 [1 - (1 - \alpha)^n]}{[1 - (1 - \alpha)^n]^3}
\]
note that for any \(0 < \alpha < 1\), \((1 - \alpha)^n > 0\), \(\ln (1 - \alpha)^n < 0\), and \([1 - (1 - \alpha)^n]^3 > 0\), so \(\frac{\partial \hat{\delta}}{\partial n} > 0\) iff 
\[
\ln (1 - \alpha)^n [1 + (1 - \alpha)^n] + 2 [1 - (1 - \alpha)^n] < 0
\]
that can be rewritten as 
\[
H (\alpha, n) \equiv \ln (1 - \alpha)^n + \frac{2 [1 - (1 - \alpha)^n]}{[1 + (1 - \alpha)^n]}
\]
Claim: For any \(0 < \alpha < 1\), \(H (\alpha, n) < 0\)

Proof: Assume \(\rho = (1 - \alpha)^n\), then \(0 < \rho < 1\) and \(H (\rho) = \ln \rho + \frac{2 [1 - \rho]}{[1 + \rho]}\). Since for any \(0 < \rho < 1\), 
\[
\frac{dH (\rho)}{d\rho} = \frac{[1 - \rho]^2}{\rho [1 + \rho]^2} > 0
\]
is increasing in \(\rho\). On the other hand \(\lim_{\rho \to 1} H (\rho) = 0\). Therefore for any \(0 < \rho < 1\), \(H (\rho) < 0\), and so for any \(0 < \alpha < 1\), \(H (\alpha, n) < 0\). 

So \(\hat{\delta} (n, \alpha)\) is strictly increasing in \(n\).

(b) 
\[
\frac{\partial \hat{\delta}}{\partial \alpha} = n (1 - \alpha)^{n-1} \times \frac{- [1 + (1 - \alpha)^n] \ln (1 - \alpha)^n - 2 [1 - (1 - \alpha)^n]}{[1 - (1 - \alpha)^n]^3}
\]
As for any \(n > 1\), \(0 < \alpha < 1\), we always have \((1 - \alpha)^{n-1} > 0\), \([1 - (1 - \alpha)^n]^3 > 0\), then \(\frac{\partial \hat{\delta}}{\partial \alpha} > 0\) iff 
\[
-[1 + (1 - \alpha)^n] \ln (1 - \alpha)^n - 2 [1 - (1 - \alpha)^n] > 0
\]
or 
\[
\ln (1 - \alpha)^n + \frac{2 [1 - (1 - \alpha)^n]}{[1 + (1 - \alpha)^n]} < 0
\]
that holds for any \(0 < \alpha < 1\), according to the Claim we proved above. So \(\hat{\delta} (n, \alpha)\) is strictly increasing in \(\alpha\) for any \(n\).

2. \(\hat{\delta} (n, \alpha, Y^H, Y^L)\) is strictly decreasing in both \(n\) and \(\alpha\):

Simply because \([1 - (1 - \alpha)^n]\) is strictly increasing in both \(n\) and \(\alpha\).
3. \( \hat{\delta} (n, \alpha) \) is increasing in both \( n \) and \( \alpha \), so

\[
\hat{\delta} (n, \alpha \to 0) < \hat{\delta} (n, \alpha) < \hat{\delta} (n, \alpha \to 1)
\]

and

\[
\hat{\delta} (n = 2, \alpha) < \hat{\delta} (n, \alpha) < \hat{\delta} (n \to \infty, \alpha)
\]

Assume \( \rho = (1 - \alpha)^n \), then

(a)

\[
\hat{\delta} (n, \alpha \to 0) = \lim_{\rho \to 1} \hat{\delta} (\rho) = \lim_{\rho \to 1} \frac{\rho \ln \rho + [1 - \rho]}{[1 - \rho]^2} H_{\text{Hopital}} \equiv \]

\[
\lim_{\rho \to 1-2[1 - \rho]} \frac{\ln \rho}{2[1 - \rho]} H_{\text{Hopital}} = \lim_{\rho \to 1/2} \frac{1 - \rho}{2} = \frac{1}{2}
\]

(b)

\[
\hat{\delta} (n, \alpha \to 1) = \lim_{\rho \to 0} \hat{\delta} (\rho) = \lim_{\rho \to 0} \frac{\ln \rho}{\rho} + 1 H_{\text{Hopital}} \equiv \lim_{\rho \to 0} \frac{1 - \rho}{\rho} = 1
\]

Therefore with changes of \( \alpha \),

\[
\frac{1}{2} \leq \hat{\delta} (n, \alpha) \leq 1
\]

(c)

\[
\hat{\delta} (n = 2, \alpha) = \frac{(1 - \alpha)^2 \ln (1 - \alpha)^2 + [1 - (1 - \alpha)^2]}{[1 - (1 - \alpha)^2]^2}
\]

(d)

\[
\hat{\delta} (n \to \infty, \alpha) = \lim_{\rho \to 0} \frac{\rho \ln \rho + [1 - \rho]}{[1 - \rho]^2} = \lim_{\rho \to 0} \rho \ln \rho = \lim_{\rho \to 0} \frac{\ln \rho}{\rho} H_{\text{Hopital}} \equiv 1
\]

Therefore with changes of \( n \),

\[
\frac{(1 - \alpha)^2 \ln (1 - \alpha)^2 + \alpha (2 - \alpha)}{\alpha^2 (2 - \alpha)} \leq \hat{\delta} (n, \alpha) \leq 1
\]
As \( \delta(n = 2, \alpha) \) is increasing in \( \alpha \), using part (a), we have

\[
\frac{1}{2} \leq \frac{(1 - \alpha)^2 \ln (1 - \alpha)^2 + \alpha (2 - \alpha)}{\alpha^2 (2 - \alpha)^2} \leq \delta(n, \alpha) \leq 1
\]

4. We know that \( \tilde{\delta}(n, \alpha, Y^H, Y^L) \) is decreasing in both \( n \) and \( \alpha \), so

\[
\tilde{\delta}(n, \alpha \to 1, Y^H, Y^L) \leq \tilde{\delta}(n, \alpha, Y^H, Y^L) \leq \tilde{\delta}(n, \alpha \to 0, Y^H, Y^L)
\]

and

\[
\tilde{\delta}(n \to \infty, \alpha, Y^H, Y^L) \leq \tilde{\delta}(n, \alpha, Y^H, Y^L) \leq \tilde{\delta}(n \to 0, \alpha, Y^H, Y^L)
\]

(a)

\[
\tilde{\delta}(n, \alpha \to 1, Y^H, Y^L) = \lim_{\alpha \to 1} \frac{Y^H}{E(Y) + \frac{n-1}{n} [1 - (1 - \alpha)^n] Y^H} = \frac{n}{2n - 1}
\]

(b)

\[
\tilde{\delta}(n, \alpha \to 0, Y^H, Y^L) = \lim_{\alpha \to 0} \frac{Y^H}{E(Y) + \frac{n-1}{n} [1 - (1 - \alpha)^n] Y^H} = \frac{Y^H}{Y^L}
\]

Therefore with changes of \( \alpha \), we have

\[
\frac{n}{2n - 1} \leq \tilde{\delta}(n, \alpha, Y^H, Y^L) \leq \frac{Y^H}{Y^L}
\]

\[
\frac{1}{2} \leq \frac{n}{2n - 1} \leq \tilde{\delta}(n, \alpha, Y^H, Y^L) \leq \frac{Y^H}{Y^L}
\]

(c)

\[
\tilde{\delta}(n \to \infty, \alpha, Y^H, Y^L) = \lim_{n \to \infty} \frac{Y^H}{E(Y) + \frac{n-1}{n} [1 - (1 - \alpha)^n] Y^H} = \frac{Y^H}{E(Y) + Y^H}
\]

(d)

\[
\tilde{\delta}(n = 2, \alpha, Y^H, Y^L) = \frac{Y^H}{E(Y) + \frac{1}{2} [1 - (1 - \alpha)^2] Y^H}
\]
Therefore with changes of $n$, we have

\[
\frac{Y^H}{\mathbb{E}(Y)} + Y^H \leq \tilde{\delta}(n, \alpha, Y^H, Y^L) \leq \frac{Y^H}{\mathbb{E}(Y) + \frac{1}{2} \left[ 1 - (1 - \alpha)^2 \right] Y^H}
\]

that can be rewritten as

\[
\frac{1}{2} \leq \tilde{\delta}(n, \alpha, Y^H, Y^L) \leq \frac{Y^H}{Y^L}
\]

because

\[
\frac{Y^H}{\mathbb{E}(Y) + Y^H} \geq \frac{Y^H}{Y^H + Y^H} = \frac{1}{2}
\]

and using part (b)

\[
\frac{Y^H}{\mathbb{E}(Y) + \frac{1}{2} \left[ 1 - (1 - \alpha)^2 \right] Y^H} \leq \frac{Y^H}{Y^L}
\]

QED

**Proof of Proposition 1:** In case of individual lending, the optimal contract $(L, R)$ is a solution to the following problem $[P1]$:

\[
\max_{R, L} V_{IL}^R = \frac{\mathbb{E}(Y) - \alpha R}{1 - \alpha \delta}
\]

subject to

1. 

\[
R \leq \delta V^R
\]

2. 

\[
R \leq Y^H
\]

3. 

\[
R \geq \frac{L}{\alpha}
\]

in which $0 \leq Y^L < Y^H$, $0 \leq \alpha \leq 1$, and $0 \leq \delta \leq 1$.  

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Since $V^R$ is decreasing in $R$, it will reach its maximum when the third constraint hold with equality, so for any $\alpha \neq 0$, $R^* = \frac{L}{\alpha}$. Substituting $R^*$ in the the objective function and constraints 1 and 2 we will have

$$\max_{L} V_{IL}^{R^*} = \frac{\mathbb{E}(Y) - L}{1 - \alpha \delta}$$

subject to

1. 
   $$L \leq \alpha \delta \mathbb{E}(Y)$$

2. 
   $$L \leq \alpha Y^H$$

$V^{R^*}$ is decreasing in $L$, so to reach the optimum answer we should set $L$ to its minimum, that is

$$L \leq \min \{ \alpha \delta \mathbb{E}(Y), \alpha Y^H \}$$

Since $\delta \mathbb{E}(Y) \leq Y^H$, the above condition can be rewritten as

$$L \leq \alpha \delta \mathbb{E}(Y)$$

QED

**Proof of Proposition 2:** In case of JL lending, the optimal contract $(L, R)$ is a solution to the following problem [$P2$]:

$$\max_{L,R} V_{JL}^{R} = \frac{\mathbb{E}(Y) - R[1 - (1 - \alpha)^n]}{1 - \delta [1 - (1 - \alpha)^n]}$$

subject to

1. 
   $$nR \leq \delta V^R$$

2. 
   $$nR \leq Y^H$$

3. 
   $$R \geq \frac{L}{1 - (1 - \alpha)^n}$$
As \( V_{jL}^R \) is decreasing in \( R \), the lender would like to set \( R \) as low as possible. The third constraint gives the minimum \( R \) required for breaking even, \( R^* = \frac{L}{1 - (1 - \alpha)^n} \), for any \( \alpha \neq 0 \). As long as \( R^* \) is limited to the upper limits expressed in constraints 1 and 2, lending is feasible, otherwise is not feasible. Replacing \( R^* = \frac{L}{1 - (1 - \alpha)^n} \), in the problem \([P2]\) we have

\[
\max_L V_{jL}^R = \frac{\mathbb{E}(Y) - L}{1 - \delta [1 - (1 - \alpha)^n]}
\]

subject to

1. \[
\frac{nL}{[1 - (1 - \alpha)^n]} \leq \frac{\delta \mathbb{E}(Y) - L}{1 - \delta [1 - (1 - \alpha)^n]}
\]

\( \Leftrightarrow \)

\[
\frac{nL}{[1 - (1 - \alpha)^n]} + \frac{\delta L}{1 - \delta [1 - (1 - \alpha)^n]} \leq \frac{\delta \mathbb{E}(Y)}{1 - \delta [1 - (1 - \alpha)^n]}
\]

\( \Leftrightarrow \)

\[
L \left[n - \delta (n - 1) [1 - (1 - \alpha)^n]\right] \leq \delta \mathbb{E}(Y) \left[1 - (1 - \alpha)^n\right]
\]

since \( [n - \delta (n - 1) [1 - (1 - \alpha)^n]] > 0 \), the above inequality can be simplified to

\[
L \leq \frac{\delta \mathbb{E}(Y) \left[1 - (1 - \alpha)^n\right]}{n - \delta (n - 1) [1 - (1 - \alpha)^n]} \equiv \hat{L} \left(n, \alpha, \delta, Y^H, Y^L\right)
\]

2. \[
\frac{nL}{[1 - (1 - \alpha)^n]} \leq Y^H
\]

\( \Leftrightarrow \)

\[
L \leq \frac{[1 - (1 - \alpha)^n] Y^H}{n} \equiv \check{L} \left(n, \alpha, Y^H\right)
\]

Any feasible answer for the above LP problem must satisfy both constraints (1) and (2). Therefore we have to have \( L \leq \min \left\{ \hat{L} \left(n, \alpha, \delta, Y^H, Y^L\right), \check{L} \left(n, \alpha, Y^H\right) \right\} \). It is either the case that \( \hat{L} \left(n, \alpha, \delta, Y^H, Y^L\right) < \check{L} \left(n, \alpha, Y^H\right) \) or \( \check{L} \left(n, \alpha, Y^H\right) \leq \hat{L} \left(n, \alpha, \delta, Y^H, Y^L\right) \):

- \( \hat{L} \left(n, \alpha, \delta, Y^H, Y^L\right) < \check{L} \left(n, \alpha, Y^H\right) \) iff

\[
\frac{\delta \mathbb{E}(Y) \left[1 - (1 - \alpha)^n\right]}{n - \delta (n - 1) [1 - (1 - \alpha)^n]} < \frac{[1 - (1 - \alpha)^n] Y^H}{n}
\]

\( \Leftrightarrow \)

\[
\frac{\delta \mathbb{E}(Y)}{n - \delta (n - 1) [1 - (1 - \alpha)^n]} < \frac{Y^H}{n}
\]
\[ \Leftrightarrow n \delta E(Y) - Y^H [n - \delta (n - 1) [1 - (1 - \alpha)^n]] < 0 \]
\[ \Leftrightarrow \delta \left[ nE(Y) + Y^H (n - 1) \right] - Y^H [n + [1 - (1 - \alpha)^n]] < 0 \]
\[ \Leftrightarrow \delta < \frac{Y^H [n + [1 - (1 - \alpha)^n]]}{nE(Y) + Y^H (n - 1)} \]
\[ \Leftrightarrow \delta < \frac{Y^H}{E(Y) + \frac{n-1}{n} [1 - (1 - \alpha)^n] Y^H} \equiv \tilde{\delta} (n, \alpha, Y^H, Y^L) \]

So for \(0 < \delta < \tilde{\delta} (n, \alpha, Y^H, Y^L)\), JL lending is feasible for any \(L < \hat{L} (n, \alpha, \delta, Y^H, Y^L)\).

- \(\hat{L} (n, \alpha, Y^H) \leq \hat{L} (n, \alpha, \delta, Y^H, Y^L)\) iff

\[ \delta \geq \frac{Y^H}{E(Y) + \frac{n-1}{n} [1 - (1 - \alpha)^n] Y^H} \equiv \tilde{\delta} (n, \alpha, Y^H, Y^L) \]

So for \(\tilde{\delta} (n, \alpha, Y^H, Y^L) \leq \delta < 1\), JL lending is feasible for any \(L \leq \tilde{L} (n, \alpha, Y^H)\).

QED

**Proof of Proposition 3:** According to Proposition 2, JL is feasible iff

\[ L \leq f (n, \alpha, \delta, Y^H, Y^L) \]

, where using Lemma 1

\[ f (n, \alpha, \delta, Y^H, Y^L) = \begin{cases} \hat{L} (n, \alpha, \delta, Y^H, Y^L) & 0 < \delta < \tilde{\delta} (n, \alpha) \\ \tilde{L} (n, \alpha, \delta, Y^H, Y^L) & \tilde{\delta} (n, \alpha) < \delta < \hat{\delta} (n, \alpha, Y^H, Y^L) \\ \tilde{L} (n, \alpha, Y^H) & \hat{\delta} (n, \alpha, Y^H, Y^L) < \delta < 1 \end{cases} \]

For given \(\alpha\) and \(\delta\), if there are some \(n\) such that

\[ \hat{\delta} (n, \alpha) < \delta < \tilde{\delta} (n, \alpha, Y^H, Y^L) \]

, then feasibility function will be strictly increasing in \(n\)(Lemma 1). Otherwise it will be strictly decreasing in \(n\).

1. First we look separately at two cases that \(\alpha\) is very small and \(\alpha\) is very large:
(a) If \( \alpha \to 0 \), define \( \rho = (1 - \alpha) \), then \( \rho \to 1 \), and

\[
\lim_{\alpha \to 0} \hat{\delta}(n, \alpha) = \lim_{\alpha \to 0} \frac{(1 - \alpha)^n \ln (1 - \alpha)^n + [1 - (1 - \alpha)^n]}{[1 - (1 - \alpha)^n]^2} =
\]

\[
\lim_{\rho \to 1} \rho^n \ln \rho^n + 1 - \rho^n \overset{\text{Hopital}}{=} \lim_{\rho \to 1} \frac{n \rho^{n-1} \ln \rho^n}{n \rho^{n-1} - 2n \rho^{n-1} \rho^n} =
\]

\[
\lim_{\rho \to 1} \frac{\ln \rho^n}{2 [1 - \rho^n]} \overset{\text{Hopital}}{=} \lim_{\rho \to 1} \frac{\rho^n}{2n \rho^{n-1}} = \lim_{\rho \to 1} \frac{1}{2 \rho^n} = \frac{1}{2}
\]

and

\[
\lim_{\alpha \to 0} \hat{\delta}(n, \alpha, Y^H, Y^L) = \lim_{\alpha \to 0} \frac{Y^H}{\alpha - 0} (Y) + \frac{n - 1}{n} [1 - (1 - \alpha)^n] Y^H = \frac{Y^H}{Y^L}
\]

So for \( \alpha \) close to zero, \( \hat{\delta}(n, \alpha) < \tilde{\delta}(n, \alpha, Y^H, Y^L) \) regardless of the magnitude of \( n \).

(b) If \( \alpha \to 1 \),

\[
\lim_{\alpha \to 1} \hat{\delta}(n, \alpha) = \lim_{\alpha \to 1} \frac{\ln (1 - \alpha)^n}{1 - (1 - \alpha)^n} + 1 \overset{\text{Hopital}}{=} \lim_{\alpha \to 1} \frac{-n (1 - \alpha)^n - 1}{n (1 - \alpha)^n} + 1 = 1
\]

and

\[
\lim_{\alpha \to 1} \hat{\delta}(n, \alpha, Y^H, Y^L) = \lim_{\alpha \to 1} \frac{Y^H}{\alpha - 1} (Y) + \frac{n - 1}{n} [1 - (1 - \alpha)^n] Y^H = \frac{n}{2n - 1}
\]

For any \( n > 1 \), \( \frac{n}{2n - 1} \) is smaller than 1. Therefore, for \( \alpha \) close to one, always \( \hat{\delta}(n, \alpha) > \tilde{\delta}(n, \alpha, Y^H, Y^L) \), for any \( n > 1 \).

From our discussions in parts (a), (b) and the fact that \( \hat{\delta}(n, \alpha) \) and \( \tilde{\delta}(n, \alpha, Y^H, Y^L) \) are monotonic (Lemma 2), we come to the conclusion that \( \hat{\delta}(n, \alpha) \) and \( \tilde{\delta}(n, \alpha, Y^H, Y^L) \) coincide only once at some critical \( \bar{\alpha} \neq 0 \). So for any given \( n \), if \( \alpha \in (0, \bar{\alpha}) \), then

\[
\hat{\delta}(n, \alpha) < \tilde{\delta}(n, \alpha, Y^H, Y^L)
\]
Therefore if $\alpha \in (0, \bar{\alpha})$ and the given $\delta$ is such that

$$\hat{\delta}(n, \alpha) < \delta < \check{\delta}(n, \alpha, Y^H, Y^L)$$

, then feasibility function will be increasing in $n$, and it will be maximized at

$$N_{\alpha, \delta} = \max \left\{ n \mid \hat{\delta}(n, \alpha) < \delta < \check{\delta}(n, \alpha, Y^H, Y^L) \right\}$$

In order to find the maximum $n$ when $\alpha$ and $\delta$ are given, our algorithm would be as follows: start with $n = 2$ and increase $n$ one by one until one of the $\hat{\delta}(n, \alpha)$ or $\check{\delta}(n, \alpha, Y^H, Y^L)$ equals $\delta$ so that $n$ can not be increased further. If $\hat{\delta}(n, \alpha) = \delta$, then $N_{\alpha, \delta} = \lfloor \hat{\delta}^{-1}(\alpha, \delta) \rfloor$ (note that we are only interested in $n$ that is a natural number) and if $\check{\delta}(n, \alpha, Y^H, Y^L) = \delta$, then $N_{\alpha, \delta} = \lfloor \check{\delta}^{-1}(\alpha, \delta, Y^H, Y^L) \rfloor$ (see Figure 1 for intuition). Thus $N_{\alpha, \delta}$ can be rewritten as

$$N_{\alpha, \delta} = \min \left\{ \lfloor \hat{\delta}^{-1}(\alpha, \delta) \rfloor, \lfloor \check{\delta}^{-1}(\alpha, \delta, Y^H, Y^L) \rfloor \right\}$$

2. A direct result from the discussion in part 2 is that for $\alpha > \bar{\alpha}$, feasibility will be decreasing in $n$ and so maximum feasibility happens at $n = 2$.

3. What is maximum $\alpha$ for which

$$\hat{\delta}(n, \alpha) < \delta < \check{\delta}(n, \alpha, Y^H, Y^L)$$

A necessary condition to satisfy is

$$\hat{\delta}(n, \alpha) < \check{\delta}(n, \alpha, Y^H, Y^L)$$

or

$$\frac{(1 - \alpha)^n \ln (1 - \alpha)^n + [1 - (1 - \alpha)^n]}{[1 - (1 - \alpha)^n]^2} < \frac{Y^H}{\alpha Y^H + (1 - \alpha) Y^L + \frac{n-1}{n} [1 - (1 - \alpha)^n] Y^H}$$

For simplicity of calculations, assume $Y^L = 0$ and $0 < \alpha < 1$, then the above inequality turns to

$$\frac{(1 - \alpha)^n \ln (1 - \alpha)^n + [1 - (1 - \alpha)^n]}{[1 - (1 - \alpha)^n]^2} < \frac{n}{n \alpha + (n - 1) [1 - (1 - \alpha)^n]}$$

- For $n = 2$, the inequality holds for $\alpha < 0.718$. Therefore for $\alpha < 0.718$, the
inequality holds at least for \( n = 2 \).

- For \( n = 3 \), the inequality holds for \( \alpha < 0.568 \). Therefore for \( \alpha < 0.568 \), the inequality holds at least for \( n = 2, 3 \).
- For \( n = 4 \), the inequality holds for \( \alpha < 0.477 \). Therefore for \( \alpha < 0.477 \), the inequality holds at least for \( n = 2, 3, 4 \).
- For \( n = 5 \), the inequality holds for \( \alpha < 0.415 \). Therefore for \( \alpha < 0.415 \), the inequality holds at least for \( n = 2, 3, 4, 5 \).
- For \( n = 10 \), the inequality holds for \( \alpha < 0.269 \). Therefore for \( \alpha < 0.269 \), the inequality holds at least for \( n = 2, 3, 4, 5, \ldots, 10 \).
- For \( n = 50 \), the inequality holds for \( \alpha < 0.082 \). Therefore for \( \alpha < 0.082 \), the inequality holds at least for \( n = 2, 3, 4, 5, \ldots, 50 \).

The largest \( \alpha \) for which the inequality holds for more than one \( n \) is \( \alpha < 0.568 \). So feasibility can be increasing in \( n \) if \( \alpha < 0.568 \), and the given \( \delta \) is such that

\[
\hat{\delta}(n, \alpha) < \delta < \tilde{\delta}(n, \alpha, Y^H, Y^L)
\]

Note that when \( Y^L \neq 0 \), then \( \alpha \) becomes a slightly smaller. For example when \( Y^L \) is as large as \( \frac{Y^H}{2} \), then \( \bar{\alpha} \) is approximately 0.458.

4. If \( n \) grow very large, feasibility of JL will be decreasing in \( n \):

\[
\hat{\delta}(n \to \infty, \alpha) = \lim_{\rho \to 0} \rho \ln \rho + \frac{[1 - \rho]}{[1 - \rho]^2} = \lim_{\rho \to 0} \rho \ln \rho = \lim_{\rho \to 0} \frac{\ln \rho}{\frac{1}{\rho}} + \frac{1}{\rho} = 1
\]

\[
\tilde{\delta}(n \to \infty, \alpha, Y^H, Y^L) = \lim_{n \to \infty} \frac{Y^H}{\mathbb{E}(Y) + \frac{n-1}{n} [1 - (1 - \alpha)^n] Y^H} = \frac{Y^H}{\mathbb{E}(Y) + Y^H}
\]

\[
\frac{Y^H}{\mathbb{E}(Y) + Y^H}
\]

is changing from \( \frac{1}{2} \) to 1. So we will always have

\[
\hat{\delta}(n \to \infty, \alpha) > \tilde{\delta}(n \to \infty, \alpha, Y^H, Y^L)
\]

, except for the case that \( \alpha \to 0 \) and \( Y^L = 0 \) simultaneously.

QED
Proof of Proposition 4:

1. When both IL and JL are feasible, then each borrower is supposed to repay \( R = \frac{L}{1 - (1 - \alpha)^n} \) under JL and \( R = \frac{L}{\alpha} \) under IL. It is not difficult to see that he pays less under JL, since \( 1 - (1 - \alpha)^n > \alpha \) for any \( n \geq 2 \).

Each borrower’s expected lifetime utility under JL is \( V_{RL} = \frac{\mathbb{E}(Y) - L}{1 - \delta [1 - (1 - \alpha)^n]} \), and under IL is \( V_{IL} = \frac{\mathbb{E}(Y) - L}{1 - \alpha \delta} \). Comparing these two lifetime utilities, JL is superior to IL iff

\[
1 - \delta [1 - (1 - \alpha)^n] < 1 - \alpha \delta
\]

, that after simplification turns to

\[
(1 - \alpha)^{n-1} < 1
\]

, and is correct for any \( n > 1 \).

2. For which one feasibility is easier to satisfy, JL or IL?

(a) If \( \delta < \tilde{\delta} (n, \alpha, Y^H, Y^L) \), then JL is feasible for all

\[
L \leq \hat{L} = \frac{\delta \mathbb{E}(Y) [1 - (1 - \alpha)^n]}{n - \delta (n - 1) [1 - (1 - \alpha)^n]}
\]

that for \( \alpha \neq 0 \) can be written as

\[
L \leq \hat{L} = \alpha \delta \mathbb{E}(Y) \times \frac{[1 - (1 - \alpha)^n]}{\alpha [n - \delta (n - 1) [1 - (1 - \alpha)^n]]}
\]

, and IL is feasible iff

\[
L \leq \alpha \delta \mathbb{E}(Y)
\]

So JL can offer larger range of loans to each member than IL if

\[
\frac{[1 - (1 - \alpha)^n]}{\alpha [n - \delta (n - 1) [1 - (1 - \alpha)^n]]} > 1
\]

or if

\[
\delta > \frac{\alpha n - [1 - (1 - \alpha)^n]}{\alpha (n - 1) [1 - (1 - \alpha)^n]}
\]

Otherwise, IL can offer larger range of loan to each member than JL. Now,
since we already have an upper bound for \( \delta \), it must be true that
\[
\frac{\alpha n - [1 - (1 - \alpha)^n]}{\alpha (n - 1) [1 - (1 - \alpha)^n]} < \delta < \tilde{\delta} (n, \alpha, Y^H, Y^L)
\]

(b) If \( \delta > \tilde{\delta} (n, \alpha, Y^H, Y^L) \), then JL is feasible for all
\[
L \leq \tilde{L}_{GL} = \frac{Y^H [1 - (1 - \alpha)^n]}{n}
\]
, and again IL is feasible if
\[
L \leq \alpha \delta \mathbb{E} (Y)
\]
So JL can offer larger range of loan to each member than IL when
\[
\alpha \delta \mathbb{E} (Y) < \frac{Y^H [1 - (1 - \alpha)^n]}{n}
\]
or
\[
\delta < \frac{Y^H [1 - (1 - \alpha)^n]}{n \alpha \mathbb{E} (Y)}
\]
Now, since we already have a lower bound for \( \delta \), it must be true that
\[
\tilde{\delta} (n, \alpha, Y^H, Y^L) < \delta < \frac{Y^H [1 - (1 - \alpha)^n]}{n \alpha \mathbb{E} (Y)}
\]
Comparing the results of part a and part b, JL is feasible for larger amount of loans than IL when
\[
\frac{\alpha n - [1 - (1 - \alpha)^n]}{\alpha (n - 1) [1 - (1 - \alpha)^n]} < \delta < \frac{Y^H [1 - (1 - \alpha)^n]}{n \alpha \mathbb{E} (Y)}
\]
We call this condition the relative feasibility condition from now on.

What are the circumstances for all parameters so that the relative feasibility condition holds? A necessary condition to satisfy will be
\[
\frac{\alpha n - [1 - (1 - \alpha)^n]}{\alpha (n - 1) [1 - (1 - \alpha)^n]} < \frac{Y^H [1 - (1 - \alpha)^n]}{n \alpha \mathbb{E} (Y)}
\]
(a) If $\alpha$ is very small, the inequality holds:

$$\lim_{\alpha \to 0} \frac{\alpha n - [1 - (1 - \alpha)^n]}{\alpha (n - 1)[1 - (1 - \alpha)^n]} = \frac{1}{2}$$

(we applied Hospital's rule for limits twice)

$$\lim_{\alpha \to 0} \frac{Y^H[1 - (1 - \alpha)^n]}{\alpha n E(Y)} = \lim_{\alpha \to 0} \frac{Y^H (1 - \alpha)^{n-1}}{E(Y)} = \frac{Y^H}{Y^L}$$

(b) If $\alpha$ is very large, the inequality does not hold:

$$\lim_{\alpha \to 1} \frac{\alpha n - [1 - (1 - \alpha)^n]}{\alpha (n - 1)[1 - (1 - \alpha)^n]} = 1$$

$$\lim_{\alpha \to 1} \frac{Y^H[1 - (1 - \alpha)^n]}{\alpha n E(Y)} = \frac{1}{n}$$

Using parts (a) and (b) and the fact that both right-hand side and left-hand side of the inequality are strictly monotonic in $\alpha \in (0, 1)$, we should conclude that there exists a critical $\alpha$ such that for any $\alpha < \alpha$, the inequality holds always.

3. A direct result of our discussion in part 2 in that if $\alpha > \alpha$, then the relative feasibility function never holds and so IL performs better than JL in terms of larger feasible maximum loans.

4. What is the maximum $\alpha$ for which

$$\frac{\alpha n - [1 - (1 - \alpha)^n]}{\alpha (n - 1)[1 - (1 - \alpha)^n]} < \frac{Y^H[1 - (1 - \alpha)^n]}{\alpha n E(Y)}$$

for simplicity of calculations assume $Y^L = 0$

$$\frac{\alpha n - [1 - (1 - \alpha)^n]}{\alpha (n - 1)[1 - (1 - \alpha)^n]} < \frac{[1 - (1 - \alpha)^n]}{\alpha n^2}$$

- If $n = 2$, the inequality holds for $\alpha < 0.764$. Thus for $\alpha < 0.764$, JL does better than IL at least groups of $n = 2$.
- If $n = 3$, the inequality holds for $\alpha < 0.634$. Thus for $\alpha < 0.634$, JL does better than IL at least groups of $n = 2, 3$. 

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• If \( n = 4 \), the inequality holds for \( \alpha < 0.552 \). Thus for \( \alpha < 0.552 \), JL does better than IL at least groups of \( n = 2, 3, 4 \).

• If \( n = 5 \), the inequality holds for \( \alpha < 0.495 \). Thus for \( \alpha < 0.495 \), JL does better than IL at least groups of \( n = 2, 3, 4, 5 \).

• If \( n = 10 \), the inequality holds for \( \alpha < 0.349 \). Thus for \( \alpha < 0.349 \), JL does better than IL at least groups of \( n = 2, 3, 4, 5, \ldots, 10 \).

• If \( n = 50 \), the inequality holds for \( \alpha < 0.150 \). Thus for \( \alpha < 0.150 \), JL does better than IL at least groups of \( n = 2, 3, 4, 5, \ldots, 50 \).

Therefore for \( \alpha < 0.764 \), JL can outperform IL if the given \( \delta \) is such that

\[
\frac{\alpha n - [1 - (1 - \alpha)^n]}{\alpha (n - 1) [1 - (1 - \alpha)^n]} < \delta < \frac{Y^H[1 - (1 - \alpha)^n]}{n\alpha E(Y)}
\]

Note that if \( Y^L \neq 0 \), then \( \alpha \) must be a slightly smaller. As an instance if \( Y^L = \frac{Y^H}{2} \), then we should have \( \alpha < 0.697 \).

5. For very large \( n \), the relative feasibility condition does not hold:

\[
\lim_{n \to \infty} \frac{(n - 1) [1 - (1 - \alpha)^n]^2}{\alpha n^2 - n [1 - (1 - \alpha)^n]} = \lim_{n \to \infty} \frac{[1 - (1 - \alpha)^n]^2}{\alpha n - [1 - (1 - \alpha)^n]^2} = 0
\]

\[
\lim_{n \to \infty} \frac{Y^H[1 - (1 - \alpha)^n]}{n\alpha E(Y)} = 0
\]

QED

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