# The core and the bargaining set for convex games

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**Abstract**

Within the class of superadditive cooperative games with transferable utility, the convexity of a game is characterized by the coincidence of its core and the steady bargaining set. As a consequence it is also proved that convexity can also be characterized by the coincidence of the core of a game and the modified Zhou bargaining set (Shimomura, 1997)

*Keywords:* cooperative game, convex games, bargaining set

## 1. Introduction

Cooperative game theory analyzes how to distribute profits arising from the cooperation of a groups of agents by proposing solutions that may consist on a unique allocation of those profits (payoff vector) or on a group of them meeting some stability conditions (set-solution). The core is the most natural set-solution concept but it might be empty. The bargaining sets (Davis and Maschler (1963, 1967), Mas-Colell (1989), Zhou (1994) and others) based on objections and counter-objections to payoff proposals offers an alternative solution to the emptiness of the core, at a cost to be rather complex to compute. For this reason, it has been interesting to define non-empty subsolutions of the bargaining sets that were more simply to describe and check, that fulfill some stability conditions and that were related to the core of the game whenever it is non-empty. In this way, the first subsolutions we can find in the literature are the notion of quasi-core, introduced by Shapley and Shubik (1966), and the concept of Kernel of a game (Davis and Maschler, 1965). Years after, Shimomura (1997) introduces the *steady bargaining set* of a game $v$, $\mathcal{SB}(v)$, which can be considered as an small modification of the Shapley and Shubik quasi-core concept. The steady bargaining set of a game includes its core and it is a subsolution of two variants of well-known bargaining sets, also introduced by Shimomura (1997): the modified Mas-Colell bargaining set $\mathcal{MB}^*(v)$ and the modified Zhou bargaining set $\mathcal{Z}^*(v)$. A sufficient condition that guarantee the non-emptiness of the steady bargaining set and the modified Zhou bargaining set of a game is its superadditivity, while weak-superadditivity suffices to check the non-emptiness of the modified Mas-Colell bargaining set. Under superadditivity conditions, the relationship among these solutions is as follows:

$$C(v) \subseteq \mathcal{SB}(v) \subseteq \mathcal{Z}^*(v) \subseteq \mathcal{MB}^*(v). \tag{1}$$

Convex or supermodular coalitional games were introduced by Shapley (1971). They are an important subclass of superadditive games and they model cooperative situations where the marginal contribution of a player to a coalition increases as the coalition becomes larger (the so called snowballing effect). Convex games satisfy important properties from a game theoretical point of view and they have been useful to analyze and capture many economic situations both in cooperative and non-cooperative frameworks.

Einy and Wettstein (1996) opened the question of characterizing the convexity of a game by comparing its bargaining sets with the core, with special reference to the stable bargaining set introduced by Greenberg (1992). Within the domain of weak-superadditive (or zero-monotonic) games, Izquierdo and Rafels (2012) gives a first answer to that question by means of the coincidence of the core of a game and its modified Mas-Colell bargaining set.

As any superadditive game is weak-superadditive, the aforementioned characterization result still holds within the subclass of superadditive games. In this paper we focus on enriching the convexity characterization results within the domain of superadditive games. The first characterization result requires the coincidence of the the core of a game and its steady bargaining set (Theorem 1). By the inclusion relationship given in (1) and taking into account the characterization result of Izquierdo and Rafels (2012), we also obtain as corollary an additional characterization of convex games in term of the coincidence of the modified Zhou bargaining set and the core of the game (Corollary 1).

## 2. Notations

Let $N = \{1, 2, \ldots, n\}$ be a set of players. For any coalition $S \subseteq N$, $|S|$ denotes the number of players in $S$. A cooperative game with player set $N$ is a function $v : 2^N \to \mathbb{R}$ assigning to each coalition $S \subseteq N$ a real number $v(S)$ such that $v(\emptyset) := 0$. The function $v$ is called the *characteristic function* of the game and $v(S)$ is the *worth* of the coalition $S$. This number is interpreted as what the coalition can obtain on its own. Let $\mathcal{G}^N$ be the class of games with player set $N$.

A game $v \in \mathcal{G}^N$ is *grand-coalition zero-monotonic* if for all $S \subseteq N$ we have $v(S) + \sum_{i \in S} v(\{i\}) \leq v(N)$. It is *superadditive* if for all $S, T \subseteq N$ with $S \cap T \neq \varnothing$ it holds $v(S) + v(T) \leq v(S \cup T)$. It is *convex* if, for all $i \in N$, $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$ for all $S \subseteq T \subseteq N \setminus \{i\}$.

Let $\mathbb{R}^N$ stand for the space of real-valued vectors $x = (x_i)_{i \in N}$ where $x_i$ is interpreted as the payoff to player $i \in N$, $x_S$ is the restriction of $x$ to the members of $S$ and $x(S)$ denotes $\sum_{i \in S} x_i$, with the convention $x(\emptyset) = 0$.

The *set of preimputations* of a game $v \in \mathcal{G}^N$ is defined by $I^*(v) := \{x \in \mathbb{R}^N \,|\, x(N) = v(N)\}$. Its *set of imputations* is defined by $I(v) := \{x \in \mathbb{R}^N \,|\, x(N) = v(N) \text{ and } x_i \geq v(\{i\}), \text{ for all } i \in N\}$ and its *core* is defined by $C(v) := \{x \in \mathbb{R}^N \,|\, x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ for all } S \subseteq N\}$. A game with a non-empty core is called a *balanced* game. Let $\mathcal{B}^N \subseteq \mathcal{G}^N$ be the subclass of balanced games with player set $N$.

Given a game $v$, a preimputation $x \in I^*(v)$ and a pair of players $i$ and $j$ we define

$$s^v_{ij}(x) = \max\{v(S) - x(S) \mid S \subseteq N,\, i \in S \text{ but } j \notin S\}.$$

We say that player $i$ outweigths player $j$ at $x$ if $s^v_{ij} > s^v_{ji}(x)$. The prekernel of the game $v$, $\mathcal{PK}(v)$, is the subset of preimputations such that no player outweights any other player at $x$. This is

$$\mathcal{PK}(v) = \{x \in I^*(v) \mid \text{ for all } i, j \in N,\ s^v_{ij}(x) = s^v_{ji}(x)\}.$$

For any game , the prekernel is always non-empty.

Shimomura (1997) considers modifications of both the Mas-Colell bargaining set (1989) and Zhou bargaining set (1994). As usual the bargaining set is defined by means of an interaction of objections and counterobjections. Let $x \in \mathbb{R}^N$ . An *objection* to $x$ is a pair $(S, y)$, $\varnothing \neq S \subseteq N$ and $y \in \mathbb{R}^S$ with $y(S) = v(S)$ such that $y_i > x_i$, for all $i \in S$. A *counterobjection* to $(S, y)$ in the sense of Mas-Colell (*à la* Shimonura) is a pair $(T, z)$, $z \in \mathbb{R}^T$ with $z(T) = v(T)$ such that $z_i > y_i$, for all $i \in T \cap S$, and $z_i > x_i$ for all $i \in T \setminus S$. A *counterobjection* to $(S, y)$ in the sense of Zhou (*à la* Shimonura) is a pair $(T, z)$, where $T \setminus S \neq \varnothing$, $S \setminus T \neq \varnothing$, $T \cap S \neq \varnothing$, and $z \in \mathbb{R}^T$ with $z(T) = v(T)$ such that $z_i > y_i$, for all $i \in T \cap S$, and $z_i > x_i$ for all $i \in T \setminus S$. Notice the bargaining process represents strictly improvements (strictly higher payoffs) for all players involved in the objections and the counterobjections.

**Definition 1.** *The* **Mas-Colell bargaining set** *(*à la Shimomura*) is defined as*

$$\mathcal{MB}^*(v) = \{x \in I(v) \mid \textit{for each objection to } x, \textit{ there is a counterobjection}\},$$

*where objection and counterobjection follow Mas-Colell's definitions.*

**Definition 2.** *The* **Zhou bargaining set** *(*à la Shimomura*) is defined as*

$$\mathcal{Z}^*(v) = \{x \in I(v) \mid \textit{for each objection to } x, \textit{ there is a counterobjection}\},$$

*where objection and counterobjection follow Zhou's definitions.*

If no confusion arises we will refer to them simply as the Mas-Colell bargaining set and the Zhou bargaining set. By definition, these sets only consist on imputations (individually rational payoff vectors) and always includes the core. Shimomura (1997) states that, in case the core of the game is empty, a sufficient condition that guarantees the Mas-Colell bargaining set to be nonempty is grand-coalition zero-monotonicity, while it is superadditivity that ensures the non-emptines of the Zhou bargaining set.

Shimomura also defines a subset of the Zhou bargaining set ( the *steady bargaining set*, $SB(v)$) by means of a dominant relationship between coalitions. He claims that the steady bargaining set can be rewritten as follows

**Definition 3.** *Let $v \in \mathcal{G}^N$. An imputation $x \in I(v)$ is in the* **steady bargaining set** *$\mathcal{SB}(v)$ if for all coalition $S \subseteq N$ with positive excess $v(S) - x(S) > 0$, there exists $M \subseteq N$, such that $S \setminus M \neq \varnothing$, $M \setminus S \neq \varnothing$, $S \cap M \neq \varnothing$ and $v(M) - x(M) \geq v(S) - x(S)$.*

For any superadditive game $v$, it can be easily proved the inclusions $C(v) \subseteq \mathcal{SB}(v) \subseteq \mathcal{Z}^*(v) \subseteq \mathcal{MB}^*(v)$.

Let $v \in \mathcal{B}^N$ be a game and $\theta = (i_1, i_2, \ldots, i_n)$ be an ordering of players in $N$. We denote by $\Theta_N$ the set of all such orderings and, for all $S \subseteq N$, we denote by $\Theta_S$ the set of all ordering of players in the subcoalition $S$. A marginal worth vector of the game $v$ relative to $\theta$, $m^\theta(v)$, is defined as

$$m^\theta_{i_k}(v) := v(\{i_1\}) \quad \text{and } m^\theta_{i_k}(v) := v(\{i_1, \ldots i_k\}) - v(\{i_1, \ldots i_{k-1}\}), \quad \text{for all } k = 1, \ldots, n.$$

It is well-known (Shapley (1971), Ichiishi (1981)) that a game is convex if and only any marginal worth vector of the game is in its core.

$$v \text{ is convex } \Leftrightarrow \; m^\theta(v) \in C(v), \text{ for all } \theta \in \Theta_N.$$

We say that $x \in \mathbb{R}^N$ lexicographically precedes $y \in \mathbb{R}^N$ with respect to $\theta$, $x \prec^\theta_\ell y$, if $x_1 < y_1$ or there exists $k \in \{1, 2, \ldots, n\}$ such that $x_{i_r} = y_{i_r}$ for all $r = 1, \ldots, k-1$ and $x_{i_k} < y_{i_k}$. The *lexmin* solution over the core of a game $v$ relative to $\theta \in \theta^N$ is defined as the (unique) payoff vector $\ell^\theta(v) \in \mathbb{R}^N$ that lexicographically precedes any other vector in the core of the game $v$, i.e. $\ell^\theta(v) \prec^\theta_\ell x$ for all $x \in C(v)$.

A formula for easy computing the lexmin solution of an arbitrary game is not available for the general case. Nevertlheless, we can define recursively a formula to obtain a payoff vector (we call it the *max-payoff vector*) that, whenever it is in the core, it coincides with the lexmin solution. To define it, given $\theta = (i_1, i_2, \ldots, i_n) \in \Theta_N$, let $P^\theta_{i_k} := \{i_1, \ldots, i_{k-1}\}$ be the set of predecessors of $i_k$ and let $F^\theta_{i_k} := \{i_k, \ldots, i_n\}$ be the set of its followers. The max-payoff vector $x^\theta(v) \in \mathbb{R}^N$ of $v$ relative to $\theta$ is defined as

$$\begin{aligned} x^\theta_{i_k}(v) &:= \max_{Q \subseteq P^\theta_{i_k}} \{v(\{i_k\} \cup Q) - x^\theta(v)(Q)\}, \quad \text{for all } k = 1, \ldots, n-1, \text{ and} \\ x^\theta_{i_n}(v) &:= v(N) - x^\theta(v)(N \setminus \{i_n\}). \end{aligned}$$

By definition, any max-payoff vector associated to $\theta = (i_1, \ldots, i_n) \in \Theta^N$, satisfies

$$x^\theta(v)(S) \geq v(S), \text{ for all } S \subseteq \{i_1, i_2, \ldots, i_{n-1}\}. \tag{2}$$

## 3. Characterization Theorem

In this section we prove the characterization of the convexity of a cooperative by means of the coincidence of its steady bargaining set with its core.

**Theorem 1.** *Let $v \in \mathcal{G}^N$ be a superadditive game. Then, the following statements are equivalent:*

*1. $\mathcal{SB}(v) = C(v)$,*

*2. $v$ is a convex game.*

Proof. *1.→ 2.*) Let $v$ be a superadditive game such that $\mathcal{SB}(v) = C(v)$. Let us first prove the following claim:

**Claim 1.** $x^\theta(v) = \ell^\theta(v)$, for all $\theta \in \Theta^N$

Since $v$ is superadditive then $\mathcal{SB}(v) \neq \varnothing$ and so, by hypothesis, $\mathcal{SB}(v) = C(v) \neq \varnothing$. Thus, the lexmin vector $\ell^\theta(v)$ is well-defined for any $\theta \in \Theta_N$.

Let $\theta = (i_1, i_2, \dots, i_n) \in \Theta_N$ be an arbitrary ordering of players. If $\ell^\theta_{i_k}(v) = x^\theta_{i_k}(v)$, for all $k = 1, \dots, n-2$, then $\ell^\theta(v) = x^\theta(v)$. To check this, first note that, by the reduced game property[1] of the core, $C(r^{\ell^\theta(v)}_{\{i_{n-1}, i_n\}}(v)) \neq \varnothing$, since $\ell^\theta(v) \in C(v)$. Note that $\ell^{(i_{n-1}, i_n)}_{i_{n-1}}(r^{\ell^\theta(v)}_{\{i_{n-1}, i_n\}}(v)) = \ell^\theta_{i_{n-1}}(v)$, since otherwise, either $\ell^\theta(v)$ would not be a core element of $v$, or $\ell^\theta(v)$ would not be the lexmin solution. Moreover, the minimum core payoff to player $i_{n-1}$ in this two-player reduced game is $\ell^{(i_{n-1}, i_n)}_{i_{n-1}}(r^{\ell^\theta(v)}_{\{i_{n-1}, i_n\}}(v)) = r^{\ell^\theta(v)}_{\{i_{n-1}, i_n\}}(v)(\{i_{n-1}\}) = \max_{Q \subseteq P^\theta_{i_{n-1}}} \{v(\{i_{n-1}\} \cup Q) - \ell^\theta(v)(Q)\} = \max_{Q \subseteq P^\theta_{i_{n-1}}} \{v(\{i_{n-1}\} \cup Q) - x^\theta(v)(Q)\} = x^\theta_{i_{n-1}}(v)$, where the third equality holds since $\ell^\theta_i(v) = x^\theta_i(v)$ for all $i \in P^\theta_{i_{n-1}}$. Hence, $x^\theta_{i_{n-1}}(v) = \ell^\theta_{i_{n-1}}(v)$ and, by efficiency, $\ell^\theta_{i_n}(v) = x^\theta_{i_n}(v)$.

Let us suppose now there exists an ordering of players $\theta = (i_1, i_2 \dots, i_n) \in \Theta_N$ such that $\ell^\theta(v) \neq x^\theta(v)$. By the above reasoning, there exists $k \in \{1, 2, \dots, n-2\}$ such that

(i) $\ell^\theta_{i_k}(v) \neq x^\theta_{i_k}(v)$, and

(ii) if $k > 1$ $\ell^\theta_{i_r}(v) = x^\theta_{i_r}(v)$, for all $r = 1, \dots, k-1$.

Let us remark that $\ell^\theta_{i_k}(v) > x^\theta_{i_k}(v)$ since, otherwise, $\ell^\theta_{i_k}(v) < x^\theta_{i_k}(v) = v(\{i_k\} \cup Q) - x^\theta(v)(Q) = v(\{i_k\} \cup Q) - \ell^\theta(v)(Q)$ for some $Q \subseteq \{i_1, \dots, i_{k-1}\}$. But in this case $\ell^\theta(Q \cup \{i_k\})(v) < v(Q \cup \{i_k\})$ getting a contradiction with $\ell^\theta(v) \in C(v)$.

Hence, let us take $\varepsilon \in \mathbb{R}$ such that

$$0 < \varepsilon < \min\{\ell^\theta_{i_k}(v) - x^\theta_{i_k}(v)\}$$

Furthermore, we define the payoff vector $\alpha \in \mathbb{R}^N$ as $\alpha_{i_k} := \ell^\theta_{i_k}(v) - \varepsilon$ and $\alpha_i := \ell^\theta_i(v)$, otherwise. Let us remark that $\alpha(N) < v(N)$. Now, let us construct the excess game $(F^\theta_{i_{k+1}}, w_\alpha)$ as follows:

$$\begin{aligned} w_\alpha(\varnothing) &:= 0, \\ w_\alpha(R) &:= \max_{Q \subseteq P^\theta_{i_{k+1}}} \{v(R \cup Q) - \alpha(R \cup Q)\} \quad \text{for all } \varnothing \neq R \subseteq F^\theta_{i_{k+1}}. \end{aligned}$$

Notice $w_\alpha(F^\theta_{i_{k+1}}) = v(N) - \alpha(N) = \varepsilon$ and $w_\alpha(R) \leq \epsilon$, for all $R \subsetneq F^\theta_{i_{k+1}}$. Moreover, for any arbitrary player $i \in F^\theta_{i_{k+1}}$, the value $v(\{i\}) - \ell^\theta_i$ turns out to be a lower bound for the

[1]Given a game $(N, v)$, a preimputation $x \in \mathbb{R}^N$ and a coalition $T \subseteq N$, the reduced game of $v$ on $T$ at $x$ is the game $(T, r^T_x(v))$ defined as $r^T_x(v)(T) = v(N) - x(N \setminus T)$ and $r^T_x(v)(R) = \max_{Q \subseteq N \setminus T}\{v(R \cup Q) - x(Q)\}$, for all $R \subsetneq T$. The reduced game property of the core implies that if $x \in C(v)$ then $x_T \in C(r^T_x(v))$.

marginal contribution of player $i$ to any subset of players $R \subseteq F^{\theta}_{i_{k+1}} \setminus \{i\}$. To check it, let us suppose $w_{\alpha}(R) = v(R \cup Q) - \alpha(R \cup Q)$ for some $Q \subseteq P^{\theta}_{i_{k+1}}$. Then,

$$\begin{aligned} w_{\alpha}(R \cup \{i\}) &\geq v(R \cup \{i\} \cup Q) - \alpha(R \cup \{i\} \cup Q) = v(R \cup \{i\} \cup Q) - \alpha_i - \alpha(R \cup Q) \\ &= v(R \cup \{i\} \cup Q) - \ell^{\theta}_i(v) - \alpha(R \cup Q) + v(\{i\}) - v(\{i\}) \\ &= [v(R \cup \{i\} \cup Q) - v(\{i\})] - \alpha(R \cup Q) + v(\{i\}) - \ell^{\theta}_i(v) \\ &\geq v(R \cup Q) - \alpha(R \cup Q) + v(\{i\}) - \ell^{\theta}_i(v) = w_{\alpha}(R) + v(\{i\}) - \ell^{\theta}_i(v) \end{aligned}$$

where the last inequality holds by the superadditivity of $v$. Therefore,

$$w_{\alpha}(R \cup \{i\}) - w_{\alpha}(R) \geq v(\{i\}) - \ell^{\theta}_i(v)$$

and the claim is proved. Now, choose an element of the prekernel of this game, say $\delta \in \mathcal{PK}(F^{\theta}_{i_{k+1}}, w_{\alpha})$. By Peleg and Südholter (2007), Theorem 5.6.1, we know that, for any player $i \in F^{\theta}_{i_{k+1}}$ and for any element in the prekernel, his payoff is bounded below by $m_i(w_{\alpha}) = \min_{S \subseteq N \setminus \{i\}} \{w_{\alpha}(S \cup \{i\}) - w_{\alpha}(S)\}$. Therefore, by the claim proved above

$$v(\{i\}) - \ell^{\theta}_i(v) \leq m_i(w_{\alpha}) \leq \delta_i, \text{ for all } i \in F^{\theta}_{i_{k+1}} \tag{3}$$

Then, define the payoff-vector $x \in \mathbb{R}^N$ as

$$x_{i_r} := \begin{cases} \alpha_{i_r} & \text{if } r \in \{1, \ldots, k\} \\ \alpha_{i_r} + \delta_{i_r} & \text{if } r \in \{k+1, \ldots, n\} \end{cases}.$$

Notice that $x$ is an efficient vector since $x(N) = v(N)$. Moreover $x_i = \ell^{\theta}_i(v) \geq v(\{i\})$, for all $i \in P^{\theta}_{i_k}$, since $\ell^{\theta}(v) \in C(v)$, $x_{i_k} = \ell^{\theta}_{i_k}(v) - \varepsilon \geq v(\{i_k\})$, where the last equality follows from the definition of $\varepsilon$, and $x_i = \ell^{\theta}_i(v) + \delta_i \geq v(\{i\})$, for all $i \in F^{\theta}_{i_{k+1}}$, where the last equality follows from (3).

However, $x \notin C(v)$ since otherwise $x \prec^{\theta}_{\ell} \ell^{\theta}(v)$ as $x_{i_r}(v) = \ell^{\theta}_{i_r}(v)$ for all $r = 1, \ldots, k-1$ and $x_{i_k} < \ell^{\theta}_{i_k}(v)$. We claim $x \in \mathcal{SB}(v)$.

Let $S \subseteq N$ such that $v(S) - x(S) > 0$. First,we claim $F^{\theta}_{i_{k+1}} \not\subseteq S$, since otherwise $\delta(F^{\theta}_{i_{k+1}}) = \varepsilon$ and $v(S) - x(S) = v(S) - \alpha(S) - \delta(F^{\theta}_{i_{k+1}}) \leq v(S) - \ell^{\theta}(v)(S) + \varepsilon - \delta(F^{\theta}_{i_{k+1}}) = v(S) - \ell^{\theta}(S) \leq 0$. Secondly, note $S \cap F^{\theta}_{i_{k+1}} \neq \varnothing$. Otherwise, i.e. $S \subseteq P^{\theta}_{i_{k+1}}$, let us consider two cases: if $i_k \notin S$, then $v(S) - x(S) = v(S) - \ell^{\theta}(v)(S) \leq 0$, where the last inequality follows since $\ell^{\theta}(v)$ is in the core of $v$; if $i_k \in S$ then $x(S) = \ell^{\theta}_{i_k}(v) - \varepsilon + \ell^{\theta}(v)(S \setminus \{i_k\}) > x^{\theta}_{i_k}(v) + \ell^{\theta}(v)(S \setminus \{i_k\}) = x^{\theta}(v)(S) \geq v(S)$, where the strict inequality follows from the definition of $\varepsilon$, the last equality follows from the the way we have selected $\theta$ and the last inequality by (2).

Therefore, let $i \in S \cap F^{\theta}_{i_{k+1}}$ and $j \in F^{\theta}_{i_{k+1}} \setminus S$ such that $s^{w_\alpha}_{ji}(\delta) \geq s^{w_\alpha}_{j'i}(\delta)$ for all $j' \in F^{\theta}_{i_{k+1}} \setminus S$. Hence,

$$\begin{aligned} 0 < v(S) - x(S) &= v(S) - \alpha(S) - \delta(S \cap F^{\theta}_{i_{k+1}}) \\ &\leq w_\alpha(S \cap F^{\theta}_{i_k+1}) - \delta(S \cap F^{\theta}_{i_{k+1}}) \leq s^{w_\alpha}_{ij}(\delta) = s^{w_\alpha}_{ji}(\delta) \\ &= w_\alpha(R) - \delta(R) = v(R \cup Q) - \alpha(R \cup Q) - \delta(R), \end{aligned}$$

for some $R \subseteq F^{\theta}_{i_{k+1}}$, with $j \in R$ and $i \notin R$, and for some $Q \subseteq P^{\theta}_{i_{k+1}}$. If we denote $M = R \cup Q$, we get

$$0 < v(S) - x(S) \leq v(M) - \alpha(M) - \delta(R) = v(M) - x(M),$$

where $i \in S \setminus M \neq \varnothing$. Moreover, $j \in M \setminus S$. Finally to prove that $S \cap M \neq \varnothing$, let us suppose to the contrary that $S \cap M = \emptyset$. Hence, by the superadditivity of the game $v$ we have

$$\begin{aligned} 0 < v(S) - x(S) &< v(S) - x(S) + v(M) - x(M) \leq v(S \cup M) - x(S \cup M) \\ &= w_\alpha((S \cup M) \cap F^{\theta}_{i_{k+1}}) - \delta((S \cup M) \cap F^{\theta}_{i_{k+1}}). \end{aligned} \tag{4}$$

Moreover, notice $F^{\theta}_{i_{k+1}} \setminus (S \cup M) \neq \varnothing$; otherwise $(S \cup M) \cap F^{\theta}_{i_{k+1}} = F^{\theta}_{i_{k+1}}$ and, by (4), we would obtain $w_\alpha(F^{\theta}_{i_{k+1}}) - \delta(F^{\theta}_{i_{k+1}}) > 0$, getting a contradiction with the fact that the vector $\delta$ is efficient in the game $\omega_\alpha$.

Let $j' \in F^{\theta}_{i_{k+1}} \setminus (S \cup M)$, then by (4)

$$\begin{aligned} s^{w_\alpha}_{j'i}(\delta) &= s^{w_\alpha}_{ij'}(\delta) \geq w_\alpha((S \cup M) \cap F^{\theta}_{i_{k+1}}) - \delta((S \cup M) \cap F^{\theta}_{i_{k+1}}) \\ &= v(S \cup M) - x(S \cup M) > v(M) - x(M) = s^{w_\alpha}_{ji}(\delta), \end{aligned}$$

which contradicts the way we have selected $j$. Hence, we conclude $S \cap M \neq \varnothing$. Therefore $x$ is in the the steady bargaining set but not in the core of the game , which contradicts the initial hypothesis. Therefore $x^{\theta}(v) = \ell^{\theta}(v)$ for all $\theta \in \Theta^N$ and Claim 1 is proved.

$\Diamond$

Taking this result into account let $v$ be a superadditive game with $\mathcal{SB}(v) = C(v)$ but let us suppose to the contrary that the game $v$ is not convex. Hence, there exists a subgame $(T, v_T)$, $T \subseteq N$, which is not convex, while all its proper subgames are convex, i.e. $(R, v_R)$ is convex for all $R \subseteq T$, $R \neq T$. By the superadditivity of the game $v$, $|T| \geq 3$. Let us denote $T = \{i_1, \ldots, i_t\}$. Moreover, and without lost of generality, we may suppose that player $i_t \in T$ is the one that breaks the snowballing convexity condition since his marginal contribution to the coalition $T \setminus \{i_t\}$ is strictly smaller than his marginal contribution to some subcoalition of $T \setminus \{i_t\}$. By the convexity of the subgame $(T \setminus \{i_t\}, v_{T \setminus \{i_t\}})$ this subcoalition is of size

$|T|-2$. Without loss of generality, let us assume that this coalition is $T \setminus \{i_{t-1}, i_t\}$ and it holds that

$$\begin{aligned} v(T) - v(T \setminus \{i_t\}) &< v(T \setminus \{i_{t-1}\}) - v(T \setminus \{i_{t-1}, i_t\}) \\ &= \max_{Q \subseteq \{i_1, i_2, \ldots, i_{t-1}\}} \{v(Q \cup \{i_t\}) - v(Q)\}. \end{aligned} \tag{5}$$

By Claim 1 we know that all the max-payoff vectors $x^\theta(v) \in C(v)$, for all $\theta \in \Theta^N$. Hence, if $|T| = n$ or $n \leq 3$, then $x^\theta(v) = m^\theta(v)$, for all $\theta \in \Theta_N$ and the proof is done since we have got $m^\theta(v) \in C(v)$, for all $\theta \in \Theta_N$. Therefore, let us assume from now on that $n \geq 4$ and $|T| \leq n-1$.

Let $\theta = (i_1, i_2, \ldots, i_{t-1}, i_t, \ldots, i_n)$ be an arbitrary ordering of $N$ where players of coalition $T$ enter first. Consider the max-payoff vector associated to $\theta$, $x^\theta(v)$. Since the subgame $v_{T \setminus \{i_t\}}$ is convex we have

$$\begin{array}{lll} x^\theta_{i_1}(v) & = m^\theta_{i_1}(v) & = v(\{i_1\}) \\ x^\theta_{i_2}(v) & = m^\theta_{i_2}(v) & = v(\{i_1, i_2\}) - v(\{i_1\}) \\ \vdots & & \vdots \\ x^\theta_{i_{t-1}}(v) & = m^\theta_{i_{t-1}}(v) & = v(\{i_1, i_2, \ldots, i_{t-1}\}) - v(\{i_1, i_2, \ldots, i_{t-2}\}) \\ & & = v(T \setminus \{i_t\}) - v(T \setminus \{i_{t-1}, i_t\}). \end{array} \tag{6}$$

Notice the vector $x^\theta(v)$ satisfies

$$x^\theta(v)(T \setminus \{i_t\}) = v(T \setminus \{i_t\}) \text{ and} \tag{7}$$

$$x^\theta(v)(T \setminus \{i_{t-1}, i_t\}) = v(T \setminus \{i_{t-1}, i_t\}). \tag{8}$$

Moreover, by the definition of max-payoff vector, we have

$$\begin{aligned} x^\theta_{i_t}(v) &= \max_{Q \subseteq T \setminus \{i_t\}} \{v(\{i_t\} \cup Q) - x^\theta(v)(Q)\} \leq \max_{Q \subseteq T \setminus \{i_t\}} \{v(\{i_t\} \cup Q) - v(Q)\} \\ &= v(T \setminus \{i_{t-1}\}) - v(T \setminus \{i_{t-1}, i_t\}) = v(T \setminus \{i_{t-1}\}) - x^\theta(v)(T \setminus \{i_{t-1}, i_t\}), \end{aligned} \tag{9}$$

where the inequality holds since $x^\theta(v) \in C(v)$, the second equality by (5) and the last equality by (8). From (9) we deduce $x^\theta(v)(T \setminus \{i_{t-1}\}) \leq v(T \setminus \{i_{t-1}\})$, but, since $x^\theta(v) \in C(v)$, we obtain

$$x^\theta(v)(T \setminus \{i_{t-1}\}) = v(T \setminus \{i_{t-1}\}). \tag{10}$$

Hence, we have

$$x^\theta_{i_t}(v) = x^\theta(v)(T \setminus \{i_{t-1}\}) - x^\theta(v)(T \setminus \{i_{t-1}, i_t\}) = v(T \setminus \{i_{t-1}\}) - v(T \setminus \{i_{t-1}, i_t\}), \tag{11}$$

where the second equality follows by (8) and (10). Combining (11) and (5) we have

$$x^\theta_{i_t}(v) = v(T \setminus \{i_{t-1}\}) - v(T \setminus \{i_{t-1}, i_t\}) > v(T) - v(T \setminus \{i_t\}) \geq v(\{i_t\}), \tag{12}$$

where the last inequality follows from the superadditivity of the game $v$. Combining (6) and (5) we obtain

$$x^\theta_{i_{t-1}}(v) = v(T \setminus \{i_t\}) - v(T \setminus \{i_{t-1}, i_t\}) > v(T) - v(T \setminus \{i_{t-1}\}) \geq v(\{i_{t-1}\}). \tag{13}$$

Finally, by (7), (11) and (5) it easily follows

$$\begin{aligned} x^\theta(v)(T) &= x^\theta(v)(T \setminus \{i_t\}) + x^\theta_{i_t}(v) = v(T \setminus \{i_t\}) + v(T \setminus \{i_{t-1}\}) - v(T \setminus \{i_{t-1}, i_t\}) \\ &> v(T \setminus \{i_t\}) + v(T) - v(T \setminus \{i_t\}) = v(T). \end{aligned} \tag{14}$$

Furthermore, we claim that if $x^\theta(v)(S) = v(S)$, for some $\varnothing \neq S \subsetneq T$, then

$$\text{there exists } i \in T \setminus S \text{ such that } x^\theta_i(v) > v(\{i\}). \tag{15}$$

To prove the claim, notice, by (14), $x^\theta(v)(T) > v(T)$. Hence, if $v(S) = x^\theta(v)(S)$, we have $x^\theta(v)(T \setminus S) = x^\theta(v)(T) - x^\theta(v)(S) > v(T) - v(S) \geq \sum_{i \in T \setminus S} v(\{i\})$, where the last inequality follows from the superadditivity of the game $v$ and proves (15).

Now, based upon the above max-payoff vector $x^\theta(v)$, we construct a new payoff vector $x$ outside the core but in the steady bargaining set reaching in this way a contradiction with the initial hypothesis $\mathcal{SB}(v) = C(v)$. To obtain the new vector, we first lower some components of the max-payoff vector $x^\theta(v)$ belonging to coalition $T$ by an appropriate epsilon, and, in a second step, we rise all the components of $N \setminus T$ in order to recover the original efficiency $v(N)$. For this purpose let us construct the set $\mathcal{I} \subseteq T$ by the following finite procedure:

**step 0:** Set $\mathcal{I}_0 = \{i_{t-1}, i_t\}$.

**step r:** If there exists[2] $\varnothing \neq S \subsetneq T$ such that $\mathcal{I}_{r-1} \subseteq S$ and $x^\theta(v)(S) = v(S)$ then, by (15), there exists[3] $i \in T \setminus S$ such that $x^\theta_i(v) > v(\{i\})$. Then, define $\mathcal{I}_r = \mathcal{I}_{r-1} \cup \{i\}$ and go to step $r + 1$. Otherwise define $\mathcal{I} = \mathcal{I}_{r-1}$ and stop.

Note that $|\mathcal{I}| \geq 2$ since players $i_{t-1}$, $i_t \in \mathcal{I}$. Let us define the following vector $\alpha \in \mathbb{R}^N$:

$$\alpha_i := \begin{cases} x^\theta_i(v) - \varepsilon, & \text{if } i \in \mathcal{I}, \\ x^\theta_i(v), & \text{if } i \in N \setminus \mathcal{I}, \end{cases} \tag{16}$$

where $\varepsilon \in \mathbb{R}$ is such that

$$0 < n \cdot \varepsilon < \min_{S \subseteq N : x^\theta(v)(S) > v(S)} \{x^\theta(v)(S) - v(S)\}. \tag{17}$$

Now to recover the original efficiency $v(N)$ let us consider the game $(N \setminus T, w_\alpha)$ defined as

$$w_\alpha(R) := \max_{Q \subseteq T} \{v(R \cup Q) - \alpha(R \cup Q)\}, \text{ for all } \varnothing \neq R \subseteq N \setminus T, \tag{18}$$

[2]If there is more than one coalition $S$, choose one of them.
[3]If there is more than one player, choose one of them.

and $w_\alpha(\varnothing) := 0$. We claim that the worth of coalitions in the game $w_\alpha$ is at most $|\mathcal{I}| \cdot \varepsilon$. That is, for any $R \subseteq N \setminus T$,

$$w_\alpha(R) \leq |\mathcal{I}| \cdot \epsilon. \tag{19}$$

To prove it, let $R$ be an arbitrary coalition in $N \setminus T$, $R \subseteq N \setminus T$. Then $w_\alpha(R) = v(R \cup Q) - \alpha(R \cup Q)$ for some $Q \subseteq T$. Since $x^\theta(v) \in C(v)$ we have $x^\theta(v)(R \cup Q) \geq v(R \cup Q)$. If $x^\theta(v)(R \cup Q) > v(R \cup Q)$ and by the definition of the parameter $\varepsilon$ – see (17) – we obtain

$$\begin{aligned} v(R \cup Q) - \alpha(R \cup Q) \quad &= v(R \cup Q) - x^\theta(v)(R \cup Q) + |\mathcal{I} \cap Q| \cdot \varepsilon \\ &\leq v(R \cup Q) - x^\theta(v)(R \cup Q) + n \cdot \varepsilon \\ &< v(R \cup Q) - x^\theta(v)(R \cup Q) + x^\theta(v)(R \cup Q) - v(R \cup Q) \\ &= 0 \leq |\mathcal{I}| \cdot \varepsilon, \end{aligned}$$

On the other hand, if $x^\theta(v)(R \cup Q) = v(R \cup Q)$ and so $v(R \cup Q) - \alpha(R \cup Q) = v(R \cup Q) - x^\theta(v)(R \cup Q) + |\mathcal{I} \cap Q| \cdot \varepsilon \leq |\mathcal{I}| \cdot \varepsilon$, as we want to prove. Moreover, the worth of the grand coalition $N \setminus T$ is

$$w_\alpha(N \setminus T) = |\mathcal{I}| \cdot \varepsilon. \tag{20}$$

To see it just take $Q = T$ in the definition of $w_\alpha(N \setminus T)$ and apply (19).

Furthermore, let $i \in N \setminus T$ and $R \subseteq N \setminus (T \cup \{i\})$ and suppose that $w_\alpha(R) = v(R \cup Q) - \alpha(R \cup Q)$, for some $Q \subseteq T$. Then the marginal contribution of player $i$ to coalition $R$ is bounded below by $v(\{i\}) - x_i^\theta(v)$

To check it notice that

$$\begin{aligned} w_\alpha(R \cup \{i\}) \quad &\geq v(R \cup \{i\} \cup Q) - \alpha(R \cup \{i\} \cup Q) \\ &= v(R \cup \{i\} \cup Q) - \alpha_i - \alpha(R \cup Q) + v(\{i\}) - v(\{i\}) \\ &= v(R \cup \{i\} \cup Q) - v(\{i\}) - \alpha(R \cup Q) + v(\{i\}) - \alpha_i \\ &\geq v(R \cup Q) - \alpha(R \cup Q) + v(\{i\}) - x_i^\theta(v) = w_\alpha(R) + v(\{i\}) - x_i^\theta(v). \end{aligned}$$

where the last inequality holds by the superadditivity of $v$ and the last equality since we are supposing $w_\alpha(R) = v(R \cup Q) - \alpha(R \cup Q)$. Hence

$$w_\alpha(R \cup \{i\}) - w_\alpha(R) \geq v(\{i\}) - x_i^\theta(v) \tag{21}$$

Now, take $\delta \in \mathcal{PK}(N \setminus T, w_\alpha)$. By Peleg and Südholter (2007), Theorem 5.6.1, we know that, for any player $i \in N \setminus T$ and for any element in the prekernel, his payoff is bounded below by $m_i(w_\alpha) = \min_{S\subseteq N\setminus\{i\}}\{w_\alpha(S\cup\{i\}) - w_\alpha(S)\}$ and bounded above by $M_i(w_\alpha) = \max_{S\subseteq N\setminus\{i\}}\{w_\alpha(S\cup\{i\}) - w_\alpha(S)\}$. Therefore, by (21), we have

$$v(\{i\}) - x_i^\theta(v) \leq m_i(w_\alpha) \leq \delta_i, \text{ for all } i \in N\setminus T. \tag{22}$$

Taking this into account, define the vector $x \in \mathbb{R}^N$ as follows:

$$x_i := \begin{cases} \alpha_i + \delta_i, & \text{for all } i \in N\setminus T, \\ \alpha_i, & \text{for all } i \in T, \end{cases} \tag{23}$$

where $\delta \in \mathcal{PK}(w_\alpha)$ is an arbitrary element of the prekernel of $(N\setminus T, w_\alpha)$.

We claim that the vector $x \in \mathbb{R}^N$ satisfies: (i) it is an imputation of $(N, v)$, (ii) it does not belong to the core of $(N, v)$ and (iii) it belongs to the steady bargaining set of $(N, v)$. In this way, we will finish the proof of the implication.

To check claim (i), this is $x \in I(v)$, first notice $x(N) = \alpha(N) + \delta(N\setminus T) = x^\theta(v)(N) - |\mathcal{I}|\cdot\varepsilon + |\mathcal{I}|\cdot\varepsilon = v(N)$. Moreover, for any $i \in N$, if $i \in \mathcal{I}$ then $x_i = \alpha_i = x_i^\theta(v) - \varepsilon > v(\{i\})$ just by the definition of $\varepsilon$; if $i \in T\setminus\mathcal{I}$ then $x_i = \alpha_i = x_i^\theta(v) \geq v(\{i\})$ since $x^\theta(v) \in C(v)$; finally if $i \in N\setminus T$ then, by (22), $x_i = x_i^\theta(v) + \delta_i \geq x_i^\theta(v) + [v(\{i\}) - x_i^\theta(v)] = v(\{i\})$.

To check claim (ii), this is $x \notin C(v)$, note that $v(T\setminus\{i_t\}) - x(T\setminus\{i_t\}) = v(T\setminus\{i_t\}) - x^\theta(v)(T\setminus\{i_t\}) + (|\mathcal{I}|-1)\cdot\varepsilon = (|\mathcal{I}|-1)\cdot\varepsilon > 0$, where the second equality comes from (7).

For proving Claim (iii), $x \in \mathcal{SB}(v)$, let us first remark some specific properties of the vector $x \in \mathbb{R}^N$. Let $S \subseteq T \subseteq N$ be a coalition in $T$ with positive excess at $x$, that is $v(S) - x(S) > 0$, then: (a) the coalition $S$ has a nonempty intersection with coalition $\mathcal{I}$; (b) the max-payoff vector $x^\theta(v)$ attains at coalition $S$ the worth $v(S)$. To check property (a), this is $S\cap\mathcal{I} \neq \varnothing$, when $v(S) - x(S) > 0$, let us suppose to the contrary that $S\cap\mathcal{I} = \varnothing$. Then we have

$$v(S) - x(S) = v(S) - x^\theta(v)(S) \leq 0,$$

where the equality follows since $\mathcal{I}\cap S = \varnothing$.

To check property (b), this is

$$x^\theta(v)(S) = v(S), \text{ whenever } v(S) - x(S) > 0 \text{ and } S \subseteq T, \tag{24}$$

let us assume to the contrary that $x^\theta(v)(S) \neq v(S)$. Since $x^\theta(v) \in C(v)$, this means $x^\theta(v)(S) > v(S)$, and then

$$v(S) - x(S) \quad = v(S) - x^\theta(v)(S) + |S\cap\mathcal{I}|\cdot\varepsilon \leq 0,$$

where the last inequality follows – see (17) – by the definition of $\varepsilon$.

To end the proof let us prove $x \in \mathcal{SB}(v)$. To check this, let $S \subseteq N$ such that $v(S) - x(S) > 0$. We show there exists a coalition $M \subseteq N$ such that $M\cap S \neq \varnothing$, $M\setminus S \neq \varnothing$, $S\setminus M \neq \varnothing$ and $v(M) - x(M) \geq v(S) - x(S)$. We consider the following cases:

(case 1) $S \subseteq T$. We know by 24 that $x^\theta(v)(S) = v(S)$.

(case 1.1) if $i_{t-1} \in S$ and $i_t \notin S$, then set $M = T\setminus\{i_{t-1}\}$. Notice $i_{t-1} \in S\setminus M$, $i_t \in M\setminus S$ and $S\cap M \neq \varnothing$, since $|S| \geq 2$. Moreover, by (10), $v(S) - x(S) \leq (|\mathcal{I}|-1)\cdot\epsilon = v(M) - x(M)$. By definition, $x \in \mathcal{SB}(v)$.

(case 1.2) if $i_{t-1} \notin S$ and $i_t \in S$, then set $M = T \setminus \{i_t\}$. Notice $i_t \in S \setminus M$, $i_{t-1} \in M \setminus S$ and $S \cap M \neq \varnothing$, since $|S| \geq 2$. Moreover, by (7), $v(S) - x(S) \leq (|\mathcal{I}| - 1) \cdot \epsilon = v(M) - x(M)$. By definition, $x \in \mathcal{SB}(v)$.

(case 1.3) if $i_{t-1} \in S$ and $i_t \in S$, recall, by (14), that $S \neq T$.

We claim, $\mathcal{I} \not\subseteq S$; otherwise, by (15) and since $x^\theta(v)(S) = v(S)$, it would exist $i \in T \setminus S$ with $x_i > v(\{i\})$ and so $i$ would belong to $\mathcal{I}$ reaching a contradiction. Hence, $v(S) - x(S) \leq (|\mathcal{I}| - 1) \cdot \varepsilon$. Then set $M = T \setminus \{i_t\}$ and notice $v(S) - x(S) \leq |\mathcal{I}| - 1) \cdot \varepsilon = v(M) - x(M)$ with $i_t \in S \setminus M$, $M \setminus S \neq \varnothing$, since $S \neq T$ and $i_t \in S$, and $i_{t-1} \in S \cap M$. By definition, $x \in \mathcal{SB}(v)$.

(case 1.4) $i_{t-1} \notin S$ and $i_t \notin S$. By property (a) discussed above we know $S \cap \mathcal{I} \neq \varnothing$. Let $k \in S \cap \mathcal{I}$ be the last player in $S$ to enter $\mathcal{I}$ according to the algorithm we use to construct $\mathcal{I}$. This means there exists $\varnothing \neq R \subsetneq T$ such that

$$\begin{array}{ll} (i) & v(R) = x^\theta(v)(R) \\ (ii) & ((S \cap \mathcal{I}) \setminus \{k\}) \cup \{i_{t-1}, i_t\} \subseteq R, \text{ and} \\ (iii) & x_k^\theta(v) > v(\{k\}) \text{ where } k \in T \setminus R. \end{array} \tag{25}$$

Hence

$$\begin{aligned} v(R) - x(R) &= v(R) - [x^\theta(v)(R) - |R \cap \mathcal{I}| \cdot \varepsilon] = |R \cap \mathcal{I}| \cdot \varepsilon \\ &= (|(R \setminus \{i_{t-1}, i_t\}) \cap \mathcal{I}| + 2) \cdot \varepsilon \geq (|(S \cap \mathcal{I}) \setminus \{k\}| + 2) \cdot \varepsilon \\ &= (|(S \cap \mathcal{I})| - 1 + 2) \cdot \varepsilon = (|(S \cap \mathcal{I})| + 1) \cdot \varepsilon > (|S \cap \mathcal{I}|) \cdot \varepsilon \\ &= v(S) - [x^\theta(v)(S) - |S \cap \mathcal{I}| \cdot \varepsilon] = v(S) - x(S), \end{aligned} \tag{26}$$

where the penultimate equality follows from (24).

If $R \cap S \neq \varnothing$ then set $M = R$ and notice that $i_t \in M \setminus S$, $k \in S \setminus M$ and, by (26), $v(M) - x(M) \geq v(S) - x(S)$.

In case $R \cap S = \varnothing$ we claim there exists $j \in S$ such that $x_j^\theta(v) = v(\{j\})$. To check it, first notice that $S \subseteq \{i_1, \ldots, i_{t-2}\}$ and

$$v(S) = x^\theta(v)(S) = m^\theta(v)(S) \geq v(S),$$

where the first equality follows by (24) since $v(S) - x(S) > 0$, and the last inequality since the subgame $v_{\{i_1, \ldots, i_{t-2}\}}$ is convex. Therefore, $m^\theta(v)(S) = v(S)$. Let $j \in S$ be the first player in $S$ according to the ordering $\theta = \{i_1, i_2, \ldots, i_n\}$, that is, take $j \in S$ such that $S \cap P_j^\theta = \varnothing$. Then,

$$v(\{j\}) \leq x_j^\theta(v) = m_j^\theta(v) = v(S) - \sum_{r \in S \setminus \{j\}} m_r^\theta(v) \leq v(S) - [v(S) - v(\{j\})] = v(\{j\}),$$

where the first inequality follows since $x^\theta(v) \in C(v)$ (see Claim 1), the first equality follows since $j \in S \subseteq \{1, 2, \ldots, t-2\}$, the second equality follows from $m^\theta(v)(S) = v(S)$ and the last inequality follows since the subgame $v_{\{i_1,\ldots,i_{t-2}\}}$ is convex and so $\sum_{r \in S\setminus\{j\}} m^\theta_r(v) \geq v(S) - v(\{j\})$. Hence, we conclude $x^\theta_j(v) = v(\{j\})$.

As a consequence, notice this player $j$ cannot be in $\mathcal{I}$, $j \notin \mathcal{I}$ (recall that $x^\theta_i(v) > v(\{i\})$, for all $i \in \mathcal{I}$), and it cannot coincide with player $k$, $j \neq k$, since $x^\theta_k(v) > v(\{k\})$ (see 25).

Finally set $M = R \cup \{j\}$ and notice that

$$v(\{j\}) + v(\{R\}) = x^\theta_j(v) + x^\theta(v)(R) = x^\theta(v)(M) \geq v(M) \geq v(\{j\}) + v(R),$$

where the last inequality follows from the superadditivity of the game and the penultimate one since $x^\theta(v) \in C(v)$. Hence, we have

$$\begin{aligned} v(M) - x(M) &= v(\{j\}) + v(R) - x_j - x(R) \\ &= v(\{j\}) + v(R) - x^\theta_j(v) - x(R) = v(R) - x(R) \geq v(S) - x(S), \end{aligned}$$

where the last inequality follows from (26). Finally notice $k \in S \setminus M$, $j \in M \cap S$ and $\{i_{t-1}, i_t\} \subseteq M \setminus S$. Therefore, by definition, $x \in \mathcal{SB}(v)$.

(case 2) $S \cap (N \setminus T) \neq \varnothing$. Notice $S \cap (N \setminus T) \neq N \setminus T$ since, otherwise, $v(S) - x(S) = v(S) - x^\theta(v)(S) + |S \cap \mathcal{I}| \cdot \epsilon - \delta(N \setminus T) = v(S) - x^\theta(v)(S) + |S \cap \mathcal{I}| \cdot \epsilon - |\mathcal{I}| \cdot \epsilon \leq v(S) - x^\theta(v)(S) \leq 0$. Therefore $|N \setminus T| \geq 2$.

Let now $i \in S \cap (N \setminus T)$ and $j \in (N \setminus T) \setminus S$ such that $s^{w_\alpha}_{ji}(\delta) \geq s^{w_\alpha}_{j'i}(\delta)$ for all $j' \in (N \setminus T) \setminus S$. Hence,

$$\begin{aligned} 0 < v(S) - x(S) &= v(S) - \alpha(S) - \delta(S \cap (N \setminus T)) \\ &\leq w_\alpha(S \cap (N \setminus T)) - \delta(S \cap (N \setminus T)) \\ &\leq s^{w_\alpha}_{ij}(\delta) = s^{w_\alpha}_{ji}(\delta) = w_\alpha(R) - \delta(R), \end{aligned}$$

where $R \subseteq N \setminus T$ is such that $j \in R$ but $i \notin R$. Moreover, by definition of $w_\alpha$ we have $w_\alpha(R) = v(R \cup Q) - \alpha(R \cup Q)$ for some $Q \subseteq T$.

Let $M = R \cup Q$ and first notice that

$$\begin{aligned} v(S) - x(S) &\leq w_\alpha(R) - \delta(R) = v(R \cup Q) - \alpha(R \cup Q) - \delta(R) \\ &= v(M) - x(M), \end{aligned} \tag{27}$$

Secondly, we claim $S \cap M \neq \emptyset$. Let us suppose on the contrary that $S \cap M = \emptyset$. Hence, by the superadditivity of the game

$$
\begin{aligned}
0 < v(M) - x(M) &< v(S) - x(S) + v(M) - x(M) \leq v(S \cup M) - x(S \cup M) \\
&\leq w_\alpha((S \cup M) \cap (N \setminus T)) - \delta((S \cup M) \cap (N \setminus T)),
\end{aligned} \tag{28}
$$

Moreover, notice $(N \setminus T) \setminus (S \cup M) \neq \varnothing$; otherwise $(S \cup M) \cap (N \setminus T) = N \setminus T$ and, by (28), we would obtain $0 = w_\alpha(N \setminus T) - \delta(N \setminus T) > 0$, getting a contradiction (the vector $\delta$ is efficient in the game $\omega_\alpha$). Let $j' \in (N \setminus T) \setminus (S \cup M)$; then by (28)

$$
s^{w_\alpha}_{j'i}(\delta) = s^{w_\alpha}_{ij'}(\delta) \geq w_\alpha((S \cup M) \cap (N \setminus T)) - \delta((S \cup M) \cap (N \setminus T)) > v(M) - x(M) = s^{w_\alpha}_{ji}(\delta),
$$

which contradicts the way we have selected $j$. Hence, we conclude $S \cap M \neq \varnothing$. Furthermore, since $i \in S \setminus M$, $j \in M \setminus S$ and $v(M) - x(M) \geq v(S) - x(S)$ (see 27) we conclude $x$ is in the steady bargaining set, i.e. $x \in \mathcal{SB}(v)$.

*2.→ 1.*) By superadditivity of $v$, recall that $C(v) \subseteq \mathcal{SB}(v) \subseteq \mathcal{MB}^*(v)$. Moreover, since the game is $v$ convex, it holds that $C(v) = \mathcal{MB}^*(v)$ (see Theorem 1 in Izquierdo and Rafels, 2012). Hence, we conclude $C(v) = \mathcal{SB}(v)$. □

As a consequence of Theorem 1 and the characterization result of Izquierdo and Rafels (2012) we get the following corollary.

**Corollary 1.** *Let $v \in \mathcal{G}^N$ be a superadditive game. Then, the following statements are equivalent:*

*1. $\mathcal{Z}^*(v) = C(v)$,*

*2. $v$ is a convex game.*

## 4. Conclusions

Bargaining sets faces the problem of distributing profits focusing on the negotiation (objections and counterobjections) between agents. Besides this, there are concepts of bargaining sets (e.g. Davis and Maschler (1963) or Shubik (1984)) that put the stress on the player who lead the objection. For these bargaining sets, there are examples of non-convex cooperative games for whom the core and the bargaining do coincide.

In Izquierdo and Rafels (2012), it has been already shown that a modification of the Mas-Colell bargaining set (Shimomura, 1994) has been useful to characterize the convexitiy of a game This notion of bargaining set considers objections and counterobjections as proposals made by a group rather than an action lead by an specific player. It is also important to remark that agents receives strictly better rewards in objections and counterobjections. Following this idea of group proposals and strictly positive incentives, we have proved in this paper that a more restricting concept of bargaining set (the modified Zhou bargaining set) also characterizes convex games within the class of superadditive games. The difference between both bargaining sets relies on the qualification of coalitions than might counter-object: while in the Mas-Colell version there are no restrictions on which are the coalitions $T$ that are allowed to react to an objection made by a coalition $S$, the Zhou's framework requires some conditions. First, there must be at least one player belonging to both coalitions; if not, $S \cap T = \varnothing$, and the counterobjection might be interpreted as a different objection rather than a proper counter-objection. Second, at least one player involved in the objection must not be involved in the counterobjection; if not, $S \subseteq T$, and the counterobjection might be interpreted as a reinforcement to the objection. Finally, the counterobjecting coalition must involved at least an agent not taking part in the objection; if not, $T \subseteq S$, but this fact might suggest that the original objection should be revised but not rejected. From the point of view of characterizing convex games, our result reveals that it is not so important if we just consider one, two, three or none of the above requirements for the counterobjecting coalitions. Objections and counter-objections made as a group and strictly positive incentives are the important keys to reach these results.

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