Abstract

Shubik’s (all-pay) dollar auction is a simple yet powerful auction model that aims to shed light on the motives and dynamics of conflict escalation. Common intuition and experimental results suggest that the dollar auction is a trap, inducing conflict by its very design. However, O’Neill [1986] proved the surprising result that, contrary to the experimental results and the intuition, the dollar auction has an immediate solution in pure strategies, i.e. theoretically it should not lead to conflict escalation. In this paper we reconsider these results following recent literature on spiteful bidders. That is, we ask the question whether the escalation in the dollar auction can be induced by meanness. Our results confirm this conjecture in various scenarios.

1 Introduction

On the surface, many social situations appear to be a trap, where it is a bad idea to move forward, but also bad to retract from the situation and lose already-invested resources. Such dilemmas are often faced by lobbyists who battle with each other in a costly and seemingly endless process of acquiring a public contract [Fang, 2002], by oligopolistic companies pressured to invest in R&D only because the competitors have just done so [Dasgupta, 1986], or by many ready-to-marry people who feel trapped in long-lasting relationships that somehow do not progress towards institutionalisation [Rhoades et al., 2010].

Shubik [Shubik, 1971] proposed a simple yet powerful model to study such situations. In his so-called dollar auction, two bidders $i$ and $j$ compete for a dollar bill. Similarly to an English auction, the highest bidder wins the prize, but, unlike in the English auction, both the winner and the loser have to pay their bids to the auctioneer. For instance, if the auction stops with the bids of $.35 and $.40, then the auctioneer receives $.35 + $.40, and the dollar is awarded to the highest bidder. Here, the bidders are not allowed to communicate, any bid must raise the bidding price by at least 5 cents, and there is no reserve price.
One might argue that it is best not to participate in the above all-pay auction. However, this is not always possible. Furthermore, the possibility that a player may choose not to bid creates a clear incentive for the other player to bid and get the prize. Matter-of-factly, this reasoning is the centrepiece of the entire dollar auction mechanism that ultimately pushes players towards conflict escalation. To illustrate this point, let us assume that the auction has started with player $i$ bidding $\$.05$, and player $j$ raising the price to $\$.10$. Player $i$ faces the following dilemma: withdraw from the auction and lose $\$.05$ with certainty, or increase the bid to $\$.15$ with the hope of gaining $\$.85$. Since the same reasoning holds at any stage during the auction, the bidding may continue well past the bill of $\$.00$ to be won. While past this point the bidders can only seek to minimize losses, they are still incentivized to increase their bids rather than drop out and lose everything.

The above dollar auction game has become an influential abstraction of conflict escalation processes. It makes for a great class-play for management students [James, 2009] but, more importantly, it offers insight into the dynamics of such processes as international conflicts, arms races, investment decisions or human relations, just to name a few. Any such situation may escalate to irrational levels despite the fact that, locally, every single participant makes a rational decision. Similar patterns of behaviour are observed in “clinical” experiments with the dollar auction—more often than not, a dollar bill is sold for considerably more than a dollar [Shubik, 1971; Kagel and Levin, 2008].

One of the key reasons behind this “paradox of escalation” [Shubik, 1971] is that a rational strategy to play this game is far from obvious. It is difficult to make an optimal choice between when “to quit” and when “to bid”, both when bidding for a dollar bill, or when facing similar real-life situations.

In his beautiful paper, O’Neill [O’Neill, 1986] offered a surprising solution to the dollar auction—he proved that, assuming finite budgets of players, in all equilibria in pure strategies, only one player bids and wins the prize. The exact amount of such a “golden” bid is a non-trivial function of the stake, the budgets, and the minimum allowable increment. In our example, if players $i$ and $j$ have equal budgets of $\$.50$ each, the first player, who has the first chance to move, should bid $\$.60$. If his opponent is rational, he should leave the game with no prize and with no losses.

Does O’Neill’s result mean that the conflict in the dollar auction does not escalate after all? O’Neill’s results were revisited by Leininger [Leininger, 1989], who showed that the escalation can be justified in this game because there exist equilibria with escalation in mixed strategies.1 Later on, Demange [Demange, 1992] proved that, if there is some uncertainty about the strength of the players, then the only stable equilibrium may entail escalation.

In this paper, we reconsider O’Neill’s results in pure equilibria from a different perspective. Following recent literature on spiteful bidders [Brandt et al., 2005; Sharma and Sandholm, 2010; Tang and Sandholm, 2012], we ask the question whether the escalation in the dollar auction may actually be caused by the meanness of some participants. Do some of us put others in an inauspicious position simply because of spite,

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1We note that the result very similar to Leininger [Leininger, 1989] were obtained more recently by Dekel et al. [Dekel et al., 2007].
rather than greed? Do we allow ourselves to be dragged along simply because we do not expect a spiteful opponent?²

To address these questions, we study the dollar auction in which a spiteful player challenges a non-spiteful one, and the non-spiteful player does not suspect the meanness of his opponent, meaning that he follows the strategy proposed by O’Neill [O’Neill, 1986]. We consider both equal and unequal budgets. Under these assumptions, we confirm our hypothesis in a number of scenarios. Some of the most important findings are as follows:

- assuming equal budgets, a strongly spiteful player is almost always able to escalate the auction and force the non-spiteful opponent to spend most of his budget. Still, it is the spiteful bidder who gets the stake at the end!
- an extreme type of the spiteful player is a malicious player who cares only about maximizing the loss of the opponent, irrespective of his own costs. In this case, if the malicious player has the bigger budget, then he is always able to force the non-spiteful player to pay almost the entire value of the prize but not get it.
- while such an advantage of the malicious player with the bigger budget is not unexpected, we obtain a surprising result for the case where the malicious player i is the one with the smaller budget b_i. In this case, he can force the non-spiteful opponent j to spend more than b_i. In other words, a weaker malicious player escalates the conflict more than a stronger malicious one!

2 Preliminaries

In this section, we formally introduce the notation and rules of the dollar auction and the concept of spitefulness.

The Dollar Auction: The auction setting proposed by Shubik [1971] consists of two players, N = {1, 2}. We will usually refer to them as i and j (if their order is not important). The players can declare bids x_i and x_j, respectively, that are multiples of a unit u ∈ N. The winner of the auction receives stake s ∈ N. Without loss generality, unless stated otherwise, we assume that u = 1. Some arbitrary mechanism chooses the player who places the first bid (from this moment on we call him player 1). Player 1 can either place a bid or pass the turn to player 2. If player 2 does not want to bid either, the auction ends. Otherwise, the turn moves back to player 1 and so on and so forth. At any moment, if a player quits, he receives nothing, while the other receives stake s. After the auction, both players must pay their bids, x_1 and x_2, to the auctioneer, regardless of the identity of the winner. Players cannot make deals, form a coalition or make threats.

O’Neill [1986] extended the above model by explicitly assuming that each player has a finite budget b_1, b_2 ∈ N, where b_1 > s > u and b_2 > s > u. The budgets naturally constrain the players, that is, if x_i ≥ b_j and player j cannot place a bid to

2 A spiteful bidder, contrary to the common assumption of self-interest, maximizes a convex combination of his own profit and the opponent’s loss. See Section 2 for more details.
overpass player \( i \), then the latter player wins. Figure 4 presents an example of the dollar auction, where \( b = 7 \). Each node represents a pair of bids \((x_1, x_2)\). In the black nodes, player 1 chooses his bid and each solid arrow corresponds to one of his possible decisions. Analogically, the white nodes are the decision nodes of player 2. The dotted arrows show his possible bids. The nodes without an outgoing arrow represent situations in which the corresponding player has spent his entire budget. The auction ends either when one of the players chooses not to make his move, or when a node is reached that has no outgoing arrows.

**Spitefulness:** Following, e.g. Brandt et al. [2005], we assume that a spiteful agent is interested in increasing his own profit while at the same time decreasing the profit of his opponent. More formally, the utility of player \( i \) is:

\[
    u_i = (1 - \alpha_i)p_i - \alpha_i p_j,
\]

where \( p_i \) and \( p_j \) are the profits of the respective players, and \( \alpha_i \in [0, 1] \) is the spite coefficient of player \( i \). This coefficient indicates how important to player \( i \) is the loss of the opponent. When \( \alpha_i = 0 \), player \( i \) simply maximizes his own profit. We call such a player a non-spiteful player. When \( 0 < \alpha_i < 1 \), we call \( i \) a spiteful player. Finally, when \( \alpha_i = 1 \), player \( i \) is not interested in his own profit at all. We call this special type of a spiteful player a malicious player.

### 3 Auction Settings

We studied various ways in which spitefulness can be introduced to the dollar auction. In particular, we considered:

(a) auctions in which one player is non-spiteful (\( \alpha = 0 \)) and the other one is spiteful/malicious (\( \alpha \in (0, 1] \)).

(b) auctions in which both players are spiteful/malicious;

Moreover, we considered two alternative assumptions about knowledge/rationality of the players. First, for setting (a) we assumed that a non-spiteful player does not suspect that his opponent is spiteful and remains so while the auction progresses. After that, for both settings (a) and (b) we considered the auctions in which both players are aware of each others’ spite coefficients. Finally, we considered all cases assuming equal and unequal budgets.

Due to space constraints we focus below on setting (a), results of which we found most interesting. However, at the end we briefly summarise the results for setting (b).

### 4 An Auction with Equal Budgets

In this section, we analyze the dollar auction between players with equal budgets. A spiteful player has a spite coefficient ranging from 1, if he is malicious, to (almost) 0. On the other hand, a non-spiteful player has a spite coefficient of 0.
Before proceeding with our analysis, let us first consider how the utility of a spiteful player \( i \) depends on \( \alpha_i \). Let us define the profit of player \( i \) as the difference between his initial budget \( b_i \) and his balance after the auction. More formally, let us denote by \( (x_1, x_2) \) the final bids of the players, where \( x_1 + x_2 > 0 \). Then, assuming that \( x_i > x_j \), the profit of the winner of the auction is \( p_i = s - x_i \), while the “profit” of the loser is \( p_j = -x_j \). Based on this, Equation (1) can be written differently as follows:

\[
\begin{align*}
\alpha_i x_j + (1 - \alpha_i)(s - x_i) & \quad \text{if } x_i > x_j, \\
\alpha_i (x_j - s) - (1 - \alpha_i)x_i & \quad \text{if } x_i < x_j.
\end{align*}
\]

### 4.1 A Malicious Player (\( \alpha_j = 1 \))

If \( \alpha_i = 0 \) and \( \alpha_j = 1 \), then player \( i \) (whom we assume to be non-spiteful and does not suspect the spitefulness of his opponent), is challenged by player \( j \) who is actually malicious, meaning that the goal of \( j \) is to maximize the loss of \( j \), no matter what the cost. In the theorem below, we show that the strategy of the malicious player is to lure the non-spiteful player to continue bidding as long as possible. Still, it is the malicious bidder who gets the stake at the end.

**Theorem 1.** Let \( i \) be a non-spiteful player (\( \alpha_i = 0 \)) who follows the strategy by O’Neill [1986], and let player \( j \) be malicious (\( \alpha_j = 1 \)). The optimal strategy of \( j \)

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3The assumption that \( x_1 + x_2 > 0 \) guarantees that the bidding has actually started, i.e., at least one bid has been made.
is to bid:

\[ x_j = \begin{cases} 
  x_i + 1 & \text{if } x_i < b - (s - 1), \\
  b & \text{otherwise.} 
\end{cases} \]

*Proof.* We begin the proof by presenting the optimal strategy of a non-spiteful player who does not suspect spitefulness from the other player. This strategy was derived by O’Neill [1986]. To this end, consider the graph corresponding to the dollar auction game (an example was illustrated earlier in Figure 4). Recall that every node in this graph represents a pair of bids. We call node \((x_1, x_2)\) a winning node for player \(i\) if, by starting the game from this node, player \(i\) is guaranteed to eventually win stake \(s\). Importantly, while the sum of all bids of player \(i\) may exceed \(s\), we note that it is irrational for any one bid to be greater than \(s\) (it is always more beneficial to simply pass and quit the auction than to place a bid greater than \(s\) in the hope of winning \(s\)). Finally, recall that the game ends if one player exceeds his budget.

We will now analyze each node of the graph to determine whether it is a winning node for player \(i\). Figure 1 highlights the parts of the graph that consist of winning nodes. In more detail, the width of area \(C_i\) is one node, while the widths of areas \(B_i, D_i\) and \(E_i\) are \(s - 1\) nodes each. Any node in \(C_j\) is a winning one for player \(i\), since player \(j\) cannot bid more than \(b\) and his only choice is to pass. Moreover, area \(E_i\) consists of winning nodes for player \(i\), since by making a bid less than \(s\), he can move to area \(C_j\), win the stake \(s\) and finish the auction. Nodes in area \(D_i\) are losing nodes for player \(i\), since by making bids less than \(s\) he can only move to area \(E_j\), which is a winning area for player \(j\). Thus, \(i\)'s optimal move in area \(D_i\) is to pass. Finally, in area \(B_i\), the only valid bid are those greater than, or equal to \(s\), so it is also optimal for \(i\) to pass. The analysis for areas \(C_j, B_j, D_j, E_j\) is identical.
Figure 3: Winning and losing areas at the beginning of the auction.

We can now move on to the analysis of the areas of the graph that are gradually closer to \((0, 0)\), as illustrated in Figure 2. The areas that we have just analyzed now play the role of areas \(C_1\) and \(C_2\). One could, of course, bid higher, but moving into \(C_i\) always optimizes cost. All arguments previously stated for areas \(B_i\), \(D_i\), \(E_i\) still stand.

We can repeat this process until we reach an area whose dimensions are less than \(s \times s\), as illustrated in Figure 3. In this case, areas \(C_1\) and \(C_2\) are winning areas for players 2 and 1, respectively. Consequently, area \(E_i\) is winning for player \(i\), where his optimal choice is to make the bid \((b - 1) \text{mod}(s - 1) + 1\) and move to area \(C_j\). As a result, the non-spiteful player who makes the first move is always able to ensure his own victory in the auction against the other non-spiteful player. Figure 5 depicts the winning (shaded) areas for player 1 and the winning moves for each such area, as well as the losing (white) areas for player 1, where the optimal choice for him is to pass.

Let us now assume that player 1 is non-spiteful and player 2 is malicious. Recall that the utility of the malicious player, \(u^{mal}_2\), depends solely on the profit of the non-spiteful player. Naturally, the malicious player cannot make the first move and arrive at an area that is outside of the winning areas of the non-spiteful player. If he did so, then the non-spiteful player would simply pass, leaving both players with zero utility.

In what follows, we will show that the malicious player maximizes his utility if he plays as the second player. Hence, given the chance to move first, he will always pass the move to the opponent. Thus, throughout the remainder of this proof, player 2 will be the malicious one.

From the formula of the utility of a malicious player, the utility of player 2, \(u^{mal}_2\), is maximized when the game ends in area \(E_1\), or in the adjacent nodes of area \(C_1\) in Figure 1. However, player 1 will not stay in area \(E_1\), but will rather make a move.
towards \( C_2 \). On the other hand, nodes from \( C_1 \) can only be reached via a move from \( B_1 \) or \( D_1 \). As visible in Figure 5, \((b - (s - 1), b)\) is the only node that player 2 is able to reach. His utility at this node is \( b - s + 1 \).

The next best group of nodes in terms of utility \( u^\text{mal}_2 \) are those in \( C_2 \) from Figure 1; player 1 wins the stake, but is forced to use all of his budget. The utility of player 2 is therefore \( b - s \), which is lower than in the node \((b - (s - 1), b)\).

A bid by a malicious player 2 that is always one unit higher than the bid of a non-spiteful player 1 keeps the game in the nodes that player 1 considers to be “winning” nodes. When the bid of player 1 reaches \( b - (s - 1) \), player 2 should bid \( b \), thereby reaching the nodes in which he achieves his highest possible utility.

### 4.2 A spiteful player \((0 < \alpha_j < 1)\)

This subsection describes strategies for player \( j \) who is spiteful but not malicious (i.e., \( 0 < \alpha_j < 1 \)). His opponent is player \( i \), who is non-spiteful (i.e., \( \alpha_i = 0 \)) and does not suspect the spitefulness of his opponent, meaning that he is following the strategy proposed by O’Neill [1986]. We divide our analysis into three parts: weakly spiteful player with \( \alpha_j \in (0, \frac{1}{2}) \), player with \( \alpha_j = \frac{1}{2} \) and strongly spiteful player with \( \alpha_j \in (\frac{1}{2}, 1) \).

First, let us start with the following lemmas, which hold for all of the above three cases.

**Lemma 1.** Let 1 be a non-spiteful player \((\alpha_1 = 0)\) who follows the strategy by O’Neill [1986], and let 2 be a spiteful player \((0 < \alpha_j < 1)\). Furthermore, let \( x \)
be an optimal initial bid of player 1, and:

\[ X_0 = (0, x), \quad X_k = (x + (k - 1)(s - 1), x + k(s - 1)), \]
\[ Y_0 = (x, 0), \quad Y_k = (x + k(s - 1), x + (k - 1)(s - 1) + 1), \]

where \( k \in [1, n] \) and \( n = \left\lfloor \frac{b}{s + 1} \right\rfloor \).

The optimal end-node for player 2 (that maximizes his utility) is among nodes \( X_0, ..., X_n, Y_0, ..., Y_n \).

Proof. First let's define areas marked in Figure 6 as:

\[ A_i = \{(x_{i,1}, d) : d \in [x_{i,2}, b]\}, \]
\[ B_i = \{(y_{i,1}, d) : d \in [y_{i,2}, y_{i,1} - 1]\}, \]

where \( X_i = (x_{i,1}, x_{i,2}) \) and \( Y_i = (y_{i,1}, y_{i,2}) \). Theses are the only possible end nodes of the auction. In particular, the auction cannot end in any of the unmarked nodes under the diagonal, because they can only be reached by a move by player 1, but he will never make such a move (see the proof of Theorem 1). The auction can end in the marked areas under the diagonal, since player 2 can choose to pass at these nodes.

The auction cannot end in any of the areas marked \( E_1 \). This is because player 1 can make moves that he considers winning from these nodes. Conversely, player 1 passes in all other areas above the diagonal. Now, we observe that the only difference between
Lemma 2. Let 1 be a non-spiteful player following the strategy by O’Neill [1986], and let 2 be a spiteful player. The end-nodes $X_1, ..., X_n$ and $Y_0, ..., Y_n$ (defined in Lemma 1) can be reached by player 2 regardless of who starts the game, while $X_0$ can be reached by player 2 if he starts the game.

Proof. Player 2 can reach any node $Y_0, ..., Y_n$ by bidding always one unit more than the non-spiteful player 1. That way, in every move player 1 will make his optimal bid to enter $Y_i$. Now, player 2 can pass in node $Y_k$, or bid $x + s - 1$ in node $Y_{k-1} = (x, y)$ to reach $X_k$ (for $k > 0$). Moreover, if player 2 starts, he can reach $X_0$ by making the bid $(b-1) \mod (s-1) + 1$. As shown in Theorem 1, in every node $X_i$ it is optimal for a non-spiteful player to pass. 

Weakly spiteful player with $\alpha_j \in (0, \frac{1}{2})$: As it turns out, a player with a small (i.e., below $\frac{1}{2}$) spite coefficient behaves like a non-spiteful player if he starts the bidding. Otherwise, after his opponent’s move, he passes or forces his opponent to pass. Figure 7 presents an example of a utility map for a player with a small spite coefficient.

Theorem 2. Let $i$ be a non-spiteful player following the strategy by O’Neill [1986], and let $j$ be a weakly spiteful player (i.e., $\alpha_j \in (0, \frac{1}{2})$). If $j$ starts the bidding, it is optimal for him to bid like a non-spiteful player. If $i$ moves first and makes bid $x$, then the optimal strategy for player $j$ is to make the bid $x + s - 1$ when $x < \frac{\alpha_j s}{1 - \alpha_j} + 1$, and pass otherwise.
Figure 7: Utility maps for a weakly spiteful player \( j \) with \( \alpha_j = 0.25 \). Blue lines represent the same utility, red lines represent zero utility. The darker the color, the higher the utility.

Proof. Let \( x \) be the optimal initial move of a non-spiteful player, i.e., \( x = (b - 1) \text{mod}(s - 1) + 1 \). Note that \( x \in [1, s - 1] \). Based on Lemma 1, the optimal end of an auction for player \( j \) is one of the nodes from the set \( \{X_0, \ldots, X_n, Y_0, \ldots, Y_n\} \). Moreover, Lemma 2 states that all of these nodes can be reached by player \( j \). To simplify notation, assume that \( j = 2 \). Consider the utility function of player \( j \) in nodes \( X_k \) and \( X_{k+1} \) for \( k \geq 1 \). If \( X_k = (a, b) \), then \( X_{k+1} = (a + (s - 1), b + (s - 1)) \). Hence,

\[
u_2(X_{k+1}) - u_2(X_k) = (s - 1)(2\alpha_j - 1) < 0.
\]

Analogously, the utility function of player 2 is lower in \( Y_{k+1} \) than in \( Y_k \). Thus, the optimal solution that can be reached by player 2 is in one of the nodes: \( X_0 = (0, x), X_1 = (x, x + s - 1), Y_0 = (x, 0), Y_1 = (x + s - 1, x + 1) \). Now, we have:

\[
\begin{align*}
u_2(X_0) - u_2(X_1) &= (s - 1)(1 - \alpha_j) - \alpha_j x > 0, \\
u_2(X_0) - u_2(Y_0) &= s - x > 0, \\
u_2(X_1) - u_2(Y_1) &= 2 - \alpha_j > 0.
\end{align*}
\]

Thus, if player 2 starts the bidding, he prefers to bid \( x \) and end the game in \( X_0 \). By comparing \( X_1 \) and \( Y_0 \) we have:

\[
u_2(X_1) - u_2(Y_0) = x(\alpha_j - 1) + \alpha_j s + 1 - \alpha_j.
\]
Thus, if \( x < \frac{\alpha_i + s}{1 - \alpha_j} + 1 \), then it is better for player 2 to bid and end the game in the state \( X_1 \); otherwise, player 2 passes.

\[ \square \]

**Spiteful player with \( \alpha_j = \frac{1}{2} \):** A player with spite coefficient \( \alpha_j = \frac{1}{2} \) acts like a non-spiteful player if he starts the auction, but can act like a malicious player otherwise. Figure 8 presents an example of utility map for a player with spite coefficient \( \alpha_j = \frac{1}{2} \).

**Theorem 3.** Let \( i \) be a non-spiteful player (i.e., \( \alpha_i = 0 \)) who is following the strategy by O’Neill [1986], and let \( j \) be a spiteful player whose spite coefficient is \( \alpha_j = \frac{1}{2} \). If \( j \) starts the bidding, it is optimal for him to bid like a non-spiteful player. If \( i \) moves first and makes a bid, the optimal strategy for player \( j \) is to outbid him by \( 1 \) for some time, and to finally make a bid \( s - 1 \) higher at any point in the auction.

**Proof.** Let us denote by \( x \) the optimal initial move of a non-spiteful player, i.e., \( x = (b - 1) \mod (s - 1) + 1 \). Note, that \( x \in [1, s - 1] \).

Again, based on Lemmas 1 and 2, we can consider only nodes \( X_1, \ldots, X_n, Y_0, \ldots, Y_n \) as the optimal end points of the auction. To simplify notation, assume that \( j = 2 \). Since \( u_2(X_{k+1}) = u_2(X_k) \) and \( u_2(Y_{k+1}) = u_2(Y_k) \) for \( k \geq 1 \), the optimal reachable solution can be found by considering the nodes \( X_0 = (0, x), X_1 = (x, x + s - 1), Y_0 = (x, 0), Y_1 = (x + s - 1, x + 1) \).

Now, we have: \( u_2(X_0) = \frac{x}{2}, u_2(X_1) = \frac{1}{2}, u_2(Y_0) = \frac{x + s}{2}, u_2(Y_1) = -1 \).

Therefore, the following inequalities hold: \( u_2(X_0) \geq u_2(X_1), u_2(X_1) > u_2(Y_1) \).
Figure 9: Utility maps for a strongly spiteful player $j$ with $\alpha_j = 0.75$. Blue lines represent the same utility, red lines represents zero utility. The darker the color, the higher the utility.

$u_2(X_1) > u_2(Y_0)$. Thus, if player 2 starts, it is optimal for him to end the auction in the state $X_0$; otherwise, he should end in any of the nodes $X_k$.

Strongly spiteful player with $\alpha_j \in (1, 1)$: A player with a high (i.e., $> \frac{1}{2}$) spite coefficient can act like a non-spiteful player if he starts the auction for very specific values of $b$ and $s$. However, in most cases he acts like a malicious player: he forces a non-spiteful player to raise his bids and wins the stake in the end. Figure 9 presents an example of a utility map for a player with a high spite coefficient.

**Theorem 4.** Let $i$ be a non-spiteful player (i.e., $\alpha_i = 0$) who is following the strategy by O’Neill [1986], and let $j$ be a strongly spiteful player (i.e., $\alpha_j \in (\frac{1}{2}, 1)$). Furthermore, let $x$ be an optimal initial bid of a non-spiteful player. If $j$ starts the auction and $x < \frac{\alpha_j}{1-\alpha_j}(s-1) - \frac{2s-1}{\alpha_j}b$, then the optimal strategy for him is to make the bid $x$.

Otherwise, $j$ should follow the strategy of a malicious player.

**Proof.** Let us denote by $x$ the optimal initial move of a non-spiteful player, i.e., $x = (b-1)mod(s-1) + 1$. Note, that $x \in [1, s-1]$. Again, we will limit our analysis to nodes from the set $\{X_1, \ldots, X_n, Y_0, \ldots, Y_n\}$ based on Lemmas 1 and 2.

To simplify notation, assume that $j = 2$. Consider the utility function of player $j$ in nodes $X_k$ and $X_{k+1}$ for $k \geq 1$. If $X_k = (a, b)$, then $X_{k+1} = (a+(s-1), b+(s-1))$. 
Here, unlike the previous cases, we have:

\[ u_2(X_{k+1}) - u_2(X_k) = (s - 1)(2\alpha_j - 1) > 0. \]

An analogous analysis can be obtained for nodes \( Y_k \) and \( Y_{k+1} \). Thus, the optimal reachable solution is one of the following nodes: \( X_0 = (0, x), X_n = (b - (s-1), b), Y_0 = (x, 0), Y_n = (b, b - (s-2)). \) Now, we have:

\[
\begin{align*}
    u_2(X_0) - u_2(Y_0) &= s - x > 0, \\
    u_2(X_n) - u_2(Y_n) &= 2 - \alpha_j > 0, \\
    u_2(X_n) - u_2(Y_0) &= 2\alpha_j b - \alpha_j s - \alpha_j x + \alpha_j s - b > 0.
\end{align*}
\]

Thus, it is always better for player 2 to end the game in node \( X_0 \) or in node \( X_n \), rather than in node \( Y_0 \) or in node \( Y_n \). Moreover,

\[ u_2(X_n) - u_2(X_0) = (2\alpha_j - 1)b + (1 - \alpha_j)x + \alpha_j(1 - s). \]

Therefore, if \( x < \frac{\alpha_j}{1 - \alpha_j}(s - 1) - \frac{2\alpha_j - 1}{1 - \alpha_j}b \) holds, then it is better for player 1 to end the game with the first bid (if player 2 starts). Otherwise, \( X_n \) is the optimal solution. \( \square \)

### 4.3 Alternative Settings

In this section, we report results for the alternative auction settings in which both players know their \( \alpha_i \) and \( \alpha_j \). We omit proofs due to space constraints.

**Theorem 5.** Let \( i \) be a non-spiteful or weakly spiteful player (\( \alpha_i \leq \frac{1}{2} \)) and let \( j \) be a spiteful player with \( \alpha_j \in (0, 1) \). The optimal strategy for player \( i \) is to either follow the strategy proposed by O’Neill (if \( j \) has low spite coefficient) or pass (if \( j \) has high spite coefficient). The optimal strategy for player \( j \) when \( \alpha_j \leq \frac{1}{2} \) is the same as for player \( i \), and when \( \alpha_j > \frac{1}{2} \) is to bid \( u \) and continue with overbidding if player \( j \) bids.

**Theorem 6.** Let \( i \) be a strongly spiteful or malicious player (\( \alpha_i > \frac{1}{2} \)) and let \( j \) be a spiteful player with \( \alpha_j \in (0, 1) \). The optimal strategy for player \( i \) is to follow the strategy of malicious player described in Theorem 1. The optimal strategy for player \( j \) when \( \alpha_j > \frac{1}{2} \) is the same as for player \( i \), and when \( \alpha_j \leq \frac{1}{2} \) is to pass.

### 5 An Auction with Unequal Budgets

We now consider auctions where player budgets are unequal.

#### 5.1 Strategies of a non-spiteful player

As shown by O’Neill [1986], a non-spiteful player has a certain strategy for an auction with unequal budgets. If he starts the game with a higher budget, he makes a bid of just one unit and expects a non-spiteful player to pass. If he starts the game with a lower budget, he makes a bid of \( s - u \) and expects a non-spiteful player to pass.
5.2 Strategies of a malicious player

Now let us consider an auction with a malicious player $j$ and a non-spiteful player $i$ (who does not suspect the spitefulness of his opponent). As it turns out, a malicious player with a higher budget can force a non-spiteful opponent to pay $s - u$ and not get the stake. Surprisingly, if a malicious player starts an auction with a lower budget $b_j$, he can force a non-spiteful opponent to pay $b_j + u$ to win the stake.

**Theorem 7.** Let $i$ be non-spiteful, and $j$ be malicious. If $b_j > b_i$, and $j$ moves second, his optimal strategy is to bid $s$ as an answer to $i$’s bid of $s - u$. Player $i$ then passes. If $j$ starts the auction, it is better for him to let $i$ move first.

**Proof.** (Sketch) Player $i$ makes the bid $s - u$ because it is an optimal bid against a non-spiteful player. He can gain the utility of $u$, when player $j$ passes. If player $j$ bids $s$ or higher, then player $i$ passes, as he cannot win due to having a smaller budget. This yields his minimal utility of $-(s - u)$, and the maximal utility of a malicious player $j$.

Suppose $j$ starts the auction and bids at least $u$. Since $i$ has a smaller budget, he knows he cannot win. To cut his losses, he passes at the very beginning, giving malicious player $j$ zero utility. On the other hand, if player $j$ lets his opponent moves first, he gets positive utility, as shown above.

**Theorem 8.** Consider the dollar auction with a malicious player $j$, a non-spiteful player $i$ and budgets $b_j < b_i$. Player $j$ can force player $i$ to pay $b_j + u$ in order to acquire the stake.

**Proof.** (Sketch) Player $j$ prefers to continue the auction, as his utility rises as the auction progresses. The optimal strategy for player $j$ is to always outbid his opponent by $u$. Since player $i$ can always win, he can make a minimum raise of $u$ to minimize his lost. When the bid of player $i$ exceeds $b_j - (s - 2u)$, then player $j$ should make the bid $b_j$. Player $i$ can then get the stake $s$ by raising his bid by $s - u$ to $b_j + u$. Since it is never rational for player $i$ to bid more than $b_j + u$, this strategy of player $j$ grants him maximal utility.

5.3 Alternative Settings

In this section, we report again results for the alternative auction settings in which both players know their $\alpha_i$ and $\alpha_j$. We omit proofs due to space constraints.

**Theorem 9.** Let $i$ be a non-spiteful or weakly spiteful player ($\alpha_i \leq \frac{1}{2}$) and let $j$ be a spiteful player with $\alpha_j \in (0, 1]$. The optimal strategy for player $i$ is to either follow the strategy proposed by O’Neill (if $j$ has low spite coefficient) or pass (if $j$ has high spite coefficient). The optimal strategy for player $j$ when $\alpha_j \leq \frac{1}{2}$ is the same as for player $i$, and when $\alpha_j > \frac{1}{2}$ is to bid $u$ and then to continue overbidding if player $i$ bids.

**Theorem 10.** Let $i$ be a strongly spiteful or malicious player ($\alpha_i > \frac{1}{2}$) and let $j$ be a spiteful player with $\alpha_j \in (0, 1]$. When $\alpha_j \geq \frac{1}{2}$ and $b_i < b_j$, the optimal strategy for player $i$ is to bid $b_i$ (regardless of whether he starts). When $\alpha_j \geq \frac{1}{2}$ and $b_i > b_j$, the optimal strategy for $i$ is to start bidding with $b_j - 1$ and answer with $b_j + 1$ to
opponent’s bid of $b_j$. When $\alpha_j < \frac{1}{2}$ player $j$ will not enter the auction, so player $i$ should simply bid 1. The optimal strategy for player $j$ when $\alpha_j > \frac{1}{2}$ is the same as for player $i$, and when $\alpha_j \leq \frac{1}{2}$ is to pass.

6 Related work

In this section we comment on the bodies of literature related to the key characteristics of our auction setting: spitefulness, all-pay format, and the assumption of the finite budget.

Spitefulness in auctions: On top of some work in game theory [Baye et al., 1996], and experimental game theory [Bolle et al., 2013] in particular, the analysis of spitefulness in simple types of auctions can be found, among others, in the works by Morgan et al. [2003], Brandt and Weiβ [2002], Brandt et al. [2005], Babaioff et al. [2007], Sharma and Sandholm [2010], Tang and Sandholm [2012]. An interesting study of vindictive behaviours in auction-like settings (where rivals engage in aggressive retaliatory behaviors) can be found in [Bolle et al., 2013]. Similarly, vindictive bidding in keyword auctions was studied by Zhou and Lukose [1].

All pay auctions: While all-pay auctions are a relatively rare auction format, they have been extensively studied [DiPalantino and Vojnovic, 2009; Lewenberg et al., 2013] as they model various realistic settings in which the prize is awarded, often implicitly, on the basis of contestants’ efforts. These include lobbying [Fang, 2002], job-promotion competitions, political campaigns, R&D competitions, to mention a few [Lev et al., 2013]. For an overview of experimental research on all-pay auctions see [Dechenaux et al., 2012].

The final budget: While in many studies on auctions the existence of the budget can be neglected, this is not necessarily so in all-pay auctions. The budget constrains in auctions in all-pay auctions were studied by Che and Gale [1996].

7 Conclusions

Our results suggest that the escalation in the real-life experiments with the dollar auction could be related not only to the desire to win but also (at least to some extent) to human meanness. In various scenarios in which a non-spiteful bidder unwittingly bids against a spiteful one, the conflict escalates. Not only can the spiteful bidder force the non-spiteful opponent to spend most of the budget but he also often wins the prize. Surprisingly, a malicious player with a smaller budget is likely to plunge the opponent more than a malicious player with a bigger budget. Thus, a malicious player should not only hide his real preferences but also the real size of his budget. Intuitively, a weak, easy-to-overcome bait may seem more attractive than a stronger one.

For the future work, it would be interesting to study how the results of the dollar auction change if bidders are not spiteful but rather altruistic [Chen et al., 2011; Chen, 2011].
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References


