Explicit solutions for interval-valued cooperative games based on quadratic programming

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Abstract: The aim of this paper is to develop a quadratic programming method for solving interval-valued cooperative games. In this method, using the least square method and distance between intervals, we construct two quadratic programming models and obtain their explicit analytical solutions which are used to determine players’ interval-valued imputations and thereby the interval-valued solutions of interval-valued cooperative games are determined in sense of minimizing the loss function. The proposed method is compared with other methods to show the validity, applicability and advantages.

Keywords: Cooperative game; Least square method; Loss function; Mathematical programming; Lagrange multiplier method

1. Introduction

Game theory is engaged in competing and strategic interaction among players (or subjects) in management, economics, finance, business, environment and engineering. It has gradually developed and formed into two main branches: cooperative games and
noncooperative games [1-7].

Due to uncertainty and information imprecision in real situations, values (or characteristic functions) of coalitions of players are usually expressed with intervals rather than real numbers. A good example may be the bankruptcy problem [8-10]. Such a type of cooperative games are called interval-valued cooperative games. Interval-valued cooperative games are different from classical cooperative games since the coalitions’ values in the former are expressed with intervals while the coalitions’ values in the latter are expressed with real numbers. Interval-valued cooperative games may come from various fields of activities in which players could predict the values of coalitions with intervals rather than real numbers owing to the lack of knowledge, the asymmetry of information and the complexity of the game problems themselves [7, 11, 12]. The imputation (or allocation) of collective gains/costs is a central question for players (or individuals, organizations) contemplating cooperation under uncertainty. The interval-valued cooperative game may provide a new theoretical angle and suitable tool for solving this question.

Recently, interval-valued cooperative games have attracted attention of researchers. Han et al. [7] introduced the notions of interval-valued cores and Shapley-like values for interval-valued cooperative games according to Moore’s subtraction operator and the newly defined order relation between intervals. Alparslan Gök et al. [9] gave an axiomatic characterization of the interval-valued Shapley value on a special subclass of interval-valued cooperative games. Mallozzi et al. [10] introduced the concept of a core-like for cooperative games in which coalitions’ values are expressed with fuzzy intervals (i.e., fuzzy numbers) [11] and a balanced-like condition which is proven to be necessary but not sufficient to guarantee the nonempty of the core-like. Branzei et al. [12] overviewed and updated the results on interval-valued cooperative games, and discussed a variety of existing and potential applications of interval-valued cooperative games in economic management situations where
probability distribution is unknown \emph{a priori}. To analyse transfer pricing in a multidivisional firm, Leng and Parlar \cite{13} constructed a cooperative game which is convex and hereby determined the Shapley value-based transfer prices for the firm. Alparslan Gök et al. \cite{14} extended the classical two-person cooperative games to the two-person cooperative games with interval uncertainty and studied the core, balanceness, superadditivity and related topics. Branzei et al. \cite{15} generalized interval-valued core solutions of interval-valued cooperative games by discussing the set of undominated core solutions, which consists of the interval-valued nondominated core, the square interval-valued dominance core and the interval-valued dominance core. Miao et al. \cite{16} proposed a cooperative differential game model and found an optimal rate control of each player to get the total minimal cost with tradeoff between the network throughput and energy efficiency of the networks. Yu and Zhang \cite{17} introduced a new class of fuzzy cooperative games with fuzzy coalitions and fuzzy characteristic functions and hereby gave the explicit form of the Shapley value. Alparslan Gök et al. \cite{18} introduced some set-valued solution concepts of interval-valued cooperative games, which include the interval-valued dominance core and the interval-valued stable sets. However, as far as we know, there is no investigation on how to solve interval-valued cooperative games. In other words, there is no specific and effective method for determining imputations of players in interval-valued cooperative games. In this paper, using the least square method and the concepts of loss functions and distances between intervals, we focus on developing a simple and an effective method for solving interval-valued cooperative games. The method proposed in this paper is remarkably different from other methods in that the former can provide analytical formulae for determining the interval-valued imputations of all players.

The rest of this paper is organized as follows. In the next section, we briefly introduced some significant interval operations and the concept of distances between two intervals.
Section 3 introduces the concept of interval-valued cooperative games and defines the loss function to measure differences between imputations and values of coalitions. Hereby two quadratic programming models are constructed to compute interval-valued solutions of any interval-valued cooperative games. In Section 4, the models and method proposed in this paper are illustrated with a real example about the optimal allocation of cooperative profits and compared with other similar methods. Section 5 concludes this paper and gives some further research directions of interval-valued cooperative games.

2. Interval operations and distances between intervals

Denote \([a_\text{L}, a_\text{R}]\) = \(\{x | x \in \mathbb{R}, a_\text{L} \leq x \leq a_\text{R}\}\), which is called an interval, where \(a_\text{L} \in \mathbb{R}\), \(a_\text{R} \in \mathbb{R}\) and \(\mathbb{R}\) is the set of real numbers. Let \(I(\mathbb{R})\) be the set of bounded intervals on \(\mathbb{R}\). Obviously, if \(a_\text{L} = a_\text{R}\), the interval \([a_\text{L}, a_\text{R}]\) degenerates to a real number, denoted by \(a\), where \(a = a_\text{L} = a_\text{R}\). Therefore, intervals is a generalization of real numbers. That is to say, real numbers are a special case of intervals.

In the following, we give some operations of intervals such as the equality, addition and the scale multiplication as follows [19-21].

**Definition 1** Let \(\overline{a} = [a_\text{L}, a_\text{R}]\) and \(\overline{b} = [b_\text{L}, b_\text{R}]\) be two intervals on \(I(\mathbb{R})\) and \(\alpha \in \mathbb{R}\) be any real number. We stipulate: (1) Equality of two intervals: \(\overline{a} = \overline{b}\) if and only if \(a_\text{L} = b_\text{L}\) and \(a_\text{R} = b_\text{R}\); (2) Addition (or sum) of two intervals: \(\overline{a} + \overline{b} = [a_\text{L} + b_\text{L}, a_\text{R} + b_\text{R}]\); (3) Scale multiplication: if \(\alpha \geq 0\), then \(\alpha \overline{a} = [\alpha a_\text{L}, \alpha a_\text{R}]\), otherwise, i.e., if \(\alpha < 0\), then \(\alpha \overline{a} = [\alpha a_\text{R}, \alpha a_\text{L}]\).

The concept of distances are defined to measure differences between intervals.

**Definition 2** Let \(\overline{a}, \overline{b}\) and \(\overline{c}\) be three intervals on the set \(I(\mathbb{R})\). If a mapping
The function $d : I(\mathbb{R}) \times I(\mathbb{R}) \to \mathbb{R}$ satisfies the three properties (1)-(3) as follows: (1) $d(\bar{a}, \bar{b}) \geq 0$, (2) $d(\bar{a}, \bar{b}) = d(\bar{b}, \bar{a})$, (3) $d(\bar{a}, \bar{b}) \leq d(\bar{a}, \bar{c}) + d(\bar{c}, \bar{b})$, then $d(\bar{a}, \bar{b})$ is called the distance between the intervals $\bar{a}$ and $\bar{b}$.

In order to elaborate the quadratic programming models for interval-valued cooperative games based on the least square method, we define (the square of) the distance between the intervals $\bar{a}$ and $\bar{b}$ as follows:

$$D(\bar{a}, \bar{b}) = (a_L - b_L)^2 + (a_R - b_R)^2 \quad (1)$$

**Theorem 1** $D(\bar{a}, \bar{b})$ defined by Eq. (1) is the distance between the intervals $\bar{a}$ and $\bar{b}$.

**Proof:** We need to validate that $D(\bar{a}, \bar{b})$ defined by Eq. (1) satisfies the three properties (1)-(3) of Definition 2, respectively. It is easy to see from Eq. (1) that $D(\bar{a}, \bar{b}) \geq 0$ and $D(\bar{a}, \bar{b}) = D(\bar{b}, \bar{a})$ for any intervals $\bar{a}$ and $\bar{b}$. Namely, $D(\bar{a}, \bar{b})$ satisfies the properties (1) and (2) of Definition 2.

For any intervals $\bar{a}$, $\bar{b}$ and $\bar{c}$ on the set $I(\mathbb{R})$, where $\bar{c} = [c_L, c_R]$, it follows from Eq. (1) that

$$D(\bar{a}, \bar{b}) = (a_L - b_L)^2 + (a_R - b_R)^2 \leq [(a_L - c_L)^2 + (c_L - b_L)^2] + [(a_R - c_R)^2 + (c_R - b_R)^2] = [(a_L - c_L)^2 + (a_R - c_R)^2] + [(c_L - b_L)^2 + (c_R - b_R)^2] = D(\bar{a}, \bar{c}) + D(\bar{c}, \bar{b})$$

i.e., $D(\bar{a}, \bar{b}) \leq D(\bar{a}, \bar{c}) + D(\bar{c}, \bar{b})$. Hence, $D(\bar{a}, \bar{b})$ satisfies the property (3) of Definition 2.

Therefore, we have proven that $D(\bar{a}, \bar{b})$ defined by Eq. (1) is the distance between the intervals $\bar{a}$ and $\bar{b}$.

Note that the square appears in Eq. (1), which is also the distance from Theorem 1. In the sequel, the distance between two intervals is referred to the square of the distance given by Eq. (1) unless special statement.

3. Quadratic programming models and methods of interval-valued cooperative games
3.1 interval-valued cooperative games and concept of solutions

An interval-valued cooperative game $\nu$ on the player set $N = \{1, 2, \ldots, n\}$ means that $\nu(S)$ is an interval for any coalition $S \subseteq N$ and $\nu(\emptyset) = 0$, where $\emptyset$ is an empty set. $\nu(S)$ is called the interval-valued characteristic function of the coalition $S$. Usually, $\nu(S)$ is denoted by the interval $\nu(S) = [\nu_L(S), \nu_R(S)]$, where $\nu_L(S) \leq \nu_R(S)$. Usually, $\nu(\{i\})$ and $\nu(\{i, j\})$ are simply denoted by $\nu(i)$ and $\nu(i, j)$, respectively.

It is easy to see that each player should receive an interval-valued imputation (or payoff) from cooperation due to the fact that each coalition’s value is an interval. Let $x_i = [x_{Li}, x_{Ri}]$ be the interval-valued imputation of the player $i \in N$. Denote $x(S) = \sum_{i \in S} x_i$, which represents the sum of the imputations of all players in the coalition $S$. Using the interval operations given above, we can express $x(S)$ as the interval $x(S) = [\sum_{i \in S} x_{Li}, \sum_{i \in S} x_{Ri}]$.

Distances are used to measure the difference between $x(S)$ and $\nu(S)$. Thus, according to Eq. (1), we define the square of the distance between the intervals $x(S)$ and $\nu(S)$ for the coalition $S$ as follows: $D(x(S), \nu(S)) = (\sum_{i \in S} x_{Li} - \nu_L(S))^2 + (\sum_{i \in S} x_{Ri} - \nu_R(S))^2$. Then, the sum of the squares of the distances between $x(S)$ and $\nu(S)$ for all coalitions $S$ in the grand coalition $N$ can be defined as follows:

$$L(x) = \sum_{S \subseteq N} D(x(S), \nu(S)) = \sum_{S \subseteq N} [(\sum_{i \in S} x_{Li} - \nu_L(S))^2 + (\sum_{i \in S} x_{Ri} - \nu_R(S))^2],$$

where $x = (x_1, x_2, \ldots, x_n)^T$ is the vector of the interval-valued imputations for all players in the grand coalition $N$. $L(x)$ may be interpreted as a type of loss functions.

3.2 The quadratic programming model and its optimal solution

It is easy to see from the concept of loss functions that an optimal allocation of all players (i.e., a solution of an interval-valued cooperative game $\nu$) is the solution of the
quadratic programming model as follows:

\[
\min \{ L(x) = \sum_{S \subseteq N} \left[ \left( \sum_{i \in S} x_{Li} - \nu_L(S) \right)^2 + \left( \sum_{i \in S} x_{RI} - \nu_R(S) \right)^2 \right] \}.
\]  

(2)

Let partial derivatives of \( L(x) \) with respect to the variables \( x_{Li} \) and \( x_{Rj} \) \((j \in S \subseteq N)\)

be equal to 0, respectively. Thus, we have

\[
\begin{align*}
\frac{\partial L(x)}{\partial x_{Lj}} &= 2 \sum_{S \subseteq N, j \in S} \left( \sum_{i \in S} x_{Li} - \nu_L(S) \right) = 0 \quad (j = 1, 2, \ldots, n) \\
\frac{\partial L(x)}{\partial x_{Rj}} &= 2 \sum_{S \subseteq N, j \in S} \left( \sum_{i \in S} x_{RI} - \nu_R(S) \right) = 0 \quad (j = 1, 2, \ldots, n)
\end{align*}
\]

which directly infers that

\[
\sum_{S \subseteq N, j \in S} \sum_{i \in S} x_{Li} = \sum_{S \subseteq N, j \in S} \nu_L(S) \quad (j = 1, 2, \ldots, n)
\]  

(3)

and

\[
\sum_{S \subseteq N, j \in S} \sum_{i \in S} x_{Ri} = \sum_{S \subseteq N, j \in S} \nu_R(S) \quad (j = 1, 2, \ldots, n).
\]  

(4)

Solving the above systems of linear equations (i.e., Eqs. (3) and (4)), we can obtain the solution of the interval-valued cooperative game \( \nu \). Thus, in the sequent, we focus on how to solve Eqs. (3) and (4), respectively.

To solve \( x_{Li} \) \((i = 1, 2, \ldots, n)\) and \( x_{Ri} \) \((i = 1, 2, \ldots, n)\), Eqs. (3) and (4) can be rewritten as follows:

\[
\begin{align*}
& a_{11} x_{L1} + a_{12} x_{L2} + a_{13} x_{L3} + \cdots + a_{1n} x_{Ln} = \sum_{S \subseteq N, i \in S} \nu_L(S) \\
& a_{21} x_{L1} + a_{22} x_{L2} + a_{23} x_{L3} + \cdots + a_{2n} x_{Ln} = \sum_{S \subseteq N, 2 \in S} \nu_L(S) \\
& \vdots \\
& a_{n1} x_{L1} + a_{n2} x_{L2} + a_{n3} x_{L3} + \cdots + a_{nn} x_{Ln} = \sum_{S \subseteq N, n \in S} \nu_L(S)
\end{align*}
\]  

(5)

and
\[
\begin{aligned}
\left\{ 
\begin{array}{l}
\sum_{S \subseteq N \mid i \in S} \nu_r(S) = a_1 x_{R_1} + a_{12} x_{R_2} + a_{13} x_{R_3} + \cdots + a_{1n} x_{R_n} \\
\sum_{S \subseteq N \mid j \in S} \nu_r(S) = a_2 x_{R_1} + a_{22} x_{R_2} + a_{23} x_{R_3} + \cdots + a_{2n} x_{R_n} \\
\vdots \\
\sum_{S \subseteq N \mid k \in S} \nu_r(S) = a_k x_{R_1} + a_{k2} x_{R_2} + a_{k3} x_{R_3} + \cdots + a_{kn} x_{R_n}
\end{array}
\right.
\end{aligned}
\]

respectively.

Let \( |S| \) be the number of all players in the coalition \( S \). According to the knowledge on permutation and combination, for the player \( i \in N \), the number of the coalitions \( S \) including \( i \) with \( |S| = 1 \) can be expressed as \( C_{n-1}^0 \). In the same way, the number of the coalitions \( S \) including \( i \) with \( |S| = 2 \) can be expressed as \( C_{n-1}^1 \). Generally, the number of the coalitions \( S \) including \( i \) with \( |S| = k \) \((k = 1, 2, \cdots, n)\) can be expressed as \( C_{n-1}^{k-1} \). It is obvious that the number of the coalitions \( S \) including \( i \) can be written as \( C_{n-1}^0 + C_{n-1}^1 + \cdots + C_{n-2}^{n-2} + C_{n-1}^{n-1} \), which is equal to \( 2^{n-1} \) by the simple observation.

Similarly, for the players \( i \in N \) and \( j \in N \) \((i \neq j)\), the number of the coalitions \( S \) including both \( i \) and \( j \) with \( |S| = 2 \) can be expressed as \( C_{n-2}^0 \), the number of the coalitions \( S \) including \( i \) and \( j \) simultaneously with \( |S| = 3 \) can be expressed as \( C_{n-2}^1 \). Generally, the number of the coalitions \( S \) including both \( i \) and \( j \) with \( |S| = k \) \((k = 2, 3, \cdots, n)\) can be expressed as \( C_{n-2}^{k-2} \). Thus, the number of the coalitions \( S \) including \( i \) and \( j \) simultaneously can be written as \( C_{n-2}^0 + C_{n-2}^1 + \cdots + C_{n-3}^{n-3} + C_{n-2}^{n-2} \), which is just about \( 2^{n-2} \).

Then, it follows from the aforementioned conclusions that

\[
a_y = \begin{cases} 
2^{n-1} & (i = j \text{ with } i, j \in \{1, 2, \cdots, n\}) \\
2^{n-2} & (i \neq j \text{ with } i, j \in \{1, 2, \cdots, n\}) 
\end{cases}
\]

Denote \( X_L = (x_{L_1}, x_{L_2}, \cdots, x_{L_n})^T \), \( X_R = (x_{R_1}, x_{R_2}, \cdots, x_{R_n})^T \).
\[
B_L = (\sum_{S \subseteq N:1 \in S} v_L(S), \sum_{S \subseteq N:2 \in S} v_L(S), \ldots, \sum_{S \subseteq N:n \in S} v_L(S))^T,
\]

\[
B_R = (\sum_{S \subseteq N:1 \in S} v_R(S), \sum_{S \subseteq N:2 \in S} v_R(S), \ldots, \sum_{S \subseteq N:n \in S} v_R(S))^T,
\]

and

\[
A = (a_{ij})_{n \times n} = \begin{pmatrix}
2^n & 2^{n-1} & \cdots & 2^n & 2^{n-1} & \cdots & 2^n \\
2^{n-1} & 2^n & \cdots & 2^{n-1} & 2^n & \cdots & 2^{n-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
2^{n-1} & 2^{n-2} & \cdots & 2^n & 2^{n-2} & \cdots & 2^n \\
\end{pmatrix}_{n \times n}.
\]

Thus, Eqs. (5) and (6) can be rewritten in the matrix format as follows:

\[
AX_L = B_L
\]

and

\[
AX_R = B_R,
\]

respectively.

Let

\[
(A, E) = \begin{pmatrix}
2^n & 2^{n-1} & \cdots & 2^n & 2^{n-1} & \cdots & 2^n \\
2^{n-1} & 2^n & \cdots & 2^{n-1} & 2^n & \cdots & 2^{n-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
2^{n-1} & 2^{n-2} & \cdots & 2^n & 2^{n-2} & \cdots & 2^n \\
\end{pmatrix}_{n \times 2n}.
\]

By using elementary linear transformation, we have

\[
(A, E) = \begin{pmatrix}
1 & 0 & \cdots & 0 & \frac{n}{2^{n-2}} & \frac{1}{n+1} & -\frac{1}{2^{n-2}} & \frac{1}{n+1} & \cdots & -\frac{1}{2^{n-2}} & \frac{1}{n+1} \\
0 & 1 & \cdots & 0 & -\frac{1}{2^{n-2}} & \frac{1}{n+1} & \frac{1}{2^{n-2}} & \frac{n}{n+1} & \cdots & -\frac{1}{2^{n-2}} & \frac{1}{n+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 1 & -\frac{1}{2^{n-2}} & \frac{1}{n+1} & -\frac{1}{2^{n-2}} & \frac{1}{n+1} & \cdots & -\frac{1}{2^{n-2}} & \frac{n}{n+1} & \cdots \end{pmatrix}_{n \times 2n}.
\]

It is obvious that matrixes \(A\) and \(E\) are row equivalent. Therefore, the matrix \(A\) is reversible. Hereby, we have
By using the multiplication of matrixes, we obtain the solutions of Eqs. (5) and (6) as follows:

\[ X_L = A^{-1}B_L \]  \hfill (7) 

and

\[ X_R = A^{-1}B_R, \]  \hfill (8)

respectively. Thus, we can obtain the optimal interval-valued imputations of the players \( i \in N \), which are expressed as \( x_i = [x_{Li}, x_{Ri}] \) \( (i = 1, 2, \ldots, n) \).

### 3.3 Specialization of the quadratic programming model

In real management situations, some constraint conditions need to be taken into consideration. In this case, the quadratic programming model (i.e., Eq. (2)) is still applicable. For example, if we consider the efficiency: \( x(N) = \nu(N) \) (i.e., \( \sum_{i=1}^{n} x_{Li}, \sum_{i=1}^{n} x_{Ri} = [\nu_L(N), \nu_R(N)] \)), then Eq. (2) can be flexibly rewritten as the following quadratic programming model:

\[
\begin{align*}
\text{min} \{ L(x) &= \sum_{S \subseteq N} [ (\sum_{i \in S} x_{Li} - \nu_L(S))^2 + (\sum_{i \in S} x_{Ri} - \nu_R(S))^2 ] \} \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_{Li} = \nu_L(N) \\
& \sum_{i=1}^{n} x_{Ri} = \nu_R(N).
\end{align*}
\]  \hfill (9)

According to the Lagrange multiplier method, the Lagrange function is constructed as follows:
\[ L(x, \lambda, \mu) = \sum_{S \subseteq N} \left[ (\sum_{i \in S} x_{Li} - \nu_L(S))^2 + (\sum_{i \in S} x_{Ri} - \nu_R(S))^2 \right] + \lambda \left( \sum_{i=1}^{n} x_{Li} - \nu_L(N) \right) + \mu \left( \sum_{i=1}^{n} x_{Ri} - \nu_R(N) \right). \]

Then, an optimal allocation of all players (i.e., a solution of the interval-valued cooperative game \( \nu \)) is the solution of the quadratic programming model as follows:

\[
\min \{ L(x, \lambda, \mu) = \sum_{S \subseteq N} \left[ (\sum_{i \in S} x_{Li} - \nu_L(S))^2 + (\sum_{i \in S} x_{Ri} - \nu_R(S))^2 \right] + \\
\lambda \left( \sum_{i=1}^{n} x_{Li} - \nu_L(N) \right) + \mu \left( \sum_{i=1}^{n} x_{Ri} - \nu_R(N) \right) \}.
\]

Let the partial derivatives of \( L(x, \lambda, \mu) \) with respect to the variables \( x_{Li}, x_{Ri} \) \( (j \in S \subseteq N), \lambda \) and \( \mu \) be equal to 0, respectively. Then, we have

\[
\begin{align*}
\frac{\partial L(x, \lambda, \mu)}{\partial x_{Li}} &= 2 \sum_{S \subseteq N, j \in S} \left( \sum_{i \in S} x_{Li} - \nu_L(S) \right) + \lambda = 0 \quad (j = 1, 2, \ldots, n) \\
\frac{\partial L(x, \lambda, \mu)}{\partial \lambda} &= \sum_{i=1}^{n} x_{Li} - \nu_L(N) = 0
\end{align*}
\]

and

\[
\begin{align*}
\frac{\partial L(x, \lambda, \mu)}{\partial x_{Ri}} &= 2 \sum_{S \subseteq N, j \in S} \left( \sum_{i \in S} x_{Ri} - \nu_R(S) \right) + \mu = 0 \quad (j = 1, 2, \ldots, n) \\
\frac{\partial L(x, \lambda, \mu)}{\partial \mu} &= \sum_{i=1}^{n} x_{Ri} - \nu_R(N) = 0,
\end{align*}
\]

which infer that

\[
\begin{align*}
\sum_{S \subseteq N, j \in S} \sum_{i \in S} x_{Li} + \frac{\lambda}{2} &= \sum_{S \subseteq N, j \in S} \nu_L(S) \quad (j = 1, 2, \ldots, n) \\
\sum_{i=1}^{n} x_{Li} &= \nu_L(N)
\end{align*}
\]

and

\[
\begin{align*}
\sum_{S \subseteq N, j \in S} \sum_{i \in S} x_{Ri} + \frac{\mu}{2} &= \sum_{S \subseteq N, j \in S} \nu_R(S) \quad (j = 1, 2, \ldots, n) \\
\sum_{i=1}^{n} x_{Ri} &= \nu_R(N),
\end{align*}
\]

respectively.
Denote \( e = (1, 1, \cdots, 1)^T \) and \( X'_L = (x'_{L1}, x'_{L2}, \cdots, x'_{Ln})^T \). Then, Eq. (11) can be rewritten as follows:

\[
AX'_L + \frac{\lambda}{2} e = B_L
\]  

(13)

and

\[
e^T X'_L = \nu_L(N).
\]

(14)

It follows from Eq. (13) that

\[
X'_L = A^{-1} B_L - \frac{\lambda}{2} A^{-1} e = X_L - \frac{\lambda}{2} A^{-1} e,
\]  

(15)

where \( X_L \) is given by Eq. (7). Then, the key of solving Eq. (11) is to calculate the value of \( \lambda \).

It is easily derived from Eqs. (14) and (15) that

\[
e^T X'_L - \frac{\lambda}{2} e^T A^{-1} e = \nu_L(N).
\]

Obviously, we have

\[
e^T X_L = \sum_{i=1}^{n} x_{Li}
\]

and

\[
e^T A^{-1} e = \frac{1}{2^{n-2}} \frac{n}{n+1}.
\]

Hence, we have

\[
\frac{\lambda}{2} = 2^{n-2} \frac{n+1}{n} \left( \sum_{i=1}^{n} x_{Li} - \nu_L(N) \right).
\]

(16)

Thus, it can be easily derived from Eqs. (15) and (16) that
\[ X'_L = X_L - 2^{n-2} \frac{n+1}{n} \left( \sum_{i=1}^{n} x_{Li} - V_L(N) \right) A^{-1} e \]
\[ = X_L - 2^{n-2} \frac{n+1}{n} \left( \sum_{i=1}^{n} x_{Li} - V_L(N) \right) \left( \frac{1}{2^{n-2}} \frac{1}{n+1} \right) e \]
\[ = X_L - \frac{1}{n} \left( \sum_{i=1}^{n} x_{Li} - V_L(N) \right) e. \]

Namely,
\[ X'_L = X_L + \frac{1}{n} (V_L(N) - \sum_{i=1}^{n} x_{Li}) e. \quad (17) \]

In a similar way to the above discussion, we can obtain the solution of Eq. (11) as follows:
\[ X'_R = X_R + \frac{1}{n} (V_R(N) - \sum_{i=1}^{n} x_{Ri}) e. \quad (18) \]

So far, we obtain the solution of Eq. (9), which consists of Eqs. (17) and (18). Thus, if the efficiency is taken into consideration, then we can determine optimal interval-valued imputations of all players (i.e., a solution of the interval-valued cooperative game \( u \)), which are expressed as \( x'_i = [x'_{Li}, x'_{Ri}] \) \((i = 1, 2, \ldots, n)\), whose lower and upper bounds are given by Eqs. (17) and (18), respectively.

4. An example analysis

There are many applications of the classical cooperative game theory to real decision problems in finance, management, business, investment and economics. To compare the method proposed in this paper with Han et al.’s method [7], we adopt the same example from Example 5.1 [7], which is given as follows. The example is an interval-valued cooperative game which is applied to determine optimal allocation strategies.

Example 1 There are three factories (i.e., players) 1, 2 and 3, who have the ability to produce separately. Denoted the set of players by \( N = \{1, 2, 3\} \). Now, they plan to work together to
develop a product. Due to the incomplete and uncertain information, they cannot precisely forecast their profits (or gains). Generally, they can estimate ranges of their profits. Namely, the profit of the coalition \( S \subseteq N \) of the factories (i.e., players) may be expressed with an interval \( \nu(S) = [\nu_L(S), \nu_R(S)] \). In this case, the optimal allocation problem of profits for the factories may be regarded as an interval-valued cooperative game \( \nu \) in which the interval-valued characteristic function of the coalition \( S \subseteq N \) is just about \( \nu(S) \). Thus, if they make the product by themselves, then their profits are expressed with the intervals \( \nu(1) = [0, 2] \), \( \nu(2) = [1/2, 3/2] \) and \( \nu(3) = [1, 2] \), respectively. Similarly, if any two factories cooperatively make the product, then their profits are expressed with \( \nu(1, 2) = [2, 3] \), \( \nu(2, 3) = [4, 4] \) and \( \nu(1, 3) = [3, 4] \), respectively. If all three factories (i.e., the greatest coalition \( N \)) cooperatively make the product, then the profit is expressed with \( \nu(1, 2, 3) = [6, 7] \).

4.1 Computational results obtained by different methods and analysis

In this section, the above numerical example is solved by the method proposed in this paper and the method proposed by Han et al. [7]. The computational results are analyzed and compared to show the validity, applicability and superiority of the quadratic programming models and method proposed in this paper.

According to the quadratic programming model (i.e., Eq. (2)), it is easily derived from Eqs. (7) and (8) that

\[
X_L = A^{-1}B_L = \begin{pmatrix}
3 & -1 & -1 \\
8 & 8 & 8 \\
-1 & 3 & 1 \\
8 & 8 & 8 \\
-1 & 1 & 3 \\
8 & 8 & 8 \\
\end{pmatrix}
\begin{pmatrix}
11 \\
25 \\
2 \\
14 \\
13 \\
16 \\
\end{pmatrix}
= \begin{pmatrix}
13 \\
16 \\
25 \\
16 \\
37 \\
16 \\
\end{pmatrix}
\]

and
\[
X_R = A^T B_R = \begin{bmatrix}
\frac{3}{8} & -\frac{1}{8} & -\frac{1}{8} \\
\frac{1}{8} & \frac{3}{8} & \frac{1}{8} \\
-\frac{1}{8} & \frac{1}{8} & \frac{3}{8}
\end{bmatrix} \begin{bmatrix} 16 \\ 31 \\ 17 \end{bmatrix} = \begin{bmatrix} 31 \\ 27 \\ 39 \\ 16 \end{bmatrix},
\]

respectively. Namely, \(x_1 = [x_{1l}, x_{1r}] = [13/16, 31/16]\), \(x_2 = [x_{2l}, x_{2r}] = [25/16, 27/16]\) and \(x_3 = [x_{3l}, x_{3r}] = [37/16, 39/16]\), which are the optimal interval-valued imputations of the factories (i.e., players) 1, 2 and 3, respectively.

As stated earlier, Eq. (2) does not consider the efficiency condition. Thus, according to the quadratic programming model with the efficiency (i.e., Eq. (9)), it easily follows from Eqs. (17) and (18) that

\[
X'_L = X_L + \frac{1}{3} (V_L(N) - \sum_{i=1}^{3} x_{Li}) = (13/16, 25/16, 37/16)^T + \frac{1}{3} (6 - \frac{75}{16})(1,1,1)^T = (5/4, 2, 11/4)^T
\]

and

\[
X'_R = X_R + \frac{1}{3} (V_R(N) - \sum_{i=1}^{3} x_{Ri}) = (31/16, 27/16, 39/16)^T + \frac{1}{3} (7 - \frac{97}{16})(1,1,1)^T = (9/4, 2, 11/4)^T,
\]

respectively. Namely, \(x'_1 = [x'_{1l}, x'_{1r}] = [5/4, 9/4]\), \(x'_2 = [x'_{2l}, x'_{2r}] = [2, 2]\) and \(x'_3 = [x'_{3l}, x'_{3r}] = [11/4, 11/4]\), which are respectively the optimal interval-valued imputations of the factories (i.e., players) 1, 2 and 3 when the efficiency is taken into consideration.

If Han et al.’s method is used to solve the above numerical example, then according to Eq. (4) given by Han et al. [7], we have

\[
\phi^*(\nu) = \sum_{S \subseteq \mathbb{N}\setminus \{1\}} \frac{|S|!(|S|-1)!}{3!} (\nu(S \cup \{1\}) - \nu(S)) 
\]

\[
= \frac{0!2!}{3!} (\nu(1) - \nu(\emptyset)) + \frac{1!1!}{3!} (\nu(1,2) - \nu(2)) + \frac{1!1!1!}{3!} (\nu(1,3) - \nu(3)) + \frac{2!1!0!}{3!} (\nu(1,2,3) - \nu(2,3)) 
\]

\[
= \frac{0!2!}{3!} ([0,2] - [0,0]) + \frac{1!1!}{3!} ([2,3] - [2,2]) + \frac{1!1!}{3!} ([3,4] - [1,2]) + \frac{2!1!0!}{3!} ([6,7] - [4,4]) 
\]

\[
= \frac{11}{12}, \frac{31}{12}
\]
In the same way, we can obtain \( \phi_2^*(\nu) = [7/6, 17/6] \) and \( \phi_3^*(\nu) = [23/12, 43/12] \).

From the above results, it is easily seen that \( \sum_{i=1}^{3} \phi_i^*(\nu) = [4, 9] \neq [6, 7] \), which shows that the interval-valued Shapley-like value \( \phi^* \) may not satisfy the efficiency condition. However, if we use another interval-valued Shapley-like value (i.e., Eq. (9) given by Han et al. [7]) which satisfies the efficiency condition, then it easily follows that \( \hat{\phi}^*(\nu) = ([19/12, 23/12], [11/6, 13/6], [31/12, 35/12]) \). Obviously, \( \sum_{i=1}^{3} \hat{\phi}_i^*(\nu) = [6, 7] \).

4.2 The comparison analysis and conclusions

In order to compare the method proposed in this paper with Han et al.’s method [7] clearly, another example is given as follow.

Example 2 There is an interval-valued cooperative game \( \nu \), where \( \nu(1) = [0.3, 1] \), \( \nu(2) = [2, 5] \) and \( \nu(1, 2) = [4, 6] \). Then, it is easily derived from Eq. (4) given by Han et al. [7] that \( \phi_1^*(\nu) = [-0.35, 2.5] \) and \( \phi_2^*(\nu) = [2.5, 5.35] \). It is obvious that the lower bound of the interval-valued imputation of player 1 is a negative number. \( \phi_1^*(\nu) \) means that player 1 may get a negative profit (or gain). In other words, player 1 may get worse if he/her cooperates with player 2. Obviously, this cooperation between players 1 and 2 will not happen due to the fact that the profit of player 1 is not smaller than 0.3 even if he/her goes it alone. Therefore, the results obtained through using Eq. (4) given by Han et al. [7] may not rational. Namely, the interval-valued Shapley-like value proposed by Han et al. [7] may result in irrational results. The main reason for this phenomenon is that the interval-type value of the coalition \( S = \{1, 2\} \) overlaps with the interval-type value of the coalition \( S = \{2\} \). That is to say, the lower bound of the interval-type value of the coalition \( S = \{1, 2\} \) is smaller than the upper bound of the interval-type value of the coalition \( S = \{2\} \). However, in many real situations,
this case is always sure that they overlap with each other.

Conversely, if we use the quadratic programming model (i.e., Eq. (2)), then according to Eqs. (7) and (8), we can obtain $x_1 = [26/30,1]$ and $x_2 = [77/30,5]$. Obviously, $x_{1L} + x_{2R} = 103/30 = 3.4333$ and $x_{1L} + x_{2R} = 6$. Thus, the sum of the lower bounds of the interval-valued imputations of players 1 and 2 is closer to 4, which is the lower bound of the interval-type value of the greatest coalition $N = \{1,2\}$.

In sum, it is not difficult to draw the following conclusions from the aforementioned modeling, solving process and computational results.

(1) The quadratic programming method proposed in this paper is simpler and more convenient from the point of view of computational complexity than Han et al.’s method [7]. In our method, Eqs. (7) and (8) (or Eqs. (17) and (18)) can be directly applied to compute interval-valued imputations of players synchronously. However, Han et al.’s method [7] is respectively used to compute interval-valued imputations of players.

(2) In the quadratic programming method, the distance is used to measure the differences between interval-valued imputations and interval-type values of coalitions. Thus, we can effectively avoid the magnification of uncertainty resulted from the subtraction of intervals. However, Han et al.’s method [7] may not overcome this disadvantage. For example, in Example 1, the interval lengths of the interval-type values of the coalitions $S$ containing player 2 are not bigger than 1. However, the interval length of the interval-valued Shapley-like value of player 2 is equal to $10/6$, which is greater than 1.

(3) Han et al.’s method [7] may obtain negative interval-valued Shapley-like values of players, which are not rational. For instance, in Example 2, $\phi^*(\nu)$ is not positive interval-valued imputation even if all coalitions’ values are positive. However, the quadratic programming method assures that the interval-valued imputations of players are always positive if all coalitions’ values are positive.
(4) Stated as in Examples 1 and 2, according to the methods proposed by Han et al. [7], the interval lengths of the interval-valued imputations of players are identical. It is noted that this conclusion may not be reasonable. In fact, in most situations, the interval lengths of interval-type values of coalitions may be different. Thus, the ranges (i.e., intervals) of marginal contributions of players are not always identical. Hereby, the interval lengths of interval-valued imputations of players may be different. From this aspect, it is also shown that Han et al.’s method [7] cannot always assure the obtained interval-valued Shapley-like values of players are rational.

5. Conclusions

In real economic management, there are many situations in which coalitions’ values are not exactly known \textit{a priori}. In this paper, we develop the quadratic programming models and method for solving interval-valued cooperative games based on the concepts of loss functions and the distance between intervals. The interval-valued imputations of players can be directly obtained through using the analytical formulae (i.e., Eqs. (7) and (8) or Eqs. (17) and (18)). The developed models and method are remarkably from other works [7-10,12,14,15,17] and have some advantages as stated previously from the aspects of the scale, solving process and computation amount.

Obviously, if all coalitions’ values degenerate to real numbers, i.e., $B = B_L = B_R$, then Eqs. (7) and (8) (or Eqs. (17) and (18)) are identical. That is to say, Eqs. (7) and (8) (or Eqs. (17) and (18)) are applicable to the classical cooperative games. Thus, the models and method developed in this paper may be regarded as an extension of that for the classical cooperative games when uncertainty and imprecision are taken into consideration.

As stated earlier, we use intervals to describe uncertainty and imprecision and hereby study how to solve interval-valued cooperative games. In fact, we also use fuzzy numbers and
intuitionistic fuzzy numbers to deal with uncertainty and imprecision. Thus, we will study and develop some effective methods for solving cooperative games with coalitions’ values represented by fuzzy numbers and/or intuitionistic fuzzy numbers in the near future.

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