Cascading to Equilibrium: 
Hydraulic Computation of Equilibria in Resource Selection Games

Yannai A. Gonczarowski∗        Moshe Tennenholtz†

March 1, 2015

“All is water” —Thales, c. 585 BC

Abstract

Drawing intuition from a (physical) hydraulic system, we present a novel framework, constructively showing the existence of a strong Nash equilibrium in resource selection games (i.e., asymmetric singleton congestion games) with nonatomic players, the coincidence of strong equilibria and Nash equilibria in such games, and the invariance of the cost of each given resource across all Nash equilibria. Our proofs allow for explicit calculation of Nash equilibrium and for explicit and direct calculation of the resulting (invariant) costs of resources, and do not hinge on any fixed-point theorem, on the Minimax theorem or any equivalent result, on linear programming, or on the existence of a potential (though our analysis does provide powerful insights into the potential, via a natural concrete physical interpretation). A generalization of resource selection games, called resource selection games with I.D.-dependent weighting, is defined, and the results are extended to this family, showing that while resource costs are no longer invariant across Nash equilibria in games of this family, they are nonetheless invariant across all strong Nash equilibria, drawing a novel fundamental connection between group deviation and I.D.-congestion. A natural application of the resulting machinery to a large class of constraint-satisfaction problems is also described.

1 Introduction

Taking which highway would allow me to arrive at my workplace as fast as possible this morning? Using which computer server would my jobs be completed the soonest? Which router would deliver my packets with least latency? And even... shopping at which fashion store would make my clothes as unique as possible? All these, and more, are dilemmas faced by players in resource selection games — games in which each player’s payoff depends solely on the quantity of players choosing the same strategy (resource) as that player — and, more generally, in congestion games — games in which each player chooses a feasible strategy set (e.g., road segments), and, roughly, aims for its intersections with other chosen strategy sets to be small.

Congestion games (Rosenthal, 1973; Monderer and Shapley, 1996) have been central to the interplay between computer science and game theory (Nisan et al., 2007). These games arise naturally in many contexts and possess various desirable properties; in particular, both atomic congestion games (where each of the finitely many players has positive contribution to

∗Einstein Institute of Mathematics, Rachel & Selim Benin School of Computer Science & Engineering and Federmann Center for the Study of Rationality, The Hebrew University of Jerusalem, Israel; and Microsoft Research, E-mail: yannai@gonch.name.

†William Davidson Faculty of Industrial Engineering and Management, Technion — Israel Institute of Technology (work carried out while at Microsoft Research), E-mail: moshet@ie.technion.ac.il.
the congestion) and nonatomic congestion games (where the singular contribution of each of the continuum of players to the congestion is negligible) possess pure-strategy Nash equilibria. It is therefore only natural that topics of major interest in the field of Algorithmic Game Theory, such as the price of anarchy (Koutsoupias and Papadimitriou, 1999; Papadimitriou, 2001; Roughgarden and Tardos, 2002), which quantifies the social loss in Nash equilibria, have been introduced in the context of such games.

From a game-theoretic perspective, much additional effort has been spent in introducing important extensions of congestion games in the context of atomic congestion games, and in particular in the context of atomic resource selection games. Such efforts include the study of strong equilibria (stability against group deviations) in various such games (Holzman and Law-Yone, 1997, 2003; Epstein, Feldman, and Mansour, 2009); player-specific congestion games (Milchtaich, 1996), where cost functions may be player-specific; and I.D.-congestion games (Monderer, 2006), where the cost of a resource may depend on the identity (rather than merely on the quantity) of the players using it. Interestingly, the challenges of dealing with such major extensions, and in particular with group deviations and with cost that depends on the identity of other players using the same resource, have not been tackled in the nonatomic case, nor have any tools been offered in order to deal with such extensions. This lacuna is especially puzzling given the centrality of nonatomic congestion games in presenting flow and communication networks, central to computer science, as well as in presenting large markets and economies, central to macroeconomics.

In this paper, we deal with such extensions of the study of nonatomic congestion games, and address the related challenges by introducing a novel computational approach, which we call hydraulic computing. Using this approach, which draws intuition from a (physical) hydraulic system, in Section 3 we show the existence of strong equilibria in nonatomic resource selection games, that strong equilibria and Nash equilibria coincide in such games, and that the cost of each given resource is invariant across all Nash equilibria. Generalizing to I.D.-congestion games, in Section 4 we show the existence of strong equilibria in resource selection games where the cost of a resource depends on the identity of the players using it, and that the cost of each given resource in such games, while interestingly no longer invariant across Nash equilibria, is nonetheless invariant across all strong equilibria, drawing a novel fundamental connection between group deviation and I.D.-congestion. Our theoretical treatment does not hinge on any fixed-point theorem, on the Minimax theorem or any equivalent result, on linear programming, or on the existence of a potential (though it does provide powerful insights into the potential when a potential exists, via a natural concrete physical interpretation — see Section 3.6), and is the first to provide explicit formulation of the resource cost obtained in equilibria of congestion games. Looking beyond the realm of games, in Section 5 we show that our framework can serve as a constructive substitute to linear-programming approaches in other contexts as well, such as that of Hall’s marriage theorem and many constraint-satisfaction problems generalizing it.

Atomic Games Atomic congestion games with finitely many players have been introduced by Rosenthal (1973), who has shown the existence of a pure-strategy Nash equilibrium in such games. Monderer and Shapley (1996) later introduced potential games, and showed that they coincide with congestion games (with finitely many players). Holzman and Law-Yone (1997, 2003) studied strong equilibria in congestion games and characterized settings in which a strong equilibrium exists; in particular, they showed that the set of strong equilibria and the set of

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1While coalitional deviations in large-scale economies, such as nonatomic games, require massive coordination to involve coalitions of nonnegligible measure, we note that such deviations are by no means purely theoretic; indeed, modern cloud-based social application such as Waze (e.g., for congestion games on graphs) allow for centralized coordination of deviations of immense scales.
Nash equilibria coincide in resource selection games (with finitely many players). Milchtaich (1996) extended congestion games to player-specific congestion games, in which players’ costs are player-specific, and showed the existence of Nash equilibrium in player-specific resource selection games (with finitely many players). Monderer (2006) introduced a general class of I.D.-congestion games, which are congestion games in which the cost of a resource depends also on the identity of the players using it. On the verge between a finite and a countable cardinality of players, Milchtaich (2000) showed the existence and uniqueness of Nash equilibria (uniqueness of strategies, not only of costs of resources) in large replications of generic finite resource selection games, as well as in the limit countable-player game.

Nonatomic Games Nonatomic congestion games, such as those that we study, have been very popular in the computer science context; see, e.g., Roughgarden and Tardos (2002) and Nisan et al. (2007). In such games, Beckmann, McGuire, and Winsten (1956) have shown the existence of Nash equilibrium and the uniqueness of costs of resources in Nash equilibria under certain differentiability assumptions; a general theorem by Schmeidler (1973) implies the existence of Nash equilibrium under the assumption of continuity (rather than differentiability) of the cost functions.

Analogies to hydraulic systems have sporadically appeared in the economics literature in the past, but seem to be anecdotal in nature. At the end of the nineteenth century, Fisher (1892) built a complex hydraulic apparatus for calculating Walrasian equilibrium prices in competitive markets with up to three goods. Kaminsky (2000) (see also Aumann, 2002) uses an analogy to a simple hydraulic system to find the nucleolus of a small special set of cooperative games. While of their own interest, we note that it does not seem that any “deep” connection exists between the ad hoc hydraulic analogies in these papers and our hydraulic framework.

Contributions The main contributions of this paper are:

1. Introducing the hydraulic computing analysis framework.
2. Providing an explicit formula (rather than an iterative procedure of computation) for calculating the cost of resources in equilibria of nonatomic resource selection games.
3. Proving the invariance of resource costs across equilibria, without any assumption of differentiability or even continuity.
4. Proving the existence of strong equilibria in resource selection games with I.D.-dependent weighting with continuous cost functions, and the invariance of resource costs across strong equilibria in such games regardless of continuity (showing by example that these costs are not invariant across all Nash equilibria), drawing a novel fundamental connection between group deviation and I.D.-congestion.
5. Applying hydraulic computing in lieu of linear-programming methods in a large class of constraint-satisfaction problems, such as generalizations of finding a perfect marriage and proving Hall’s theorem.

2 Notation

Definition 1 (Notation).

- (Naturals). We denote the strictly-positive natural numbers by \( \mathbb{N} \triangleq \{1, 2, 3, \ldots\} \).
• ([n]). For every \( n \in \mathbb{N} \), we define \([n] \triangleq \{1, 2, \ldots, n\}\).

• (Reals). We denote the real numbers by \( \mathbb{R} \).

• (Nonnegative Reals). We denote the nonnegative reals by \( \mathbb{R}_\geq \triangleq \{ r \in \mathbb{R} \mid r \geq 0 \} \).

• (Maximizing Arguments). Given a set \( S \) and a function \( f : S \to \mathbb{R} \) that attains a maximum value on \( S \), we denote the set of arguments in \( S \) maximizing \( f \) by \( \arg \max_{s \in S} f(s) \triangleq \{ s \in S \mid f(s) = m \} \), where \( m \triangleq \max_{s \in S} f(s) \).

• (Simplex). For a set \( R \subseteq S \), we define:

\[
\Delta^R = \{ s \in [0,1]^S \mid \sum_{j \in R} s_j = 1 \text{ and } \forall j \in S \setminus R : s_j = 0 \}.
\]

(The set \( S \) will be clear from context.)

• (Nonempty Subsets). For a set \( S \), we define \( 2_{\neq \emptyset}^S \triangleq 2^S \setminus \emptyset \) — the nonempty subsets of \( S \).

**Definition 2** (Plateau Height). Let \( f : \mathbb{R} \to \mathbb{R} \) be a nondecreasing function. We say that \( h \in \mathbb{R} \) is a plateau height of \( f \) if there exist \( x \neq y \in \mathbb{R} \) s.t. \( f(x) = f(y) = h \).

**Remark 1.** A strictly increasing function has no plateau heights.

3 \ “Standard” Resource Selection Games

3.1 Setting

**Definition 3** (Resource Selection). Let \( n \in \mathbb{N} \). An \( n \)-resource selection game is defined by a pair \( ((f_j)_{j=1}^n; (\mu^R)_{R \in 2_{\neq \emptyset}^n}) \), where \( f_j : \mathbb{R}_\geq \to \mathbb{R} \) is a nondecreasing function for every \( j \in [n] \), and \( \mu^R \in \mathbb{R}_\geq \) for every \( R \in 2_{\neq \emptyset}^n \).

In a resource selection game, each \( R \in 2_{\neq \emptyset}^n \) indicates a player type. Each player of type \( R \) may consume only from resources in \( R \); the total mass of the continuum of players of type \( R \) is \( \mu^R \). For each resource \( j \in [n] \), \( f_j \) is a function from the consumption amount of this resource to the cost of consuming from the resource. We now formally define these concepts.

**Definition 4** (Consumption Profile; \( \mu^*_j \); \( h^*_j \); Nash Equilibrium). Let \( G = ((f_j)_{j=1}^n; (\mu^R)_{R \in 2_{\neq \emptyset}^n}) \) be a resource selection game.

• A consumption (strategy) profile in \( G \) is a function \( s : 2_{\neq \emptyset}^n \to \mathbb{R}_\geq^n \) s.t. \( s(R) \in \mu^R \cdot \Delta^R \) for every \( R \in 2_{\neq \emptyset}^n \).

• Given a consumption profile \( s \) in \( G \), for every \( j \in [n] \) we define \( \mu^*_j \triangleq \sum_{R \in 2_{\neq \emptyset}^n} s_j(R) \) — the load on (i.e., total consumption from) resource \( j \). Furthermore, we define \( h^*_j \triangleq f_j(\mu^*_j) \) — the cost of resource \( j \).

• A Nash equilibrium in \( G \) is a consumption profile \( s \) s.t. for every \( R \in 2_{\neq \emptyset}^n \) and for every \( k \in \text{supp}(s(R)) \) and \( j \in R \), it is the case that \( h^*_k \leq h^*_j \).

**Example 1** (Home Internet / Cellular Market). Consider a scenario in which the resources are internet service providers (ISPs), and the players are customers on the market for home internet. (Alternatively, one could think of resources as cellular operators, and of players as customers on the market for cellular service.) Each customer may choose between the providers available in this customer’s geographical area, and would like to get a connection with the largest bandwidth
possible given this constraint. \( \mu^R \) in this case is proportional to the amount of customers with possible ISPs \( R \), and for each \( j \in [n] \), we choose \( f_j \) s.t. \( h^R_j = f_j(\mu^R_j) \) is inversely proportional to the effective bandwidth of each subscriber of ISP \( j \), when there are \( \mu^R_j \) subscribers to this ISP.

If each ISP has the same total (i.e., overall) bandwidth, then the speed of the connection of a single customer subscribed to an ISP is inversely proportional to this ISP’s number of subscribers, and so obtaining the fastest connection possible is equivalent to subscribing to a least-subscribed-to ISP, and so this case is captured by setting \( f_j \triangleq \text{id} \) for every \( j \in [n] \). Generalizing, we may imagine that, say, some ISPs may have different total bandwidths than others (which may be captured by setting \( f_j(x) \triangleq x/b_j \), where \( b_j \) is the total bandwidth of ISP \( j \)), or that some ISPs may even purchase some additional total bandwidth as their subscriber pool grows; in either scenario, in order to surf with greatest speed, each customer would prefer to subscribe not necessarily to a least-subscribed-to ISP (i.e., one with minimal \( \mu^R_j \)), but rather to an ISP from which the customer would receive the fastest connection, i.e., one with minimal \( h^R_j = f_j(\mu^R_j) \).

The study of stability against group deviations was initiated by Aumann (1959), who considered deviations from which all deviators gain. Recently, the CS literature considers a considerably stronger solution concept, according to which a deviation is considered beneficial even if only some of the participants in the deviating coalition gain, as long as none of the participants lose (see, e.g., Rozenfeld and Tennenholtz, 2006). While stability against the classical all-gaining coalition deviation is termed strong equilibrium, this more demanding concept is referred to as super-strong equilibrium; there are very few results showing its existence in nontrivial settings. We now formally define both concepts.

**Definition 5** (Strong / Super-Strong Nash Equilibrium). Let \( G = ((f_j)_{j=1}^n; (\mu^R)_{R \in 2^{[n]}}) \) be a resource selection game and let \( s \) be a Nash equilibrium in \( G \). For every \( R \in 2^{[n]} \setminus \emptyset \) with \( \mu^R > 0 \), let \( h^R \triangleq h^R_j \) for every \( j \in \text{supp}(s(R)) \). (\( h^R \) is well defined by definition of Nash equilibrium.)

- \( s \) is a strong Nash equilibrium if there exists no consumption profile \( s' \neq s \) s.t. for every \( R \in 2^{[n]} \setminus \emptyset \) and \( k \in \text{supp}(s'(R)) \) s.t. \( s'_k(R) > s_k(R) \), it is the case that \( h^R_k < h^R \).

- \( s \) is a super-strong Nash equilibrium if there exists no consumption profile \( s' \) s.t. for every \( R \in 2^{[n]} \setminus \emptyset \) and \( k \in \text{supp}(s'(R)) \) s.t. \( s'_k(R) > s_k(R) \), it is the case that \( h^R_k' \leq h^R \), with \( h^R_k < h^R \) for at least one pair of type \( R \in 2^{[n]} \setminus \emptyset \) and resource \( k \in \text{supp}(s'(R)) \).

**Remark 2.** Every super-strong Nash equilibrium is a strong Nash equilibrium.

### 3.2 Formal Results

In Sections 3.4 and 3.5, we constructively prove the following three theorems and corollary.

**Theorem 1** (\( \exists \) Strong Nash Equilibrium). Let \( G = ((f_j)_{j=1}^n; (\mu^R)_{R \in 2^{[n]}}) \) be a resource selection game. If \( f_1, \ldots, f_n \) are continuous, then a strong Nash equilibrium exists in \( G \).

**Theorem 2** (Resource Costs are Independent of Nash Equilibrium). Let \( G \) be an \( n \)-resource selection game. \( h^R_j = h^R_{j'} \) for every \( j \in [n] \) and every two Nash equilibria \( s, s' \) in \( G \).

\( ^2 \)The minimal coalition that can cause a deviation from \( s \) to \( s' \) is the coalition containing, for every type \( R \in 2^{[n]} \setminus \emptyset \) and resource \( k \in \text{supp}(s'(R)) \) s.t. \( s'_k(R) > s_k(R) \), a mass of \( s'_k(R) - s_k(R) \) players of type \( R \) who consume from \( k \) in \( s' \) but not in \( s \).

\( ^3 \)We require that no member of the minimal coalition described in Footnote 2 lose, but allow the gaining member to be any player, i.e., even one whose consumption is not necessarily changed. (Indeed, we do not require that \( s'_k(R) > s_k(R) \) for the pair \( R \) and \( k \) for which \( h^R_k < h^R \).)
Corollary 1. Let \( G = ((f_j)_{j=1}^n; (\mu^R)_{R \in 2^{[n]} R \neq \emptyset}) \) be a resource selection game.

a. (Players are Indifferent between Nash Equilibria). \( h^s_k = h^{s'}_{k'} \) for every \( k \in \text{supp}(s(R)) \) and \( k' \in \text{supp}(s'(R)) \), for every \( R \in 2^{[n]} \) \( R \neq \emptyset \) and every two Nash equilibria \( s, s' \) in \( G \).

b. (Resource Loads are Independent of Nash Equilibrium). If no two of \( (f_j)_{j=1}^n \) share any plateau height, then \( \mu^s_j = \mu^{s'}_j \) for every \( j \in [n] \) and every two Nash equilibria \( s, s' \) in \( G \).

Theorem 3 (All Nash Equilibria are Strong / Super-Strong). Let \( G = ((f_j)_{j=1}^n; (\mu^R)_{R \in 2^{[n]} R \neq \emptyset}) \) be a resource selection game.

a. All Nash equilibria in \( G \) are strong.

b. If \( h^s_j \) is not a plateau height of \( f_j \) for each \( j \in [n] \) in any/every Nash equilibrium \( s \), then all Nash equilibria in \( G \) are super-strong.

We significantly generalize all of these results in Section 4. Theorem 1 significantly strengthens a corollary of Schmeidler (1973) that shows existence of a (not-necessarily-strong) Nash equilibrium for continuous cost functions; Theorem 2 strengthens a result of Beckmann et al. (1956) that requires differentiability. We emphasize that unlike Schmeidler’s proof, our proof of Theorem 1 allows for explicit calculation of an equilibrium. Indeed, none of our proofs hinge on any fixed-point theorem, on the Minimax theorem or any equivalent result, on the existence of a potential, or on linear programming. (Moreover, as shown in Section 5, our results can even replace nonconstructive techniques such as linear programming in certain problems that are traditionally viewed as unrelated to games.) Similarly, unlike Beckmann et al.’s proof, our proof of Theorem 2 gives an explicit formula for \( h^s_j \) for every \( j \in [n] \). See Section 6 for a discussion of the benefits of such explicit formulations.

3.3 Construction and Hydraulic Intuition

In this section, we intuitively survey the construction underlying our results, as a prelude to the formal analysis given in Sections 3.4 and 3.5. We start with the special case in which \( f_j = \text{id} \) for every \( j \in [n] \), i.e., \( h^s_j = \mu^s_j \) for every \( j \in [n] \) and consumption profile \( s \). Our hydraulic construction for this case, from which our analysis draws intuition, consists of a system of containers, interconnected balloons, and pistons, which is illustrated in Figs. 1 and 2.\(^4\)

The intuition underlying our results draws from a number of key observations regarding this construction (we generalize and formalize these observations in Sections 3.4 and 3.5):

I. If the pistons in a set \( S \) (e.g., \( S = \{1, 3\} \) or \( S = \{4\} \)) of containers stop simultaneously, then at the time of their stopping, no liquid under them can escape to any container in which the piston has not yet stopped (or else it would do so and the piston above it would not stop).

II. By Observation I, and as pistons that stop later in time stop at a lower height, in the resulting consumption profile no player type has any incentive to deviate, and so it is indeed a Nash equilibrium.

\(^4\)This construction significantly generalizes an ad hoc construction that appears as a secondary auxiliary result in a previous discussion paper by the authors (Gonczarowski and Tennenholtz, 2014). That discussion paper deals with combining a highly restricted form of resources selection games (with degenerate strategy sets, i.e., where \( \mu^R = 0 \) for all sets \( R \) but those of a very specific form) with facility location games, drawing conclusions regarding the possibility of false appearance of collusion in internet markets and various food markets.
A set of 5 open-top hollow box containers, corresponding, from left to right, to resources 1,...,5, respectively. For each player type $R$ with $\mu^R > 0$ (each such type is assigned a distinct color in the illustration), a balloon, or plastic bag, is placed in each container $j \in R$. Balloons corresponding to the same type $R$ are connected via a thin tube emerging from a narrow slit (not shown) running vertically along the back of each container, and are jointly filled with $\mu^R$ liquid.

Pistons are simultaneously lowered through the top sides of all the containers. As the piston in the first container reaches the balloons in this container, they are compressed, causing the balloons connected to them (i.e., the purple balloon in the third container, the blue balloons in the third and fourth containers, and the light blue balloons in the second, third, and fifth containers) to inflate.

As the piston in the third container reaches the balloons in this container, they start to compress as well, causing, e.g., the interconnected blue balloon in the fourth container to inflate even faster.

At a certain point in time, no balloon in the first or third containers can be compressed any further, as all the liquid in these containers that could have escaped to other containers has been depleted. The pistons in the first and third containers halt, and the remaining pistons continue their descent.

At some later point in time, no balloon in the fourth container can be compressed any further, as all the liquid in this container that could have escaped to any container other than the first or the third ones has been depleted.

Eventually, no balloon in the second or fifth containers can be compressed any further, and the process concludes.

Figure 1: (See Fig. 2 for an animated version, which unfortunately cannot be printed.) Illustration of the construction underlying our analysis, for $n = 5$ and for $f_j = \text{id}$ for every $j \in [5]$. E.g., as exactly 87.5% of the red liquid in Fig. 1(f) is in the second container and the remaining 12.5% is in the fifth container, the strategy for the player type corresponding to the red color (i.e., $R = \{2, 5\}$) in the (super-)strong Nash equilibrium that we construct is $0.875 \cdot \mu^R$ consumption, solely from resource 4.
III. If we initially distribute the liquid of each “color” (among the various balloons corresponding to this color) according to some Nash equilibrium (e.g., if we initially distribute the liquid as in Fig. 1(f)), then the liquid distribution would not change during the entire process of descent of the pistons. Therefore, each Nash equilibrium may be attained from some initial liquid distribution.

IV. After pistons 1 and 3 (in Fig. 1) stop, we effectively start over, solving a 3-resource (2, 4, 5) selection game between all player types whose original acceptable resources were not merely resource 1 and/or 3.

V. In Fig. 1, pistons 1 and 3 are the earliest to stop. By Observation I above, no part of the liquid under these pistons when they stop can ever, regardless of the initial liquid distribution, end up in any container other than 1 or 3. Therefore, these pistons always stop having under them at least the liquid that is under them in Fig. 1(d), and accordingly at least at the height at which they stop in Fig. 1(d). By the same observation, the pistons stopping earliest always stop having under them solely liquid that cannot escape to any other container, and so, regardless of the initial liquid distribution, if this set were not pistons 1 and 3, then it would stop below the stopping height of pistons 1 and 3. Therefore, pistons 1 and 3 always stop earliest, and at the same height. Using Observation IV, an inductive argument can show that the height at which each piston stops (and the stopping order) is independent of the initial liquid distribution, and so by Observation III, $h_j^s$ for every $j \in [n]$ is independent of the choice of Nash equilibrium. Furthermore, by the same argument, each player always consumes from resources with the same $h_j^s$, independently of the choice of Nash equilibrium $s$. 

Figure 2: Animated version of Fig. 1; requires Adobe Reader. Click the ▶ button to start the animation.
We note that while the final piston heights (i.e., values of \( h^*_j \)) are independent of the initial distribution of liquid among connected balloons (i.e., of the choice of Nash equilibrium \( s \)), the final liquid distribution (i.e., players’ strategies) is not; in Fig. 1(f), e.g., any amount of light blue liquid may be transferred from the second to the fifth container “in exchange for” an identical amount of red liquid.\(^5\)

For the general case of arbitrary \( f_j \), we intuitively think of replacing the \( j \)th box container, for every \( j \in [n] \), with a container shaped so that whenever it is filled with any amount \( \mu_j \in \mathbb{R}_\geq \) of liquid, the resulting surface level would be precisely \( f_j(\mu_j) \). See Fig. 3 for an illustration. We emphasize that while the actual construction of such vessels requires differentiability (almost everywhere) of the cost functions \( f_j \), our formal proof of Theorem 1 only requires continuity of the cost functions, while our formal proofs of Theorems 2 and 3 and Corollary 1 do not require even that. We note that this continuity assumption (in Theorem 1) is in fact not superfluous; indeed, if even one of the cost functions is discontinuous, then a Nash equilibrium need not necessarily exist; see Fig. 4 for an example and an illustration.

3.4 Definitions for Formalizing the Observations from Section 3.3

Building upon the intuition of Section 3.3, we formally derive the results of Section 3.2 in Section 3.5, with proofs in Appendices A.1 and A.2. In this section, we review the formal definitions\(^5\)While this nonuniqueness seemingly contradicts a uniqueness theorem of Orda et al. (1993), we note that their setting in fact differs from ours; they deal with finitely many players with splittable demand, while we deal with a continuum of nonatomic players. We furthermore note that their analysis requires that the cost functions \( f_j \) be strictly increasing; in their setting, this guarantees uniqueness of both equilibrium loads \( \mu_j^* \) and consumptions \( s \), while in our setting, this assumption guarantees solely the uniqueness of equilibrium loads \( \mu_j^* \) (without this assumption, we have only uniqueness of equilibrium resource costs \( h^*_j \)); see Corollary 1(b).
underlying this derivation. Full proofs of all claims given below are given in Appendix A.1.

3.4.1 Communicating-Vessel Equalization

Let $S$ be the set of pistons stopping earliest during the process depicted in Fig. 1. Assume that when these pistons stop, the total amount of liquid in the respective containers is $\mu$. At what height did the pistons stop? In this section we formalize the answer to this question.

**Definition 6** (Nondecreasing Function to $\mathbb{R} \cup \{\text{undefined}\}$). Let $f : \mathbb{R}_+ \to \mathbb{R} \cup \{\text{undefined}\}$. We say that $f$ is nondecreasing if $f|_{f^{-1}(\mathbb{R})}$ is nondecreasing; i.e., if for every $\mu < \mu' \in \mathbb{R}_+$, if both $f(\mu) \in \mathbb{R}$ and $f(\mu') \in \mathbb{R}$, then $f(\mu) \leq f(\mu')$.

**Definition 7** (Communicating-Vessel Equalization). Let $m \in \mathbb{N}$ and let $f_1, \ldots, f_m : \mathbb{R}_+ \to \mathbb{R} \cup \{\text{undefined}\}$ be nondecreasing functions. We define the function Equalize$_{f_1, \ldots, f_m} : \mathbb{R} \to \mathbb{R} \cup \{\text{undefined}\}$ by

$$
\mu \mapsto \begin{cases} 
\exists \mu_1, \ldots, \mu_m \in \mathbb{R}_+ : \sum_{j=1}^m \mu_j = \mu & \text{if } f_1(\mu_1) = f_2(\mu_2) = \cdots = f_m(\mu_m) 
\text{in } \mathbb{R} \subseteq \mathbb{R} \cup \{\text{undefined}\} \\
\text{undefined} & \text{otherwise.}
\end{cases}
$$

**Remark 3** (Equalizing Multiple Identical Functions). If $f_1 = f_2 = \cdots = f_m$ and this function is defined on all $\mathbb{R}_+$, then Equalize$_{f_1, \ldots, f_m}(\mu) = f_1(\frac{\mu}{m})$.

For $f_1, \ldots, f_m : \mathbb{R}_+ \to \mathbb{R}$, one may intuitively think of Equalize$_{f_1, \ldots, f_m}(\mu)$ as exactly the answer to the question raised above: if $f_1, \ldots, f_m$ are the functions corresponding (see Fig. 3) to the containers of the pistons stopping earliest during the process depicted in Fig. 1, and if the total amount of liquid in the respective containers when these pistons stop is $\mu$, then Equalize$_{f_1, \ldots, f_m}(\mu)$ is the height at which these pistons stop; Equalize$_{f_1, \ldots, f_m}(\mu) = \text{undefined}$ if it is impossible that all these pistons simultaneously stop when the total amount of liquid in these containers is $\mu$. Alternatively and equivalently, if empty containers corresponding (see Fig. 3) to $f_1, \ldots, f_m$ are connected at their base and the resulting system of communicating vessels is jointly filled with $\mu$ liquid, then Equalize$_{f_1, \ldots, f_m}(\mu)$ is the resulting liquid surface level; see Fig. 5 for an illustration.

When two of the functions $f_1, \ldots, f_m$ share a plateau height (cf. Corollary 1(b)), then the liquid distribution $\mu_1, \ldots, \mu_m$ may not be well defined; see Fig. 6 for an illustration. Nonetheless, the following lemma shows that the resulting surface level Equalize$_{f_1, \ldots, f_m}$ is well defined, i.e., independent of the chosen liquid distribution $\mu_1, \ldots, \mu_m$.

**Lemma 1** (Equalization is Well Defined and Nondecreasing). Let $m \in \mathbb{N}$ and let $f_1, \ldots, f_m : \mathbb{R}_+ \to \mathbb{R} \cup \{\text{undefined}\}$ be nondecreasing functions. Equalize$_{f_1, \ldots, f_m}$ is a well-defined nondecreasing function from $\mathbb{R}_+$ to $\mathbb{R} \cup \{\text{undefined}\}$.

In the remainder of this section, we derive some additional properties of communicating-vessel equalization. The following lemma notes that “connecting a single vessel with itself” has no effect, while connecting several vessels may be done by first connecting subsets of these vessels into “intermediate vessels”, and only then connecting all “intermediate vessels” together; it is for the sake of the latter that we have allowed the functions $f_1, \ldots, f_m$ in Definition 7 to assume the value undefined.

**Lemma 2** (Composition of Equalizations). Let $m \in \mathbb{N}$ and let $f_1, \ldots, f_m : \mathbb{R}_+ \to \mathbb{R} \cup \{\text{undefined}\}$ be nondecreasing functions.

a. Equalize$_{f_1} \equiv f_1$. 

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6We allow the functions $f_1, \ldots, f_m$ to assume the value undefined for technical reasons that become apparent in Lemma 2 below. The reader may intuitively think of $f_1, \ldots, f_m$ as real functions until reaching that lemma.
Equalize $f_1, \ldots, f_6(\mu)$ equals the liquid surface level when the containers are jointly filled with $\mu$ liquid.

Equalize $f_1, \ldots, f_n(\mu) = \text{undefined}$, as no distribution of $\mu$ liquid between the containers results in an even liquid surface level across all containers (recall that the fifth container corresponds to the function $x + c$ for some constant $c$, and therefore if it is empty, then its liquid surface level is defined as the level $c$ of its bottom side).

Recall that Theorem 1 requires that $f_1, \ldots, f_n$ be continuous. (An example in which one of these functions is discontinuous and no Nash equilibrium exists was given in Fig. 4.) We therefore conclude this section with an analysis of the equalization of continuous functions.

**Definition 8** (Function to $\mathbb{R} \cup \{\text{undefined}\}$: Continuous / Defined on a Suffix of $\mathbb{R}_\geq$). Let $f : \mathbb{R}_\geq \to \mathbb{R} \cup \{\text{undefined}\}$.

- We say that $f$ is **continuous** if $f|_{f^{-1}(\mathbb{R})} : f^{-1}(\mathbb{R}) \to \mathbb{R}$ is continuous.
- We say that $f$ is **defined on a suffix of $\mathbb{R}_\geq$** if for every $\mu < \mu' \in \mathbb{R}_\geq$, if $f(\mu) \in \mathbb{R}$, then $f(\mu') \in \mathbb{R}$ as well.

**Lemma 3** (Equalization of Continuous Functions). Let $m \in \mathbb{N}$ and let $f_1, \ldots, f_m : \mathbb{R}_\geq \to \mathbb{R} \cup \{\text{undefined}\}$ be nondecreasing functions.

a. If at least one of $f_1, \ldots, f_m$ is continuous, then Equalize $f_1, \ldots, f_m$ is continuous.

b. If each of $f_1, \ldots, f_m$ is continuous and defined on a suffix of $\mathbb{R}_\geq$, then Equalize $f_1, \ldots, f_m$ is continuous and defined on a suffix of $\mathbb{R}_\geq$ as well.

**Remark 4** (Equalization of Lipschitz Functions). Let $m \in \mathbb{N}$ and let $f_1, \ldots, f_m : \mathbb{R}_\geq \to \mathbb{R} \cup \{\text{undefined}\}$ be nondecreasing functions. A proof virtually identical to that of Lemma 3(a) can be used to show that if at least one of $f_1, \ldots, f_m$ is Lipschitz, then Equalize $f_1, \ldots, f_m$ is Lipschitz with the same Lipschitz constant.

The following corollary shows that for continuous real functions, the only “reason” for their equalization to be undefined is of the type depicted in Fig. 5(b), i.e., an uneven bottom of the corresponding containers.
Figure 6: Equalization of two copies of the function from Fig. 3(f), via two distinct liquid distributions. Formally, when \( \mu > 2d \), there exists a continuum of pairs \( \mu_1, \mu_2 \in \mathbb{R}_+ \) s.t. \( \mu_1 + \mu_2 = \mu \) and \( \min\{\mu_1, d\} = \min\{\mu_2, d\} \). For all such \( \mu_1, \mu_2 \), it is nonetheless always the case that \( \min\{\mu_1, d\} = d = \min\{\mu_2, d\} \), and so \( \text{Equalize}_{f_1, f_2}(\mu) = d \) is well defined.

**Corollary 2.** Let \( m \in \mathbb{N} \) and let \( f_1, \ldots, f_m : \mathbb{R}_+ \to \mathbb{R} \) be nondecreasing continuous real functions. \( \text{Equalize}_{f_1, \ldots, f_m} \) is a real function iff \( f_1(0) = f_2(0) = \cdots = f_m(0) \).

### 3.4.2 An Explicit Formula for the Highest-Costing Resources and their Cost

Following the discussion in the previous section, if the set of highest-costing resources (highest-stopping pistons) is \( P \), then by Observation I from Section 3.3, we expect them to cost (stop at height) \( E_G(P) \), where \( E_G \) is defined as follows.

**Definition 9** \( (E_G) \). Let \( G = ((f_j)_{j=1}^n; (\mu^R)_{R \in 2^\{\neq \emptyset\}_R}) \) be a resource selection game. We define \( E_G(S) \triangleq \text{Equalize}_{f_{1:k} \in S} \left( \sum_{R \in 2^\{\neq \emptyset\}_R} \mu^R \right) \), for every \( S \in 2^{[n]} \).

A main challenge that remains before moving on to prove the results of Section 3.2, therefore, is to find an expression for \( P \), as given such an expression, we could find \( E_G(P) \) and proceed inductively via a formalization of Observation IV from Section 3.3. A natural first candidate for the role of \( P \) may be to take a set of resources with maximal \( E_G \), i.e., some element of \( \arg \max_{S \in 2^{[n]}_R} E_G(S) \), where the value undefined is here and henceforth treated as \( -\infty \) for comparisons by the Max operator. Noticing that for many natural choices of resource selection games \( G = ((f_j)_{j=1}^n; (\mu^R)_{R \in 2^\{\neq \emptyset\}_R}) \), the set of all such maximizing sets of resources \( \arg \max_{S \in 2^{[n]}_R} E_G(S) \) is closed under set union, and therefore contains a greatest element (namely, \( \bigcup \arg \max_{S \in 2^{[n]}_R} E_G(S) \)), a natural candidate for the role of \( P \) would be this greatest element. While for many natural choices of resource selection games \( G = ((f_j)_{j=1}^n; (\mu^R)_{R \in 2^\{\neq \emptyset\}_R}) \), this greatest element indeed exists and coincides with the set of highest-costing resources (indeed, this is the case when all \( f_j \) are strictly increasing), this need not generally be the case. To see that these need not coincide, consider the following example.

**Example 2** \( (\bigcup \arg \max_{S \in 2^{[n]}_R} E_G(S) \) is Not the Set of Highest-Costing Resources). Consider the game \( G = ((f_j)_{j=1}^n; (\mu^R)_{R \in 2^\{\neq \emptyset\}_R}) \), for \( n = 2 \), \( f_1 = \text{id}, f_2(x) = \min\{x, 2\} \), \( \mu^{1} = 1 \), \( \mu^{[2]} = 3 \), and \( \mu^{[1,2]} = 0 \). In this game, consumption of each player type \( \{i\} \) solely from resource \( i \) is the unique consumption profile and hence the unique Nash equilibrium — denote it by \( s \). Note that \( h^*_1 = 1 \) and \( h^*_2 = 2 \), and so \( 2 \) is the unique highest-costing resource, albeit \( \bigcup \arg \max_{S \in 2^{[n]}_R} E_G(S) = \bigcup \{2\} \cup \{1, 2\} = \{1, 2\} \). Indeed, in this case while the set of highest-costing resources \( P = \{2\} \) is an element in \( \arg \max_{S \in 2^{[n]}_R} E_G(S) = \{2\} \cup \{1, 2\} \), it is not the greatest element of this set.

In fact, as shown in the following example, the set \( \arg \max_{S \in 2^{[n]}_R} E_G(S) \) need not even contain a greatest element.

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Example 3 (arg Max_{S \subseteq [n]} E_G(S) Has No Greatest Element). Consider the game \( G = \left( (f_j)_{j=1}^n; (\mu^R)_{R \subseteq [n]} \right) \), for \( n = 3 \), \( f_1 = f_2 = \text{id} \), \( f_3(x) = \min\{x, 2\} \), \( \mu^{(1)} = \mu^{(2)} = 1 \), \( \mu^{(3)} = 3 \), and \( \mu^R = 0 \) for all nonsingleton \( R \in [n] \). It is easy to verify that arg Max_{S \subseteq [n]} E_G(S) = \{ \{3\}, \{1, 3\}, \{2, 3\} \}, and so this set contains no greatest element.

Removing any hope of representing \( P \) using some other function of arg Max_{S \subseteq [n]} E_G(S), the following example shows that the set of highest-costing resources cannot be inferred from arg Max_{S \subseteq [n]} E_G(S) alone.

Example 4 (arg Max_{S \subseteq [n]} E_G(S) Does Not Determine the Set of Highest-Costing Resources). Consider the game \( G = \left( (f_j)_{j=1}^n; (\mu^R)_{R \subseteq [n]} \right) \), for \( n = 2 \), \( f_1 = f_2 = \text{id} \), \( \mu^{(2)} = \mu^{(1, 2)} = 1 \), and \( \mu^{(1)} = 0 \). In this game, the unique Nash equilibrium \( s \) is for all players of type \( \{2\} \) to consume from resource 2 and for all players of type \( \{1, 2\} \) to consume from resource 1. Note that \( h^*_1 = 1 \) and \( h^*_2 = 1 \), and so the set of highest-costing resources is \( \{1, 2\} \). We note that arg Max_{S \subseteq [n]} E_G(S) = \{ \{2\}, \{1, 2\} \}, just as in Example 2, even though the set of highest-costing resources in that examples is different.

Examining Example 2, we note that while, indeed, the total mass of all players who cannot consume from any resource outside \{1, 2\} in that example, when “equalized” among the resources in \{1, 2\}, yields a “height” of \( 2 = \text{Max}_{S \subseteq [n]} E_G(S) = E_G(P) \), in fact no consumption profile corresponds to this equalization, as not enough of these players are allowed to consume from resource \( 1^7 \). To derive a general formula for the set of highest-costing resources, we therefore need to exclude such “problematic” sets of resources.

Definition 10 \( (M_G; D_G; P_G; h_G) \). Let \( G = \left( (f_j)_{j=1}^n; (\mu^R)_{R \subseteq [n]} \right) \) be a resource selection game. We define:

- \( M_G(S) \triangleq \left\{ S' \subseteq [n] \mid \forall \mu \leq \sum_{R \subseteq [n]} \mu^R : \text{Equalize}(\mu) \neq E_G(S) \right\} \subseteq [n] \setminus \{S\} \), for every \( S \in [n] \).

A set \( S \in M_G(S) \) is not allowed for consumption by enough players (out of those who can consume only from \( S \)) to create a consumption profile s.t. only consumers who can consume solely from \( S \) are allowed to consume from resource \( S \). In Example 2, \( M_G(\{1, 2\}) = \{\{1\}\} \) while \( M_G(\{1\}) = \emptyset \).

- \( D_G \triangleq \left\{ S \in [n] \mid E_G(S) \in \mathbb{R} \& M_G(S) = \emptyset \right\} \) — these are the sets termed “nonproblematic” above.

- \( P_G \triangleq \bigcup_{S \in D_G} \text{arg Max}_{S \in D_G} E_G(S) \) — we show below that this is precisely the set of highest-costing resources.

- \( h_G \triangleq \text{Max}_{S \in P_G} E_G(S) \) — we show below that this is precisely the cost of every resource in \( P_G \).

Remark 5. Let \( G = \left( (f_j)_{j=1}^n; (\mu^R)_{R \subseteq [n]} \right) \) be a resource selection game. We show in Appendix A.1 that in the cases that we study (i.e., where \( G \) has a Nash equilibrium or where \( f_1, \ldots, f_n \) are continuous), \( P_G \) is the greatest element of arg Max_{S \subseteq D_G} E_G(S), and so \( h_G = E_G(P_G) \). Furthermore, we show that in these cases \( h_G = \text{Max}_{S \subseteq [n]} E_G(S) \); i.e., \( D_G \) may be replaced by \( 2_{[n]} \) in the definition of \( h_G \). (But not in the definition of \( P_G \) if even one of \( f_j \) is not strictly increasing, by Example 2.)

Lemma 4. In every resource selection game \( G \), \( P_G \neq \emptyset \), and \( h_G \in \mathbb{R} \) is well defined.

\(^7\)If both \( f_1 \) and \( f_2 \) were strictly increasing, then this would imply that \( E_G(\{2\}) > E_G(\{1, 2\}) \); such an argument may be used to show that \( \bigcup \text{arg Max}_{S \subseteq [n]} E_G(S) \) indeed coincides with the set of highest-costing resources when all \( f_j \) are strictly increasing.
3.4.3 Resource Removal

The following definition will be useful when formalizing Observation IV from Section 3.3.

**Definition 11 (Resource Removal).** Let $G = (\{f_j\}_{j=1}^n; (\mu^R)_{R \in 2^{[n]}_{\neq \emptyset}})$ be a resource selection game and let $S \subseteq [n]$ be a subset of the resources in $G$.

- For every $R' \in 2^{[n] \setminus S}_{\neq \emptyset}$, we define $\mathcal{R}(R', G - S) : = \{ R \in 2^{[n]}_{\neq \emptyset} \mid R \setminus S = R' \} \subseteq 2^{[n]}_{\neq \emptyset} \setminus 2^S$ — the set of player types in $G$ for whom the allowed resources outside $S$ are precisely $R'$.\textsuperscript{8}

- We define $G - S : = (\{f_j\}_{j \in [n] \setminus S}; (\sum_{R \in \mathcal{R}(R', G - S)} \mu^R)_{R' \in 2^{[n] \setminus S}})$ — the $|[n] \setminus S|$-resource selection game obtained from $G$ by disallowing any consumption from resources in $S$ and removing all players who cannot consume from any resource outside $S$.

**Lemma 5 (Fundamental Properties of Resource Removal).** Let $G$ be a resource selection game.

a. $G - \emptyset = G$.

b. $G - S - S' = G - (S \cup S')$, for every two disjoint subsets $S, S'$ of the resources in $G$.

3.5 Formal Derivation of the Results of Section 3.2

In this section, we present our analysis formalizing the observations from Section 3.3 via the definitions of Section 3.4 and leading to the results of Section 3.2. Full proofs of all the results of this section are given in Appendix A.1; the subsequent proofs of the results of Section 3.2 are given in Appendix A.2.

3.5.1 Uniqueness and Strength

At the heart of our proof of Theorem 2 lies Lemma 6, formalizing Observations I and III through V from Section 3.3. We note that unlike Theorem 1, neither Lemma 6 nor Theorem 2 or 3 require the continuity of $f_1, \ldots, f_n$.

**Lemma 6 (Uniqueness of Highest-Costing Resources and their Cost).** Let $s$ be a Nash equilibrium in a resource selection game $G = (\{f_j\}_{j=1}^n; (\mu^R)_{R \in 2^{[n]}_{\neq \emptyset}})$, and let $P^* : = \arg \max_{j \in [n]} h_j$.

a. $P^* = P_G$.

b. $h_j^* = h_G$, for every $j \in P^*$.

c. $s_j(R) = 0$ for every $R \in 2^{[n]}_{\neq \emptyset} \setminus 2^{P^*}_{\neq \emptyset}$ and $j \in P^*$.

d. The function $s_j^* : 2^{[n]}_{\neq \emptyset} \setminus P^* \to \mathbb{R}^{[n]}_{\geq 0}$, defined by $s_j^*(R') : = \sum_{R \in \mathcal{R}(R', G - P^*)} s_j(R)$ for every $j \in [n] \setminus P^*$ and $R' \in 2^{[n]}_{\neq \emptyset} \setminus P^*$, constitutes a Nash equilibrium in the game $G - P^*$. Furthermore, $h_j^* = h_j^*$ for every $j \in [n] \setminus P^*$.

**Remark 6.** In Lemma 6, The r.h.s. of Parts a and b, and therefore also the quantifications in Parts b through d and the game defined using resource removal in Part d, are independent of the choice of $s$.

The proof of Theorem 2 using Lemma 6 is given in Appendix A.2. This proof effectively follows Algorithm 1, a succinct algorithm (based upon Lemma 6), which, if any Nash equilibrium
exists, directly and explicitly calculates $h^*_j$ for all $j$ in every Nash equilibrium $s$ (without the need to first calculate players’ strategies, which are dependent on $s$).

Full proofs of Corollary 1 and Theorem 3 are given in Appendix A.2. The former is based on Theorem 2 (as explained in Observation V from Section 3.3), and the latter on the analysis of Lemma 6, following and formalizing an extension of Observation II from Section 3.3. We conclude this section by demonstrating that, as suggested by the manner in which Theorem 3 is stated, a Nash equilibrium is not necessarily super-strong when the condition of Part b of this theorem (regarding the plateau heights of the cost functions) is not met.

**Example 5** (A Not-Super-Strong Equilibrium). Consider the game $G = ((f_j)_{j=1}^n; (\mu^R)_{R \in 2^{\{1,2\}}} \setminus \emptyset)$, for $n = 2$, $f_1 = \text{id}$, $f_2(x) = \min\{x, 3\}$, $\mu^{(1)} = 1$, $\mu^{(2)} = 2$, and $\mu^{(1,2)} = 3$. In this game, a (strong) Nash equilibrium $s$ is given by $\{1\} \mapsto (1,0)$, $\{2\} \mapsto (0,2)$, $\{1,2\} \mapsto (2,1)$. (Note that $h^*_2 = 3$ is a plateau height of $f_2$.) This Nash equilibrium is not super-strong, since a coalition of players of types $\{1\}$ and $\{1,2\}$ can deviate with $\{1\} \mapsto (1,0)$ (no change) and $\{1,2\} \mapsto (0,3)$ (more players consuming from resource 2), from which no coalition member is harmed, while coalition members of type $\{1\}$ benefit. A different deviation showing that this Nash equilibrium is not super-strong and worth mentioning is of a coalition consisting solely of players of type $\{1,2\}$, which can deviate with $\{1\} \mapsto (1,2)$ (by, e.g., some coalition members switching to consume from resource 2 instead of resource 1 while the others do not change strategies, or, e.g., by each player of type $\{1,2\}$ switching resources), from which no coalition member is harmed, while the coalition members consuming from resource 1 (whether they have actually changed strategies or not) benefit.

### 3.5.2 Existence

We proceed to the proof of existence of Nash equilibrium. A full proof of Theorem 1 is given in Appendix A.2. This proof formalizes Observations I and II from Section 3.3, effectively following the construction of Fig. 1 and showing that in each step, the pistons stopping are those computed in Algorithm 1. This is done using the following lemma, constructively showing, even in the absence of prior knowledge of existence of Nash equilibrium, that the liquid that by Lemma 6(c)

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*We emphasize that $R(R', G - S)$ is defined as a function of three parameters: $R'$, $G$, and $S$, rather than as a function of two parameters ($R'$ and $G - S$, the latter of which we have not yet defined). We use the notation $R(R', G - S)$ rather than $R(R', G, S)$ solely for readability.
should be under the pistons $P$ when they stop can be distributed appropriately among them, and that Algorithm 1 indeed finds the sets $P$ in decreasing order of stopping height.

**Lemma 7** ($P_G$ and $h_G$ are Viable as Highest-Costing Resources and their Cost). Let $G = ((f_j)_{j=1}^n; (\mu^R)_{R \in 2^{[n]}})$ be a resource selection game s.t. $f_1, \ldots, f_n$ are continuous.

a. (Liquid Distribution under $P_G$). There exists a consumption profile $s$ in the $|P_G|$-resource selection game $((f_j)_{j \in P_G}; (\mu^R)_{R \in 2^{P_G}})$, s.t. $h^s_j = h_G$ for every $j \in P_G$.

b. (Pistons Stopping Order). If $P_G \neq [n]$, then $h_G > h_{G-P_G}$.

We once again emphasize that none of the results presented in this paper hinge on any fixed-point theorem, on the Minimax theorem or any equivalent result, on linear programming, or on the existence of a potential. Nonetheless, as we show in the following section, our analysis provides powerful insights into the potential, via a natural concrete physical interpretation.

### 3.6 Abstract Game-Theoretic Potential as Physical Gravitational Potential

A popular nonconstructive method for proving the existence of Nash equilibrium in resource selection games (and, more generally, in congestion games) is to define an appropriate potential function, so named due to its properties, which resemble those of an abstract physical potential. This approach is due to Monderer and Shapley (1996), who used it in the context of atomic congestion games; see Nisan et al. (2007) for details regarding the generalization to nonatomic congestion games. In the notation of this paper, the proof defines the following scalar function of consumption profiles:

$$P^*(s) \triangleq \sum_{j \in [n]} \int_0^{\mu_j^s} f_j(x)dx,$$

and shows that when a single player deviates from one strategy to another, the change in this player’s cost equals, roughly speaking, the derivative of $P^*$ in the direction of the deviation. The conclusion (under certain assumptions) is that there exists a consumption profile minimizing $P^*$, and that this profile is therefore a Nash equilibrium; in fact, it can be shown that a consumption profile is a Nash equilibrium iff it minimizes $P^*$. A ramification of this result is that every construction that finds a Nash equilibrium in a resource selection game must minimize $P^*$; in particular, even though it does not explicitly aim to do so (indeed, $P^*$ does not appear in any of our proofs or definitions), so does our hydraulic construction. It is interesting to note, though, that our hydraulic construction does not merely minimize $P^*$ as a “side effect” of finding a potential, but also gives a natural physical interpretation to $P^*$, which justifies the name “potential function” not only abstractly, but also concretely.

The reader may recall from high-school physics class that the gravitational potential energy of a point mass of mass $m$ near the surface of the earth is given by $mgh$, where $h$ is the height of the mass, and $g$ is the standard acceleration due to gravity. More generally, the gravitational potential energy of a non-point-mass system may be expressed by the Riemann–Stieltjes integral $\int_{-\infty}^{\infty} gh \, dm(h)$, where $m(h)$ is the cumulative mass in the system up to height $h$. In our hydraulic system, we have

$$\int_{-\infty}^{\infty} gh \, dm(h) = g \sum_{j \in [n]} \int_0^{\mu_j^s} f_j(x)dx = g \cdot P^*(s),$$

and so $P^*$, up to a multiplicative constant, is precisely the gravitational potential energy of our system (which our construction therefore turns out to minimize). Perhaps more intuitive to

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9 As is customary, we ignore the negligible effect of small changes in $h$ on the value of $g$. 

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nonphysicists would be to reason not about the gravitational potential energy of our hydraulic system, but rather about the height of the center of mass of the system, given by:

$$\frac{1}{\mu} \int_{-\infty}^{\infty} h \, dm(h) = \frac{1}{\mu} \sum_{j \in [n]} \int_{0}^{\mu_j} f_j(x) \, dx = \frac{P^*(s)}{\mu},$$

where $\mu \triangleq \lim_{h \to \infty} m(h) = \sum_{R \in R} h^R$ is the total mass of the system. Once again, the height of the center of mass, which our construction turns out to minimize, equals $P^*$ up to a multiplicative constant.\(^{10}\)

We conclude this discussion by noting that the generalization of resource selection games studied in the next section also demonstrates the ability of hydraulic analysis to generalize beyond what can be proven regarding resource selection games via the game-theoretic potential approach.\(^{11}\) Indeed, even though the games studied in the next section are generally not potential games, our hydraulic construction naturally extends to solving them. (Indeed, in that generalized setting, in which the total mass of the system is no longer constant, our construction no longer necessarily minimizes the gravitational potential energy, nor the height of the center of mass, of the system.)

## 4 Resource Selection Games with I.D.-Dependent Weighting

In this section, we describe an extension of the results of Section 3 to a model where the cost of a resource may depend on the identity, rather than merely the quantity, of players using it. While such major extensions have been studied in the context of atomic games, no tools have been previously offered to tackle them in nonatomic settings. For $n, k \in \mathbb{N}$, an $n$-resource/$k$-player-type resource selection game with I.D.-dependent weighting is defined by a triple $((f_j)_{j=1}^n; (f^i_j)_{j \in [n]}; (R^i)_{i=1}^k)$, where $f_j : \mathbb{R}_+ \to \mathbb{R}$ is a nondecreasing function for every $j \in [n]$, $f^i_j : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function for every player-type/resource pair $(i, j) \in [k] \times [n]$, and $R^i \subseteq 2^{[n]}$ and $\mu^i \in \mathbb{R}_+$ for every player type $i \in [k]$. For each player type $i \in [k]$, $R^i$ specifies the set of resources from which this player type may consume, while $\mu^i$ is the amount to be consumed by all players of this type. As before, for each resource $j \in [n]$, $f^i_j$ is a function from the consumption amount of this resource to its cost. The newly introduced functions $f^i_j$ indicate the weighting of the consumption of player type $i$ from resource $j$ (see below).

A consumption (strategy) profile in this game is a function $s : [k] \to \mathbb{R}_+^{[n]}$ s.t. $s(i) \in \mu^i \cdot \Delta^{R^i}$ for every $i \in [k]$. Given a consumption profile $s$ in this game, for every $j \in [n]$ we define $\mu^i_j \triangleq \sum_{i=1}^{k} f^i_j(s_j(i))$ (note the newly introduced weighting) — the weighted load on (i.e., total weighted consumption from) resource $j$. As before, we define $h^i_j \triangleq f^i_j(\mu^i_j)$ for every $j \in [n]$ — the cost of resource $j$. A Nash equilibrium in this game is a consumption profile $s$ s.t. for every $i \in [k]$ and for every $\ell \in \supp(s(i))$ and $j \in R^i$, it is the case that $h^i_\ell \leq h^i_j$.

**Example 6** (Computing in the Cloud). Consider a scenario in which the resources are computer servers, and each of the many player wishes to run a relatively small computing job, where jobs corresponding to players of the same type are of a similar nature. A player of type $i \in [k]$

\(^{10}\) The reader is referred once again to the special case of our construction that is given in Gonczarowski and Tennenholtz (2014), which can be more easily and intuitively shown to minimize gravitation potential energy and height of center of mass.

\(^{11}\) One may say that with respect to resource selection games, hydraulic analysis has more potential than potential has.
may choose between the machines $R^i$, whose hardware is compatible with jobs of players of this type, and would like for her job to be completed as soon as possible given this constraint. $\mu^i$ in this case is proportional to the quantity of players of type $i$, and $f_j^i(x)$ is a linear function s.t. $f_j^i(x)$ is proportional to the number of cycles of machine $j$ required to compute $x$ jobs of players of type $i$. (The hardware of each machine may run jobs of some nature more efficiently than jobs of another nature; e.g., machine 2 may run image-processing jobs faster than text-analysis ones, while machine 3 may run the latter faster than the former.) For each $j \in [n]$, we choose $f_j$ s.t. $h_j^s = f_j(\mu_j^s)$ is proportional to the number of seconds required for $\mu_j^s$ cycles of machine $j$ to complete. (Assume that the resources of each machine are parallelized between its different users, so that their jobs are all completed at the same time.)

We note that by setting $f_j^i = \text{id}$ for all $i,j$, we obtain a resource selection game as in Section 3, and so this setting is a strict generalization of resource selection games as defined there. Intuitively, the hydraulic construction from Figs. 1 and 2 may be adapted to this generalized framework by inserting “compressors/expanders” into the tubes between balloons corresponding to the same player type. E.g., if $\mu^1 = \{1, 2\}$, $f_1^1(x) = x$ and $f_2^1(x) = 2x$, then the balloon system corresponding to player type 1 consists of two balloons, one in container 1 and the other in container 2, connected by a compressor/expander tube s.t. for each drop of liquid that enters the tube from the balloon in container 1, two drops exit into the balloon in container 2, and for every two drops of liquid that enter the tube from the balloon in container 2, one drop exits into the balloon in container 1.

The first thing that we note about this generalized game is that it no longer holds that $h_j^s$ is independent of the choice of Nash equilibrium $s$; see Fig. 7 for an illustration. Nonetheless, if

\begin{figure}[h]
\centering
\begin{subfigure}{.24\textwidth}
\centering
\includegraphics[width=\textwidth]{figure_a.png}
\caption{(a)}
\end{subfigure}
\begin{subfigure}{.24\textwidth}
\centering
\includegraphics[width=\textwidth]{figure_b.png}
\caption{(b)}
\end{subfigure}
\begin{subfigure}{.24\textwidth}
\centering
\includegraphics[width=\textwidth]{figure_c.png}
\caption{(c)}
\end{subfigure}
\begin{subfigure}{.24\textwidth}
\centering
\includegraphics[width=\textwidth]{figure_d.png}
\caption{(d)}
\end{subfigure}
\caption{Liquid distributions among balloons, corresponding to a plethora of Nash equilibria $s$ with distinct $h_j^s$, when $n = 2$, $k = 2$ (blue corresponding to $i = 1$, and red — to $i = 2$), $f_1 = f_2 = \text{id}$, $\mu^1 = \mu^2 = 1$, $f_1^1(x) = f_2^1(x) = x$ and $f_2^2(x) = f_1^2(x) = 2x$. Only the Nash equilibrium depicted in Fig. 7(d) is strong (in fact, it is super-strong); this is the unique equilibrium that our hydraulic construction finds.}
\end{figure}

we accept the physical intuition that when compressed via pistons, each of the liquid distributions given in Fig. 7 eventually reaches the liquid distribution depicted in Fig. 7(d), then it is intuitively clear why our construction can be formally shown to yield a strong (and under the conditions of Theorem 3(b), super-strong) Nash equilibrium. Consequently, uniqueness of $h_j^s$ can still be shown to hold among strong Nash equilibria; we know of no analogy to these results (drawing a fundamental connection between group deviation and I.D.-congestion) in atomic resource selection games. Formally, Theorem 3(a) no longer holds in the setting of I.D.-dependent weighting, while Theorems 1, 2, and 3(b) and Corollary 1 still hold in this setting when replacing every occurrence of “Nash” by “strong Nash”, and when adding the analogous assumptions regarding $(f_j^j)_{j \in [n]}$ (e.g., continuity for Theorem 1); we emphasize that all of these results are novel. The formal analysis is similar to that of Section 3.5.
5 Beyond Games

As we have been pointing out above, the development of the machinery of this paper is free of any fixed-point theorem, of the Minimax theorem and any equivalent result, and of linear programming. As such nonconstructive techniques are traditionally the tools used when attempting to establish the existence of equilibria, one may claim that in a sense, hydraulic analysis “replaces” such techniques in our analysis. It is therefore only natural to ask whether other results that are traditionally obtained via linear-programming methods can be also be derived as consequences of our machinery.\footnote{This question also naturally arises from noting that a naive in silico computation of whether a descending piston is blocked bears a striking resemblance to a search for an augmenting path.} In this section, we show that hydraulic analysis can indeed serve as a constructive substitute to linear-programming approaches also outside the realm of games, shedding new light on several flow/linear-programming problems. We start by deducing a novel, surprisingly intuitive, proof of Hall’s theorem using hydraulic analysis.

5.1 Case Study: Application to Hall’s Fractional Marriage Theorem

For this section, let \( n \in \mathbb{N} \), and for every \( i \in [n] \), let \( R^i \subseteq [n] \). We consider a scenario involving \( n \) women and \( n \) men, where for every \( i \in [n] \), we interpret \( R^i \) as the set of men acceptable to woman \( i \); a perfect marriage is a one-to-one correspondence \( \mathcal{M} : [n] \rightarrow [n] \), where \( \mathcal{M}(i) \in R^i \) for every \( i \in [n] \). For every subset \( I \subseteq [n] \), which we interpret as a set of women, we define \( R^I = \cup_{i \in I} R^i \) — the set of men acceptable to at least one woman in \( I \). A well-known result in graph theory is the following characterization of the conditions for the existence of a perfect marriage.

**Theorem 4** (Hall’s Marriage Theorem (Hall, 1935)). A perfect marriage exists iff \(|I| \leq |R^I|\) for every \( I \subseteq [n] \).

We now use the machinery of this paper to prove a slightly weaker form of Theorem 4. A perfect fractional marriage is a function \( s : [n] \rightarrow \mathbb{R}^{|R^i|}_\geq \) s.t. \( s(i) \in \Delta R^i \) for every \( i \in [n] \) and s.t. \( \sum_{i=1}^n s_j(i) = 1 \) for every \( j \in [n] \).

**Theorem 5** (Fractional Version of Hall’s Marriage Theorem). A perfect fractional marriage exists iff \(|I| \leq |R^I|\) for every \( I \subseteq [n] \).

To prove Theorem 5, we analyze the underlying scenario as a resource selection game. Let \( f_j \triangleq \text{id} \) for every \( j \in [n] \), let \( \mu^R \triangleq \bigl\{ i \in [n] \mid R^i = R \bigr\} \) for every \( R \in \mathbb{R}^{2^n}_{\geq} \), and define \( G \triangleq ((f_j)_{j=1}^n; (\mu^R)_{R \in \mathbb{R}^{2^n}_{\geq}}) \) — an \( n \)-resource selection game. The following two lemmas are obtained directly from definitions.

**Lemma 8.** A function \( s : [n] \rightarrow \mathbb{R}^{|R^i|}_\geq \) is a perfect fractional marriage iff \( s \) is a Nash equilibrium in \( G \) with \( \Delta^j_s = 1 \) for every \( j \in [n] \).

**Proof.** By definition, \( s \) is a perfect fractional marriage iff \( s \) is a consumption profile in \( G \) with \( \Delta^j_s = 1 \) for every \( j \in [n] \). A consumption profile \( s \) in \( G \) with \( \Delta^j_s = 1 \) for every \( j \in [n] \) is, by definition, a Nash equilibrium.

**Lemma 9.** A perfect fractional marriage exists iff \( \Delta^j_s = 1 \) for every \( j \in [n] \) and every Nash equilibrium \( s \) in \( G \).

**Proof.** Immediate from Lemma 8 and Theorems 1 and 2.
Lemmas 8 and 9, in conjunction with the analysis of Section 3, give rise to the following hydraulic algorithm for finding a perfect fractional marriage (or disproving its existence): set up the hydraulic system corresponding to \(G\) (as in Section 3.3), and start lowering the pistons until a Nash equilibrium is obtained. If the resulting stopping heights are all 1, then this equilibrium is a perfect fractional marriage; otherwise, no perfect fractional marriage exists. We use this algorithm to outline what we consider to be a surprisingly intuitive proof for Theorem 5; we now focus on the “harder” direction of this theorem, i.e., that lack of a perfect fractional marriage implies that \(|I| > |R^I|\) for some \(I \subseteq [n]\); the other (trivial) direction is left to the reader. A succinct formalization of the following argument is given in Appendix A.3.

Proof sketch. Assume that not all stopping heights are 1. Hence, the earliest-stopping pistons, \(P_G\), stop at a height higher than 1. Therefore, there is more than \(|P_G|\) liquid under the pistons \(P_G\) when they stop. Thus, there exists a set \(I\) (of all players corresponding to this mass of liquid) s.t. \(R^I = P_G\) even though \(|I| > |P_G| = |R^I|\), as required. \(\square\)

5.2 Beyond Hall’s Theorem

We note that, in fact, the argument presented in Section 5.1 does not require hydraulic analysis, and can also be carried out using Theorems 1 and 2 (and our explicit formula for \(P_G\)) as black boxes. While we find that phrasing it in terms of hydraulic analysis makes for far more tangible intuition, this is not the only reason why we have chosen this presentation. Indeed, as we show in this section, while several generalizations of Hall’s (fractional) theorem can still be analyzed directly via hydraulic analysis, and, moreover, even further generalizations can be hydraulically analyzed while it is not clear how to analyze them using resource selection games as defined in this paper. (We stress that for ease of exposition, we intentionally do not describe the most general class of flow/linear-programming problems solvable using our machinery.)

For the remainder of this section, let \(n, k \in \mathbb{N}\), for every \(i \in [k]\) let \(0 \leq \mu^i \leq \mathcal{M}^i\), for every \(j \in [n]\) let \(0 \leq t_j \leq T_j\), and for every \(i \in [k]\) and \(j \in [n]\), let \(0 \leq m^i_j \leq \mathcal{M}^i_j\). A solution to the triple \(\left((\mu^i, \mathcal{M}^i)\right)_{i=1}^k, \left([t_j, T_j]\right)_{j=1}^n, \left([m^i_j, \mathcal{M}^i_j]\right)_{j=1}^n\) is a matrix \((q^i_j)_{i=1}^k_{j=1}^n\) satisfying all of the following.

- \(m^i_j \leq q^i_j \leq \mathcal{M}^i_j\) for every \(i \in [k]\) and \(j \in [n]\).
- \(\mu^i \leq \sum_{j=1}^n q^i_j \leq \mathcal{M}^i\) for every \(i \in [k]\).
- \(t_j \leq \sum_{i=1}^k q^i_j \leq T_j\) for every \(j \in [n]\).

We note that Hall’s theorem deals with the question of the existence of a solution for \(\mu^i = \mathcal{M}^i = t_j = T_j = 1, m^i_j = 0\) and \(\mathcal{M}^i_j = 1_{R^i}(j)\). (The attentive reader may note that the scenario described in this section is also a strict generalization of the problem of the satisfiability of distribution constraints from Appendix A.1.3.)

While it is not hard to find conditions for the existence of such a solution, as well as methods for efficiently finding such a solution, by formulating an equivalent flow problem, we now analyze this problem using the hydraulic machinery of this paper.

We note that w.l.o.g. we may assume that \(m^i_j = 0\) for all \(i\) and \(j\) (otherwise, \(m^i_j\) may be subtracted from \(m^i_j, \mu^i, \mathcal{M}^i, t_j, \) and \(T_j\)). We first consider an “intermediate” case in which \(\mu^i = \mathcal{M}^i\) for every \(i \in [k]\), \(t_j = T_j\) for every \(j \in [n]\), and \(\mathcal{M}^i_j \in \{0, \mu^i\}\) (i.e., either forcing \(q^i_j = 0\) or not enforcing any limitation thereon) for every \(i \in [k]\) and \(j \in [n]\). (We assume in this case that \(\sum_{i=1}^k \mu^i = \sum_{j=1}^n t_j\); otherwise, no solution can possibly exist.)
While this case may be easily solved using the same machinery as in the previous section, by setting \( f_j(\mu) \triangleq \frac{\mu}{T_j} \) and setting \( \mu^R \triangleq \sum_{i \in [k]: R^i = R} \mu^i \), where \( R^i \triangleq \{ j \in [n] \mid M_j > 0 \} \), it may also be directly analyzed using hydraulic analysis without the need to vary the shape of containers.

The main observation that we now make use of is that in the hydraulic algorithm presented in Section 5.1, it is in fact not necessary to lower the pistons simultaneously. In fact, lowering them using any timing, as long as all of them eventually reach a height of 1, results in a perfect fractional marriage, while failing to do so (using any timing) proves the absence of a perfect fractional marriage. Using this observation, we readily obtain a simpler hydraulic algorithm for solving the above “intermediate case”: set up a hydraulic system with \( n \) containers (all corresponding to the identity function, as in Section 3.3) and \( k \) “liquid colors”, where the \( i \)th liquid color has \( \mu^i \) volume and has balloons in all containers \( j \in [n] \) s.t. \( M_j > 0 \). Start lowering the pistons in any order (say, sequentially) so that for every \( j \in [n] \), piston \( j \) eventually reaches height \( t_j \) (or gets blocked from reaching this height). If all pistons successfully reach their respective desired stopping heights, then the liquid distribution is a solution as required; otherwise, no solution exists.\(^{13}\)

Let us now consider arbitrary \( 0 \leq \mu^i \leq M^i \) and \( 0 \leq t_j \leq T_j \) (but not yet arbitrary \( M_j^i \)). While this scenario may still be analyzed as a resource selection game, the transformation into such a game becomes increasingly complex: (the verification of the following transformation into an \((n+1)\)-resource selection game is left to the reader)

\[
f_j(\mu) \triangleq \begin{cases} \frac{\mu}{T_j} & \mu < t_j \\ 1 & t_j \leq \mu \leq T_j \\ \frac{\mu}{T_j} & T_j < \mu \end{cases}, \quad f_{n+1} \equiv 1, \quad \mu^R = \sum_{i \in [k]: R^i = R} \mu^i, \quad \mu^{R \cup [n+1]} = \sum_{i \in [k]: R^i = R} M^i - \mu^i.
\]

Nonetheless, using an argument similar to the one above, the hydraulic algorithm for solving such a case, while also more complex and with an additional element, is considerably easier to visualize than the one solving this resource selection game; indeed, it does not require oddly shaped containers corresponding to functions that are not strictly increasing (as in Fig. 3(f)), nor does it require the description of how precisely pistons interact with such containers. This algorithm may be described as follows: set up a hydraulic system with \( n+1 \) containers (all corresponding to the identity function, as in Section 3.3) and \( 2k \) “liquid colors”, where the \( i \)th liquid color has \( \mu^i \) volume and has balloons in all containers \( j \in [n] \) s.t. \( M_j^i > 0 \) and where the \((k+i)\)th liquid color has \( M^i - \mu^i \) volume and has balloons in the same containers as the \( i \)th liquid color, and, additionally, in container \( n+1 \). Imagine also that container \( n+1 \) has no associated piston, but is “pressurized” so that liquid flows into it only if a currently descending piston would otherwise get stuck (i.e., only if all space in all other relevant containers is already occupied). Start lowering the pistons in any order (say, sequentially) so that for every \( j \in [n] \), piston \( j \) eventually reaches height \( T_j \) (or gets stuck in the process). If any piston gets stuck during this process, then no solution exists. We note that at the end of this process, the liquid distribution may not (yet) constitute a solution as required, since it is possible that some container \( j \) contains less than \( t_j \) liquid. Hence (say, one by one), we lower each piston \( j \) from height \( T_j \) to height \( t_j \), or until it gets stuck in the process. At the end of this process (with each piston \( j \) either at height \( t_j \) or blocked from reaching this height), if each container \( j \in [n] \) contains at least \( t_j \) liquid (i.e., if each piston touches the liquid surface of its container), then the liquid distribution is a solution as required; otherwise, no solution exists.

Finally, generalizing to arbitrary \( M_j^i \) (once again assuming w.l.o.g. that \( m_j^0 = 0 \)), while the corresponding resource selection game would have to be generalized beyond the definition of a

\(^{13}\)While colored liquids are often solutions, they are not the type of solutions we are interested in.
resource selection game as presented in either Section 3 or Section 4, the hydraulic algorithm may be modified by adding one additional very intuitive “physical” constraint: for every $i \in [k]$ and $j \in [n]$, all balloons of liquid color $i$ and of liquid color $k+i$ in container $j$ are wrapped together in an outer balloon that may not inflate to a height greater than $M_j^i$.

We conjecture that hydraulic analysis may indeed yield many more intuitive and visually appealing proofs for various other linear-programming problems. Can it be applied even beyond linear-programming problems?

6 Discussion

In addition to proving a gamut of novel theoretical results (in Sections 3 and 4), including some results that significantly strengthen age-old central theorems, and in addition to constructively reproving old results (in Section 5), a significant feature of our machinery is that it provides an explicit expression (see Section 3.4, as well as Lemma 6 in Section 3.5) for the highest resource cost (and for the set of highest-costing resources) in equilibrium, which can be sequentially used (see Algorithm 1 in Section 3.5) to compute all resource costs in equilibrium.

While the benefits of having an explicit expression for resource costs are by far not limited merely to actual computation, the question of the complexity of such actual computation is a valid one. The complexity of calculating this expression in practice depends critically on our ability to compute the equalization $\text{Equalize}_{f_j : f \in S}$ of any of the cost functions defining the resource selection game at hand; as with our ability to compute the cost functions themselves, our ability to compute their equalization strongly depends on the way they are specified. Indeed, in situations where the mere evaluation of some of the cost functions $f_j$ may be costly, it is hard to expect the calculation of their equalization to be any less costly; on the other hand, in many naturally occurring scenarios, the calculation of this equalization can be undertaken easily and efficiently, as is demonstrated in Remark 3 (see, e.g., our proof of Theorem 5). Assuming for a moment that the computation of this equalization can be carried out efficiently, then by Remark 5, the complexity of calculating the maximum cost (and when $f_j$ is strictly increasing — also the highest-costing resources) is linear in the size of the input $(\mu^R)_{R \in 2^{[n]} \setminus \emptyset}$. It should be noted, though, that in some real-life scenarios, the input is sparse and can be efficiently encoded into a size considerably smaller than $\Theta(2^n)$; when no prior information is known regarding the structure of the input, this may render the complexity of this computation exponential in the encoded input size.

Nonetheless, as emphasized above, the benefits of having an explicit expression for resource costs are by far not limited merely to actual computation. Indeed, in Section 5 this explicit expression is used to phrase an extremely concise proof of Hall’s theorem. For a more elaborate example, we turn to Gonczarowski and Tennenholtz (2014), where we analyze the dynamics of a complex two-stage game: in the first stage, merchants choose store locations (some store locations are accessible to more customers than others, but in turn are associated with higher real-estate prices), and the second stage is a resource selection game, where each customer aims to purchase from a least crowded store; the payoffs to the merchants are determined according to the Nash equilibrium loads in the second-stage resource selection game (these are well defined by Theorems 1 and 2). When analyzing game dynamics between the merchants in this complex multistage game, we must somehow quantify the effect of dynamic changes in merchants’ strategies in the first stage of the game (i.e., the effect of changes in the availability of a certain merchant to some customers) on the Nash equilibrium loads in the second-stage resource selection game. Our explicit expression for resource costs offers a concise way to do precisely this. (A proof of Proposition 1 that directly uses our explicit expression for resource costs and is free of any reasoning about incentives or deviations is given in Appendix A.4.)
Proposition 1 (Properties of $h_j$ as a Function of $\mu^R$). Let $G = \left( (f_j)_{j=1}^n; (\mu^R)_{R \in \mathcal{P}[n]} \right)$ be a resource selection game s.t. $f_1, \ldots, f_n$ are continuous. For every $j \in [n]$ and $R \in \mathcal{P}[n] \neq \emptyset$, both of the following hold, where $h_j$ is the cost of resource $j$ in all Nash equilibria of $G$.

a. $h_j$ is continuous and nondecreasing as a function of $\mu^R$.

b. If $f_j$ is Lipschitz, then $h_j$ is Lipschitz as a function of $\mu^R$, with the same Lipschitz constant.

Indeed, a special case of Proposition 1 allows us (in Gonczarowski and Tennenholtz, 2014) to prove powerful results regarding convergence of dynamics in this complex multistage game.

We conclude with a note about hydraulics. As can be seen when examining the extensions of our machinery in Section 4 and in Section 5.2, our hydraulic analysis framework is both flexible and robust; indeed, we conjecture that the full extent of its power is yet to be discovered, both within the realm of games and beyond. The results of this paper, as well as the earlier results of Fisher (1892), show not only that physical hydraulic systems may be a fruitful source of intuition for proofs regarding equilibria, but furthermore that they may be used to naturally “calculate” a variety of flavors of equilibria. It would be interesting to rigorously define a “hydraulic” calculation, and to study its strength and limitations.

Acknowledgments

The first author was supported in part by ISF grants 230/10 and 1435/14 administered by the Israeli Academy of Sciences, by the European Research Council under the European Community’s Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement no. [249159], and by an Adams Fellowship of the Israeli Academy of Sciences and Humanities. We would like to thank Sergiu Hart for pointing out Kaminsky’s construction, and Noam Nisan and Sergiu Hart for pointing out Fisher’s construction. We would like to thank Binyamin Oz for a question inspiring the definition of a resource selection game with I.D.-dependent weighting given in Section 4, and Noga Alon for a discussion that inspired Section 5.

References


A Proofs and Auxiliary Results

A.1 Proofs of Lemmas and Corollaries from Sections 3.4 and 3.5, and Auxiliary Results

A.1.1 Definitions for Formalizing the Observations from Section 3.3

We begin with an immediate consequence of Definition 6.

**Lemma 10.** Let \( m \in \mathbb{N} \) and let \( f_1, \ldots, f_m : \mathbb{R}_\geq \to \mathbb{R} \cup \{ \text{undefined} \} \) be nondecreasing functions. Let \( \mu_1, \ldots, \mu_m \in \mathbb{R}_\geq \) s.t. \( f_1(\mu_1) = f_2(\mu_2) = \cdots = f_m(\mu_m) \in \mathbb{R} \) and let \( \mu'_1, \ldots, \mu'_m \in \mathbb{R}_\geq \) s.t. \( \sum_{j=1}^m \mu'_j \geq \sum_{j=1}^m \mu_j \).

1. If \( f_1(\mu'_1), f_2(\mu'_2), \ldots, f_m(\mu'_m) \in \mathbb{R} \), then there exists \( j \in [m] \) s.t. \( f_j(\mu'_j) \geq f_j(\mu_j) \).
2. If \( f_1(\mu'_1) = f_2(\mu'_2) = \cdots = f_m(\mu'_m) \in \mathbb{R} \), then \( f_1(\mu'_1) \geq f_1(\mu_1) \).

**Proof.** For Part a, since \( \sum_{j=1}^m \mu'_j \geq \sum_{j=1}^m \mu_j \), there exists \( j \in [m] \) s.t. \( \mu'_j \geq \mu_j \). As \( f_j \) is nondecreasing and as \( \mu_j, \mu'_j \in f_j^{-1}(\mathbb{R}) \), we have \( f_j(\mu'_j) \geq f_j(\mu_j) \), as required. For Part b, by Part a there exists \( j \in [m] \) s.t. \( f_j(\mu'_j) \geq f_j(\mu_j) \); therefore, \( f_1(\mu'_1) = f_2(\mu'_2) = \cdots = f_m(\mu'_m) \in \mathbb{R} \), as well. This follows directly from Lemma 10(b), as both \( \sum_{j=1}^m \mu_j \leq \sum_{j=1}^m \mu'_j \), and both \( \sum_{j=1}^m \mu_j = \sum_{j=1}^m \mu'_j \). The fact that \( \text{Equalize}_{f_1, \ldots, f_m}(\mu) \) is nondecreasing follows directly from Lemma 10(b) as well.

**Proof of Lemma 2.** Part a follows directly by definition, as when \( m = 1 \), we always have \( \mu_1 = \mu \). We move on to prove Part b; let \( k \in [m] \) and \( 1 \leq j_1 < j_2 < \cdots < j_k < m \); define \( j_0 = 0 \) and \( j_{k+1} = m \).

If \( h = \text{Equalize}_{f_1, \ldots, f_m}(\mu) \in \mathbb{R} \), then there exist \( \mu_1, \ldots, \mu_m \in \mathbb{R}_\geq \) s.t. \( \sum_{j=1}^m \mu_j = \mu \) and \( f_1(\mu_1) = f_2(\mu_2) = \cdots = f_m(\mu_m) = h \). Let \( i \in [k+1] \); as \( f_{j_i-1+1}(\mu_{j_i-1+1}) = f_{j_i-1+2}(\mu_{j_i-1+2}) = \cdots = f_{j_i}(\mu_{j_i}) \), we have that \( \text{Equalize}_{f_{j_i-1+1}, \ldots, f_{j_i}}(\sum_{j=j_i-1+1}^{j_i} \mu_j) = f_{j_i-1+1}(\mu_{j_i-1+1}) = h \). Hence, we have \( h = \text{Equalize}_{f_1, \ldots, f_{j_1}} \cdots \text{Equalize}_{f_{j_k+1}, \ldots, f_m}(\mu) \), as required.

Conversely, if \( h = \text{Equalize}_{f_1, \ldots, f_{j_1}} \cdots \text{Equalize}_{f_{j_k+1}, \ldots, f_m}(\mu) \in \mathbb{R} \), then there exist \( \tilde{\mu}_1, \ldots, \tilde{\mu}_{k+1} \) s.t. \( \sum_{i=1}^{k+1} \tilde{\mu}_i = \mu \) and \( \text{Equalize}_{f_{j_1-1+1}, \ldots, f_{j_1}}(\tilde{\mu}_i) = h \) for every \( i \in [k+1] \). Therefore, for every \( i \in [k+1] \), there exist \( \mu_{j_i-1+1}, \ldots, \mu_{j_i} \) s.t. \( \sum_{\ell=j_i-1+1}^{j_i} \mu_{\ell} = \tilde{\mu}_i \) and \( f_{j_i-1+1}(\mu_{j_i-1+1}) = \cdots = f_{j_i}(\mu_{j_i}) = h \). As \( \sum_{j=1}^m \mu_j = \sum_{i=1}^{k+1} \tilde{\mu}_i = \mu \) and \( h = f_1(\mu_1) = f_2(\mu_2) = \cdots = f_m(\mu_m) \), we have that \( \text{Equalize}_{f_1, \ldots, f_m}(\mu) = h \), as required.

**Proof of Lemma 3.** By Lemma 2, when proving either part it is enough to consider the case in which \( m = 2 \). (The case \( m = 1 \) follows from Lemma 2(a), while the case \( m > 2 \) follows from the case \( m = 2 \) by iteratively applying Lemma 2(b).)

We start by proving Part a. Let \( \mu \in \mathbb{R}_\geq \) s.t. \( h = \text{Equalize}_{f_1, f_2}(\mu) \in \mathbb{R} \) and let \( \varepsilon > 0 \); assume w.l.o.g. that \( f_1 \) is continuous. By definition of \( h \), there exists \( \mu_1 \in [0, \mu] \) s.t. \( f_1(\mu_1) = f_2(\mu_1) = h \). By continuity of \( f_1 \), there exists \( \delta > 0 \) s.t. \( |f_1(\mu') - h| < \varepsilon \) for every \( \mu' \in (\mu - \delta, \mu + \delta) \cap f_1^{-1}(\mathbb{R}) \). Let \( \mu' \in (\mu - \delta, \mu + \delta) \cap \text{Equalize}_{f_1, f_2}(\mathbb{R}) \); by definition, there exists \( \tilde{\mu}_1 \in [0, \mu'] \) s.t. \( f_1(\tilde{\mu}_1) = f_2(\mu' - \mu'_1) = h' = \text{Equalize}_{f_1, f_2}(\mu') \). If \( h' = h \), then we trivially have \(|h' - h| = 0 < \varepsilon\), as
required; assume, therefore, that $h' \neq h$. We show that $\mu'_1 \in (\mu_1 - \delta, \mu_1 + \delta)$ by considering two cases. If $h' > h$, then as $f_1, f_2$ are nondecreasing and as $f_1(\mu_1) = h < h' = f_1(\mu'_1)$ and $f_2(\mu_1 - \mu_1) = h < h' = f_1(\mu'_1)$, we have $\mu_1 < \mu'_1$ and $\mu - \mu_1 < \mu' - \mu'_1$; combining these, we have that $\mu'_1 \in (\mu_1, \mu_1 + \mu - \mu_1) \subseteq (\mu_1, \mu_1 + \mu - \mu_1) \subseteq (\mu_1 - \delta, \mu_1 + \delta)$ in this case. If $h' < h$, then similarly, as $f_1, f_2$ are nondecreasing and as $f_1(\mu_1) = h > h' = f_1(\mu'_1)$ and $f_2(\mu_1 - \mu_1) = h > h' = f_1(\mu'_1)$, we have $\mu_1 > \mu'_1$ and $\mu - \mu_1 > \mu' - \mu'_1$; combining these, we have that $\mu'_1 \in (\mu_1 + \mu - \mu_1, \mu) \subseteq (\mu_1 + \mu - \mu_1, \mu) \subseteq (\mu_1 - \delta, \mu_1 + \delta)$ in this case as well. By definition of $\delta$ and as $f_1(\mu'_1) = h' \in \mathbb{R}$, we obtain $|h' - h| = |f_1(\mu'_1) - h| < \epsilon$, as required.

We proceed to the proof of Part b. By Part a, $\text{Equalize}_{f_1, f_2}$ is continuous; it therefore remains to show that $\text{Equalize}_{f_1, f_2}$ is defined on a suffix of $\mathbb{R}_{>2}$. Recall that for every $\mu \in \mathbb{R}$, by definition $\text{Equalize}_{f_1, f_2}(\mu) \in \mathbb{R}$ iff there exists $\mu_1 \in [0, \mu]$ s.t. $f_1(\mu_1) = f_2(\mu - \mu_1) \in \mathbb{R}$. Let $\mu \in \mathbb{R}_{>2}$ s.t. $\text{Equalize}_{f_1, f_2}(\mu) \in \mathbb{R}$; therefore, there exists $\mu_1 \in [0, \mu]$ s.t. $f_1(\mu_1) = f_2(\mu - \mu_1) \in \mathbb{R}$. Let $\mu' > \mu$; note that as $\mu_1 \leq \mu$, we have $\mu_1 + \mu' - \mu \leq \mu'$. Since $f_1(\mu_1), f_2(\mu - \mu_1) \in \mathbb{R}$ and as $\mu' > \mu$, we have, by $f_1$ and $f_2$ being defined on a suffix of $\mathbb{R}_{>2}$, that $f_1(\mu_1 + \mu' - \mu), f_2(\mu - \mu_1) \in \mathbb{R}$ as well. Furthermore, as $f_1$ and $f_2$ are nondecreasing, we have $f_1(\mu_1) = f_2(\mu - \mu_1) \leq f_2(\mu - \mu_1)$ and $f_1(\mu_1 + \mu' - \mu) \geq f_1(\mu_1) = f_2(\mu - \mu_1) = f_2(\mu' - (\mu_1 + \mu' - \mu))$. By continuity of $f_1$ and $f_2$ and as $[\mu_1, \mu_1 + \mu' - \mu] \subseteq f_2^{-1}(\mathbb{R})$ and $[\mu' - (\mu_1 + \mu' - \mu), \mu' - \mu_1] = [\mu_1 - \mu_1, \mu' - \mu_1] \subseteq f_2^{-1}(\mathbb{R})$, we have by the intermediate value theorem that there exists $\mu'_1 \in [\mu_1, \mu_1 + \mu' - \mu] \subseteq [0, \mu']$ s.t. $f_1(\mu'_1) = f_2(\mu' - \mu'_1) \in \mathbb{R}$, as required.

\textbf{Proof of Corollary 2.} By Lemma 3(b), $\text{Equalize}_{f_1, \ldots, f_m}$ is a real function iff $\text{Equalize}_{f_1, \ldots, f_m}(0) \in \mathbb{R}$, which by definition holds iff $f_1(0) = f_2(0) = \cdots = f_m(0)$. \hfill \square

\textbf{Proof of Lemma 4.} By definition, $S \notin M_G(S)$ for every $S \in 2^{[n]}_{\neq 0}$ (by taking $\mu = \frac{1}{\sum_{R \in 2^{[n]}_{\neq 0}} \mu^R}$). Therefore, $M_G(\{1\}) = \emptyset$; furthermore, by Lemma 2(a), $E_G(\{1\}) = \text{Equalize}_{f_1}(\mu^{(1)}) = f_1(\mu^{(1)}) \in \mathbb{R}$. Therefore, $\{1\} \in D_G$. In particular, we have that $D_G \neq \emptyset$, and so, by finiteness of $D_G$, we have that $P_G \neq \emptyset$ and that $h_G \in \mathbb{R}$ is well defined. \hfill \square

\textbf{Proof of Lemma 5.} Both parts of the lemma follow straight from definition. \hfill \square

\subsection{Uniqueness and Strength}

\textbf{Proof of Lemma 6.} We start by proving Part c. Let $R \in 2^{[n]}_{\neq 0}$ s.t. there exists $j \in P^s$ s.t. $s_j(R) > 0$; it is enough to show that $R \in 2^{P^s}_{\neq 0}$. By definition of $s$, $h^s_j \leq h^s_k$ for every $k \in R$, and as $h^s_j = \max_{i \in [n]} h^s_i \geq h^s_k$, we have $h^s_k = h^s_j$ and so $k \in P^s$ for every $k \in R$. Therefore, $R \in 2^{P^s}_{\neq 0}$ as required.

We proceed to the proof of Part d. We first show that $s'$ is a consumption profile in the game $G' \equiv G - P^s$. Let $R' \in 2^{[n]}_{\neq 0} P^s$. By definition of $s'$, we have that $s'_j(R') = \sum_{R \in \mathcal{R}(R', G')} s_j(R) \geq 0$ for every $j \in [n] \setminus P^s$. Furthermore, for every $j \in ([n] \setminus P^s) \setminus R'$, we have by definition that $j \notin R$ for every $R \in \mathcal{R}(R', G')$, and so $s'_j(R') = \sum_{R \in \mathcal{R}(R', G')} s_j(R) = 0$. Finally, we have that $\sum_{j \in [n] \setminus P^s} s'_j(R') = \sum_{j \in [n] \setminus P^s} \sum_{R \in \mathcal{R}(R', G')} s_j(R) = \sum_{R \in \mathcal{R}(R', G')} \sum_{j \in [n] \setminus P^s} s_j(R) = \sum_{R \in \mathcal{R}(R', G')} \mu^R$, where the penultimate equality is by Part c.

We move on to show that $h^s_j = h^s_j$ for every $j \in [n] \setminus P^s$. By definition of $s'$, we have for every $j \in [n] \setminus P^s$ that $\mu^s_j = \sum_{R \in 2^{[n]} \setminus P^s} s'_j(R') = \sum_{R \in 2^{[n]} \setminus P^s} \sum_{R \in \mathcal{R}(R', G')} s_j(R) = \sum_{R \in 2^{[n]} \setminus P^s} \sum_{R \in \mathcal{R}(R', G')} (R) = \sum_{R \in 2^{[n]} \setminus P^s} \sum_{R \in \mathcal{R}(R', G')} (R) = \mu^s_j$ (where the penultimate equality is since $j \notin R$ for every $R \in 2^{[n]} \setminus P^s$), and hence $h^s_j = f_j(\mu^s_j) = f_j(\mu^s_j) = h^s_j$, as required.

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We conclude by showing that $s'$ is indeed a Nash equilibrium in $G'$. Let $R' \in 2^{[n]\setminus P_s}$, and let $k \in \text{supp}(s'(R'))$ and $j \in R'$. As $0 < s'_k(R') = \sum_{R \in \mathcal{R}(R', G')} s_k(R)$, we have that there exists $R \in \mathcal{R}(R', G')$ s.t. $k \in \text{supp}(s(R))$. As $j \in R' \subseteq R$, since $s'$ is a Nash equilibrium in $G$, we have that $h^*_k \leq h^*_j$; therefore, $h^*_k = h^*_j = h^*_j = h^*_j$ and so $s'$ is a Nash equilibrium in $G'$, as required.

Before proceeding to prove Parts a and b, we prove a few auxiliary results. We first show that

$$\forall j \in P^s : h^*_j = E_G(P^s) = \text{Equalize}_{f_{j,k} \in P_s} \left( \sum_{R \in 2^{P_s}} \mu^R \right). \quad (1)$$

By definition of $P^s$, $f_j(\mu^*_j) = h^*_j = h^*_j = f_k(\mu^*_k)$ for every $j,k \in P^s$. Therefore, $h^*_j = \text{Equalize}_{f_{j,k} \in P_s} (\sum_{k \in P^s} \mu^*_k)$ for every $j \in P^s$. It is therefore enough to show that $\sum_{k \in P^s} \mu^*_k = \sum_{R \in 2^{P_s}} \mu^R$. Indeed, we have $\sum_{k \in P^s} \mu^*_k = \sum_{k \in P^s} \left( \sum_{R \in 2^{P_s}} s_k(R) \right) = \sum_{R \in 2^{P_s}} \sum_{k \in P^s} s_k(R) = \sum_{R \in 2^{P_s}} \mu^R$, where the penultimate equality is by Part c, and the last equality is because $s(R) \in \mu^R$, $\Delta^R \subseteq \mu^R$, $\Delta^P$ for every $R \in 2^{P_s}$.

Next, we show that for every $S \in 2^{[m] \setminus P_s}$ s.t. $E_G(S) = \text{Equalize}_{f_{j,k} \in S} \left( \sum_{R \in 2^{P_s} \setminus P_s} \mu^R \right) \in \mathbb{R}$, there exists $k \in S$ s.t. $f_k(\mu^*_k) \geq E_G(S)$. Indeed, since $s(R) \in \mu^R$, $\Delta^R \subseteq \mu^R$, $\Delta^S$ for every $R \in 2^{P_s}$, we have that $\sum_{R \in 2^{P_s} \setminus P_s} \mu^R = \sum_{R \in 2^{P_s} \setminus P_s} \sum_{k \in S} s_k(R) \leq \sum_{R \in 2^{P_s} \setminus P_s} \sum_{k \in S} s_k(R) = \sum_{R \in 2^{P_s}} \sum_{k \in S} s_k(R) = \sum_{k \in S} \mu^*_k$ and so, by Lemma 10(a), there exists $k \in S$ s.t. $f_k(\mu^*_k) \geq \text{Equalize}_{f_{j,k} \in S} \left( \sum_{R \in 2^{P_s} \setminus P_s} \mu^R \right) = E_G(S)$, as required.

We now show that $P^s \in \arg \text{Max}_{S \in 2^{[n] \setminus P_s}} E_G(S)$, where the value undefined is treated as $-\infty$ for comparisons by the Max operator. Let $S \in 2^{[n] \setminus P_s}$ s.t. $E_G(S) \in \mathbb{R}$. As shown above, there exists $k \in S$ s.t. $f_k(\mu^*_k) \geq E_G(S)$. Therefore, by Eq. (1) and by definition of $P^s$ we obtain that $E_G(P^s) = \text{Max}_{j \in [n]} h^*_j \geq h^*_k = f_k(\mu^*_k) \geq E_G(S)$, and so indeed $P^s \in \arg \text{Max}_{S \in 2^{[n] \setminus P_s}} E_G(S)$.

Finally, we show that $M_G(P^s) = \emptyset$. We have to show that for every $S \in 2^{P_s} \setminus P_s$ there exists $\mu \leq \sum_{j \in S \setminus P_s} \mu^R$ s.t. $\text{Equalize}_{f_{j,k} \in S}(\mu) = E_G(P^s)$. Let, therefore, $S \in 2^{P_s} \setminus P_s$ and define $\mu \neq \sum_{j \in S \setminus P_s} \mu^*_j$. By Eq. (1) and by definition of $P^s$, it is enough to show that both $\text{Equalize}_{f_{j,k} \in S}(\mu) = \text{Max}_{j \in [n]} h^*_j$ and $\mu \leq \sum_{j \in S \setminus P_s} \mu^R$. Since $S \subseteq P^s$, we have $f_k(\mu^*_k) = h^*_k = \text{Max}_{j \in [n]} h^*_j$ for every $k \in S$, and so, by definition, $\text{Equalize}_{f_{j,k} \in S}(\mu) = \text{Equalize}_{f_{j,k} \in S}(\sum_{j \in S} \mu^*_j) = \text{Max}_{j \in [n]} h^*_j$. For every $j \in S$, we have $\mu^*_j = \sum_{j \in S \setminus P_s} s_j(R) = \sum_{R \in 2^{P_s \setminus P_s}} s_j(R) = \sum_{R \in 2^{P_s \setminus P_s} \setminus 2^{P_s \setminus S}} s_j(R)$, where the penultimate equality is by Part c since $j \in S \subseteq P^s$, and the last inequality is since $j \notin R$ for every $R \in 2^{P_s \setminus S}$. Therefore, $\sum_{j \in S} \mu^*_j = \sum_{j \in S} \sum_{R \in 2^{P_s \setminus P_s} \setminus 2^{P_s \setminus S}} s_j(R) = \sum_{R \in 2^{P_s} \setminus 2^{P_s \setminus S}} \sum_{j \in S} s_j(R) \leq \sum_{R \in 2^{P_s} \setminus 2^{P_s \setminus S}} \sum_{j \in S} s_j(R) = \sum_{R \in 2^{P_s}} \mu^R$, as required, and so $M_G(P^s) = \emptyset$.

We proceed to prove Part a by showing mutual containment between the two sides of the equality.

1. It is enough to show that $P^s \in \arg \text{Max}_{S \in D_G} E_G(S)$. As $M_G(P^s) = \emptyset$ and as by Eq. (1) $E_G(P^s) \in \mathbb{R}$, we have $P^s \in D_G$. As $E_G(P^s) = \text{Max}_{S \in 2^{[n] \setminus P_s}} E_G(S) \geq \text{Max}_{S \in D_G} E_G(S)$, we therefore have $P^s \in \arg \text{Max}_{S \in D_G} E_G(S)$, as required.

2. We must show that $S \subseteq P^s$ for every $S \in \arg \text{Max}_{S \not= P^s} E_G(S')$. Define $S' \triangleq S \cap P^s \in 2^{S'}$ and assume by way of contradiction that $S' \not= \emptyset$. It is enough to show that $\text{Equalize}_{f_{j,k} \in S'}(\mu) \not= E_G(S')$ for every $\mu \leq \sum_{R \in 2^{P_s \setminus S'}} \mu^R$, since this implies $S' \in M_G(S')$ — a contradiction, as $S \in D_G$. Let, therefore, $\mu \leq \sum_{R \in 2^{P_s \setminus S'}} \mu^R$; as by Lemma 4, $E_G(S) = h_G \in \mathbb{R}$, it is enough to show that if $\text{Equalize}_{f_{j,k} \in S'}(\mu) \in \mathbb{R}$, then $\text{Equalize}_{f_{j,k} \in S'}(\mu) < E_G(S)$. Recall from the proof
of the other direction ("⊆") that \( P^s \in \arg\max_{D_G} E_G(S') \); therefore, by definition of \( S \), by Eq. (1) and by definition of \( P^s \), we obtain that \( E_G(S) = E_G(P^s) = \max_{k \in [n]} h^s_k \). It is thus enough to show that Equalize \( f_{k,k'} \in S'(\mu) \leq \max_{k \in [n]} h^s_k \).

By definition of \( S' \) and \( P^s \), we have that \( h^s_j < \max_{k \in [n]} h^s_k \) for every \( j \in S' \) and \( h^s_j = \max_{k \in [n]} h^s_k \) for every \( j \in S' \setminus S' \); ergo, \( s_j(R) = 0 \) for every \( j \in S' \setminus S' \) and \( R \in 2^{S' \setminus S'} \). Hence, \( \sum_{j \in S'} \mu^s_j = \sum_{j \in S'} \sum_{R \in \mathcal{P}_2^{[n]} \setminus \emptyset} s_j(R) \geq \sum_{j \in \mathcal{S}'} \sum_{R \in \mathcal{P}_2^{[n]} \setminus \emptyset} s_j(R) = \sum_{R \in \mathcal{P}_2^{[n]} \setminus \emptyset} \sum_{j \in \mathcal{S}'} s_j(R) = \sum_{R \in \mathcal{P}_2^{[n]} \setminus \emptyset} \sum_{j \in \mathcal{S}'} s_j(R) = \sum_{R \in \mathcal{P}_2^{[n]} \setminus \emptyset} \sum_{j \in \mathcal{S}'} s_j(R) = \sum_{R \in \mathcal{P}_2^{[n]} \setminus \emptyset} \mu^s_j \geq \mu \). Therefore, by Lemma 10(a) there exists \( j \in S' \) s.t. \( f_j(\mu^s_j) \geq \text{Equalize}_{f_{k,k'} \in S'}(\mu) \), and thus \( \text{Equalize}_{f_{k,k'} \in S'}(\mu) \leq f_j(\mu^s_j) = h^s_j < \max_{k \in [n]} h^s_k \), as required.

We conclude by proving Part b. Recall from the proof of the first direction (⊆) of Part a that \( E_G(P^s) = \max_{S \in D_G} E_G(S) \). Therefore, by Eq. (1), \( h^s_j = E_G(P^s) = \max_{S \in D_G} E_G(S) = h_G \) for every \( j \in P^s \), as required.

### A.1.3 Constrained Distribution

Before proving Lemma 7, we first formulate and prove a combinatorial result that we use in the proof of this lemma.

**Definition 12** (Distribution Constraint).

- A distribution constraint is a pair \( (\mu^R)_{R \in 2^{[n]} \setminus \emptyset}, ([t_j, T_j])_{j=1}^n \), where \( n \in \mathbb{N}, \mu^R \in \mathbb{R}_\geq \) for every \( R \in 2^{[n]} \setminus \emptyset \), and \( t_j \leq T_j \in \mathbb{R}_\geq \) for every \( j \in [n] \).
- We say that a distribution constraint \( C = (\mu^R)_{R \in 2^{[n]} \setminus \emptyset}, ([t_j, T_j])_{j=1}^n \) is satisfiable if there exist \( (\mu^R_j)_{j \in [n]} \) s.t. \( (\mu^R_j)_{j \in [n]} \in \mu^R \cdot \Delta^R \) for every \( R \in 2^{[n]} \setminus \emptyset \) and \( \sum_{R \in 2^{[n]} \setminus \emptyset} \mu^R_j \in [t_j, T_j] \) for every \( j \in [n] \).
- Given a distribution constraint \( C = (\mu^R)_{R \in 2^{[n]} \setminus \emptyset}, ([t_j, T_j])_{j=1}^n \), for every \( S \in 2^{[n]} \setminus \emptyset \) we define \( m_C(S) = \sum_{R \in 2^{[n]} \setminus \emptyset} \mu^R \cdot \Delta^R \), \( M_C(S) = \sum_{R \in 2^{[n]} \setminus \emptyset} \mu^R \cdot \Delta^R \), \( t_C(S) = \sum_{j \in S} t_j \) and \( T_C(S) = \sum_{j \in S} T_j \). We say that \( C \) is normal if both \( t_C(S) \leq M_C(S) \) and \( m_C(S) \leq T_C(S) \) for every \( S \in 2^{[n]} \setminus \emptyset \).

We note that it is trivial to show that every satisfiable distribution constraint is normal.

In this section, we constructively show (without the use of, e.g., linear programming) that the other direction holds as well, and give a procedure for explicitly finding a solution to (i.e., a witness to the satisfiability of) any given normal distribution constraint.

**Lemma 11.** Every normal distribution constraint is satisfiable.

Before proving Lemma 11, we first develop some machinery.

**Lemma 12.** Let \( C = (\mu^R)_{R \in 2^{[n]} \setminus \emptyset}, ([t_j, T_j])_{j=1}^n \) be a normal distribution constraint.

a. \( M_C(S \cup S') = t_C(S \cup S') \), for every \( S, S' \in 2^{[n]} \setminus \emptyset \), s.t. \( M_C(S) = t_C(S) \) and \( M_C(S') = t_C(S') \).

b. \( m_C(S \cap S') = T_C(S \cap S') \), for every \( S, S' \in 2^{[n]} \setminus \emptyset \), s.t. \( m_C(S) = T_C(S) \) and \( m_C(S') = T_C(S') \) and \( S \cap S' \neq \emptyset \).
Proof. For every $S, S' \in 2^{[n]} \setminus \emptyset$ s.t. $M_C(S) = t_C(S)$ and $M_C(S') = t_C(S')$, we have $M_C(S \cup S') \leq M_C(S) + M_C(S') - M_C(S \cap S') = t_C(S) + t_C(S') - M_C(S \cap S') \leq t_C(S) + t_C(S') - t_C(S \cap S') = t_C(S \cup S')$, as required. (The other side of the inequality follows from normality of $C$.)

For every $S, S' \in 2^{[n]} \setminus \emptyset$ s.t. $m_C(S) = T_C(S)$, $m_C(S') = T_C(S')$ and $S \cap S' \neq \emptyset$, we have $m_C(S \cap S') \geq m_C(S) + m_C(S') - m_C(S \cup S') = T_C(S) + T_C(S') - m_C(S \cup S') \geq T_C(S) + T_C(S') - T_C(S \cup S') = T_C(S \cap S')$, as required. (Once again, the other side of the inequality follows from normality of $C$.)

Lemma 13 (Moving Mass from $R$ to $\{n\}$). Let $C = \left( (\mu^R)_{R \in 2^{[n]} \setminus \emptyset}, ([t_j, T_j])_{j=1}^{m} \right)$ be a normal distribution constraint. For every $R \in 2^{[n]} \setminus \emptyset$ s.t. $\{n\} \subseteq R$, we define

$$Q^R_C \triangleq \min \left\{ \min_{S \in 2^{[n]} \setminus \emptyset : \emptyset \cap R \neq \emptyset} \left( M_C(S) - t_C(S) \right), \min_{S \in 2^{[n]} \setminus \emptyset : \emptyset \subseteq S \cap \emptyset \neq \emptyset} \left( T_C(S) - m_C(S) \right) \right\}.$$ 

a. $Q^R_C \geq 0$ for every $R \in 2^{[n]} \setminus \emptyset$ s.t. $\{n\} \subseteq R$.

b. If $t_n > \mu^{[n]}$, then there exists $R \in 2^{[n]} \setminus \emptyset$ s.t. $\{n\} \subseteq R$, $\mu^R > 0$ and $Q^R_C > 0$.

Let $R \in 2^{[n]} \setminus \emptyset$ s.t. $\{n\} \subseteq R$ and let $\mu \in [0, \mu^R]$. For every $R' \in 2^{[n]} \setminus \emptyset$, $\{R, \{n\}\}$, let $\mu^{R'} \triangleq \mu^{R'}$ and let $\mu^{R'} \triangleq \mu - \mu^R$ and $\mu^{[n]} \triangleq \mu^{[n]} + \mu$. Define $C' \triangleq ((\mu^{R'}))_{R \in 2^{[n]} \setminus \emptyset}, ([t_j, T_j])_{j \in [n]}$.

c. If $\mu \leq Q^R_C$, then $C'$ is normal. Furthermore, in this case $Q^R_C = Q^R_C - \mu$, and $Q^R_C \leq Q^R_{C'}$ for every $R' \in 2^{[n]} \setminus \emptyset$ s.t. $\{n\} \subseteq R'$.

d. If $C'$ is satisfiable, then $C$ is satisfiable.

Remark 7. The condition of Lemma 13(c) is actually also necessary; i.e., $C'$ is normal iff $\mu \leq Q^R_C$.

Proof of Lemma 13. Part a follows directly from the fact that $C$ is normal, and so $M_C(S) - t_C(S) \geq 0$ and $T_C(S) - m_C(S) \geq 0$ for every $S \in 2^{[n]} \setminus \emptyset$.

To prove Part b, let $S_1 \triangleq \bigcup \{S \in 2^{[n-1]} \setminus \emptyset \mid M_C(S) = t_C(S)\} \subseteq 2^{[n-1]} \setminus \emptyset$ and $S_2 \triangleq [n] \cap \bigcap \{S \in 2^{[n]} \setminus \emptyset \mid S \subseteq S \cap \{n\} \subseteq \{n\}\}$ (the intersection with $[n]$ has an effect only if $[n]$ is the sole element in the intersection defining $S_2$). We first show that there exists $R \subseteq S_2 \setminus S_1$ s.t. $\{n\} \subseteq R$ and $\mu^R > 0$.

For ease of notation, we extend the definition of $m_C(S)$, $M_C(S)$, $t_C(S)$, and $T_C(S)$ also to the case $S = \emptyset$, via the same definition; we note that these all equal zero when $S = \emptyset$, as they are all defined by empty sums in this case. We note that if $S_1 \neq \emptyset$, then $M_C(S_1) = t_C(S_1)$ by Lemma 12(a), and if $S_1 = \emptyset$, then $M_C(S_1) = 0 = t_C(S_1)$ by definition.

We first consider the case where $S_2 \neq [n]$. In this case, by Lemma 12(b), $T_C(S_2) = m_C(S_2)$. Let $S \triangleq S_1 \cap S_2 \subseteq [n-1]$. We note that $t_C(S_1) - t_C(S) = t_C(S_1 \setminus S) \leq M_C(S_1 \setminus S) = M_C(S_1) - \sum_{R_1}^{[n]} m_C(R_1 \setminus \emptyset) \leq \mu^R \leq M_C(S_1) - \sum_{R_2 \in 2^{[n]} \setminus \emptyset} m_C(S_2 \setminus S_2 \setminus \emptyset) \mu^R = t_C(S_1) - \sum_{R_2 \in 2^{[n]} \setminus \emptyset} m_C(S_2 \setminus S_2 \setminus \emptyset) \mu^R$; therefore, $\sum_{R_2 \in 2^{[n]} \setminus \emptyset} m_C(S_2 \setminus S_2 \setminus \emptyset) \mu^R \leq t_C(S) \leq T_C(S)$. Hence, and as $T_n \geq t_n > \mu^{[n]}$, we have that $m_C(S_2) - \sum_{R_1 \in 2^{[n]} \setminus \emptyset} m_C(R_1 \setminus \emptyset) \mu^R - \mu^{[n]} - \sum_{R_2 \in 2^{[n]} \setminus \emptyset} m_C(S_2 \setminus S_2 \setminus \emptyset) \mu^R = m_C(S_2 \setminus (S \cup \emptyset)) \leq T_C(S_2 \setminus (S \cup \emptyset)) = T_C(S_2) - T_C(S) - T_C(\{n\}) = m_C(S_2) - T_C(S) - T_n < m_C(S_2) - \sum_{R_1 \in 2^{[n]} \setminus \emptyset} m_C(R_1 \setminus \emptyset) \mu^R - \mu^{[n]}$. 29
Therefore, $\sum_{R \subseteq 2^{\mathcal{S}} \setminus \emptyset} \mu^R > 0$, and so there exists $R \subseteq S_2 \setminus S = S_2 \setminus S_1$ s.t. $\{n\} \subseteq R$ and $\mu^R > 0$, as required.

We now consider the case in which $S_2 = [n]$. Note that $M_C(S_1 \cup \{n\}) \geq t_C(S_1 \cup \{n\}) = t_C(S_1) + t_n = M_C(S_1) + t_n > M_C(S_1) + \mu^{(n)} = M_C(S_1 \cup \{n\}) - \sum_{R \subseteq 2^{\mathcal{S}} \setminus \emptyset, \{n\} \subseteq R} \mu^R$; therefore, $\sum_{R \subseteq 2^{\mathcal{S}} \setminus \emptyset, \{n\} \subseteq R} \mu^R > 0$, and so there exists $R \subseteq [n] \setminus S_1 = S_2 \setminus S_1$ s.t. $\{n\} \subseteq R$ and $\mu^R > 0$, as required.

Either way, there exists $R \subseteq S_2 \setminus S_1$ s.t. $\{n\} \subseteq R$ and $\mu^R > 0$. Therefore, for every $S \in 2^{[n]} \setminus \emptyset$ s.t. $S \cap R \neq \emptyset$, we have $S \not\subseteq S_1$ and so $M_C(S) \neq t_C(S)$ and by normality of $C$, $M_C(S) > t_C(S)$; for every $S \in 2^{[n]} \setminus \emptyset$ s.t. $n \in S$ and $R \not\subseteq S$, we have $S_2 \not\subseteq S$ and so $T_C(S) \neq m_C(S)$ and by normality of $C$, $T_C(S) > m_C(S)$. By both of these, $Q_C^R > 0$ and the proof of Part b is complete.

We move on to Part c; let $S \in 2^{[n]} \setminus \emptyset$. If $R \subseteq S$ (and so also $n \in S$) or both $R \not\subseteq S$ and $n \notin S$, then by normality of $C$,

$$T_{C'}(S) = T_C(S) \geq m_C(S) = \sum_{R \in 2^{\mathcal{S}} \setminus \emptyset} \mu^R = \sum_{R \subseteq 2^{\mathcal{S}} \setminus \emptyset} \mu^R = m_{C'}(S);$$

otherwise, $R \not\subseteq S$ and $n \in S$, and by definition of $\mu$ and of $Q_C^R$,

$$T_{C'}(S) = T_C(S) \geq m_C(S) + Q_C^R \geq m_C(S) + \mu = \sum_{R \subseteq 2^{\mathcal{S}} \setminus \emptyset} \mu^R = m_{C'}(S).$$

If $S \cap R = \emptyset$ (and so also $n \notin S$) or both $S \cap R \neq \emptyset$ and $n \in S$, then by normality of $C$,

$$t_{C'}(S) = t_C(S) \leq M_C(S) = \sum_{R \subseteq 2^{[n]} \setminus \emptyset} \mu^R = \sum_{R \subseteq 2^{[n]} \setminus \emptyset} \mu^R = M_C(S);$$

otherwise, $S \cap R \neq \emptyset$ and $n \notin S$, and by definition of $\mu$ and of $Q_C^R$,

$$t_{C'}(S) = t_C(S) \leq M_C(S) - Q_C^R \leq M_C(S) - \mu = \sum_{R \subseteq 2^{[n]} \setminus \emptyset} \mu^R = M_C(S).$$

Therefore, $C'$ is normal.

For every $S \in 2^{[n]} \setminus \emptyset$ s.t. $S \cap R \neq \emptyset$, we have that $M_{C'}(S) = M_C(S) - \mu$; for every $S \in 2^{[n]} \setminus \emptyset$ s.t. $R \not\subseteq S$ and $n \in S$, we have that $m_{C'}(S) = m_C(S) + \mu$. Therefore,

$$Q_{C'}^R = \min \left\{ \min_{S \in 2^{[n]} \setminus \emptyset, S \cap R \neq \emptyset} \left( M_{C'}(S) - t_{C'}(S) \right), \min_{S \in 2^{[n]} \setminus \emptyset, R \not\subseteq S \& n \in S} \left( T_{C'}(S) - m_{C'}(S) \right) \right\} =$$

$$= \min \left\{ \min_{S \in 2^{[n]} \setminus \emptyset, S \cap R \neq \emptyset} \left( M_C(S) - \mu - t_C(S) \right), \min_{S \in 2^{[n]} \setminus \emptyset, R \not\subseteq S \& n \in S} \left( T_C(S) - m_C(S) - \mu \right) \right\} =$$

$$= Q_C^R - \mu.$$

For every $S \in 2^{[n]} \setminus \emptyset$, we have $M_{C'}(S) \in \{ M_C(S), M_C(S) - \mu \}$ (as shown above, depending on whether or not both $S \cap R \neq \emptyset$ and $n \notin S$); for every $S \in 2^{[n]} \setminus \emptyset$ s.t. $n \in S$, we have that
m_C'(S) ∈ \{m_C(S), m_C(S) + \mu\} (as shown above, depending on whether or not both R ⊈ S and n ∈ S). Therefore, for every R' ∈ 2^{[n]}_{\neq \emptyset} s.t. \{n\} ⊈ R',

\[
Q_C^{R'} = \min\left\{ \min_{S \in 2^{[n]}_{\neq \emptyset}} \left( M_{C'}(S) - t_{C'}(S) \right), \min_{S \in 2^{[n]}_{\neq \emptyset}} \left( T_{C'}(S) - m_{C'}(S) \right) \right\} \leq \min\left\{ \min_{S \in 2^{[n-1]}_{\neq \emptyset}} \left( M_{C'}(S) - t_{C'}(S) \right), \min_{S \in 2^{[n-1]}_{\neq \emptyset}} \left( T_{C'}(S) - m_{C'}(S) \right) \right\} = Q_C^{R'}.
\]

Therefore, the proof of Part c is complete.

We conclude by proving Part d. As C' is satisfiable, by definition there exist (μ'^{R'})_{j \in [n]} s.t. (μ'^{R'})_{j \in [n]} ∈ μ'^{R'} · Δ^R for every R' ∈ 2^{[n]}_{\neq \emptyset} and \sum_{R' \in 2^{[n]}_{\neq \emptyset}} μ'^{R'} ∈ [t_j, T_j] for every j ∈ [n]. For every (j, R') ∈ [n] × 2^{[n]}_{\neq \emptyset}, if j ≠ n or R' ∉ \{R, \{n\}\}, we let μ'^{R'} \triangleq μ'^{R'}; let μ'_n = μ'_n + μ and μ'^{n} \triangleq μ'^{n} - μ. (As μ'^{n} = μ'^{n} = μ' + μ, we have that μ'^{n} ∈ R_{≥}.)

For every R' ∈ 2^{[n]}_{\neq \emptyset} \setminus \{R, \{n\}\}, by definition (μ'^{R'})_{j \in [n]} = (μ'^{j})_{j \in [n]} ∈ μ'^{R'} · Δ^R = μ'^{R'} · Δ^R. Furthermore, as (μ'^{R'})_{j \in [n]} ∈ μ'^{R'} · Δ^R and by definition of (μ'^{R'})_{j \in [n]} and as n ∈ R, we have that (μ'_j)_{j \in [n]} ∈ (μ' + μ) · Δ^R = μ' · Δ^R. Similarly, as (μ'^{n})_{j \in [n]} ∈ μ'^{n} · Δ^R and by definition of (μ'^{n})_{j \in [n]}, we have that (μ'_j)_{j \in [n]} ∈ (μ'^{n} - μ) · Δ^R = μ'^{n} · Δ^R. Finally, \sum_{R' \in 2^{[n]}_{\neq \emptyset}} μ'^{R'} = \sum_{R' \in 2^{[n]}_{\neq \emptyset}} μ'^{R'} ∈ [t_j, T_j] for every j ∈ [n], and the proof is complete.

**Lemma 14** (Distributing All Mass but μ' among [n − 1]). Let C = \left( (μ'^{R'})_{R \in 2^{[n]}_{\neq \emptyset}}; ([t_j, T_j])_{j \in [n]} \right) be a normal distribution constraint s.t. μ'^{n} ≥ t_n. We say that condition DC holds if \( T_{C'}(S) ≥ m_C(S \cup \{n\}) - μ'^{n} \) for every S ∈ 2^{[n-1]}_{\neq \emptyset}.

a. If T^n = μ'^{n}, then condition DC holds.

b. If Q_C^R = 0 for every R ∈ 2^{[n]}_{\neq \emptyset} s.t. \{n\} ⊈ R and μ'^R > 0, then condition DC holds.

For every R ∈ 2^{[n-1]}_{\neq \emptyset}, let μ'^{R} = μ'^{R} + μ'^{R,j} \{n\}. Define C' = \left( (μ'^{R})_{R \in 2^{[n]}_{\neq \emptyset}}; ([t_j, T_j])_{j \in [n-1]} \right).

c. If condition DC holds, then C' is normal.

d. If C' is satisfiable, then the C is satisfiable.

**Remark 8.** Once again, the condition of Lemma 14(c) is actually also necessary; i.e., C' is normal iff condition DC holds.

**Proof of Lemma 14.** Part a holds as for every S ∈ 2^{[n-1]}_{\neq \emptyset}, \( T_{C'}(S) = T_{C'}(S \cup \{n\}) - T^n = m_C'(S \cup \{n\}) - T^n \geq m_C'(S \cup \{n\}) - T^n = m_C'(S \cup \{n\}) - μ'^{n} \).

To prove Part b, define S_1 and S_2 as in the proof of Lemma 13(b); as in that proof, it suffices to show that if condition DC does not hold, then there exists R ⊆ S_2 \ S_1 s.t. \{n\} ⊈ R and μ'^R > 0. As in that proof, we extend the definition of m_C'(S), M_C'(S), t_C'(S) and T_{C'}(S) also to the case S = \emptyset. By Part a, T_n > μ'^{n}. If S_2 ≠ [n], then the proof follows as in the proof of Lemma 13(b) (as that proof only uses the fact that μ'^{n} < T_n when S_2 ≠ [n], and does not rely
on the inequality $\mu^{(n)} < t_n$ for this case). It therefore remains to consider the case in which $S_2 = \{n\}$.

Recall that if $S_1 \neq \emptyset$, then $M_C(S_1) = t_C(S_1)$ by Lemma 12(a), and if $S_1 = \emptyset$, then $M_C(S_1) = 0 = t_C(S_1)$ by definition. As condition $D_C$ does not hold, there exists $S \in 2^{[n-1]} \setminus 2_{\emptyset}^{[n-1]}$ s.t. $T_C(S) < m_C(S \cup \{n\}) - \mu^{(n)}$.

Let $S' \triangleq S_1 \cap S \subseteq [n-1]$. We note that $t_C(S_1) - t_C(S') = t_C(S_1 \setminus S') \leq M_C(S_1 \setminus S') - \sum_{R \in 2^{[n-1] \setminus 2_{\emptyset}^{[n-1]}}}^\mu \cdot \sum_{R \in 2^{[n-1] \setminus 2_{\emptyset}^{[n-1]}}}^\mu \cdot \mu^{(n)} = M_C(S \setminus S') \leq T_C(S \setminus S') = T_C(S) - T_C(S') \leq m_C(S \cup \{n\}) - T_C(S') - \mu^{(n)} \leq m_C(S \cup \{n\}) - \sum_{R \in 2^{[n-1] \setminus 2_{\emptyset}^{[n-1]}}}^\mu \cdot \mu^{(n)}$. Therefore, $\sum_{R \in 2^{[n-1] \setminus 2_{\emptyset}^{[n-1]}}}^\mu \cdot \mu^{(n)} > 0$, and so there exists $R \subseteq (S \cup \{n\}) \setminus S'_2 \subseteq S_2 \setminus S_1$ s.t. $\{n\} \subseteq R$ and $\mu^R > 0$, as required, and the proof of Part b is complete.

We move on to Part c. For every $S \in 2^{[n-1]} \setminus 2_{\emptyset}^{[n-1]}$, as condition $D_C$ holds,

$$T_{C'}(S) = T_C(S) \geq m_C(S \cup \{n\}) - \mu^{(n)} = \sum_{R \in 2_{\emptyset}^{[n-1]}}^\mu \cdot \mu^{(n)} = m_C(S);$$

furthermore,

$$t_{C'}(S) = t_C(S) \leq M_C(S) = \sum_{R \in 2^{[n-1] \setminus 2_{\emptyset}^{[n-1]}}}^\mu \cdot \mu^{(n)} = \sum_{R \in 2^{[n-1] \setminus 2_{\emptyset}^{[n-1]}}}^\mu \cdot \mu^{(n)} = M_C(S).$$

Therefore, $C'$ is normal, as required.

We conclude by proving Part d. As $C'$ is satisfiable, by definition there exist $(\mu_j^R)_{R \in 2^{[n-1] \setminus 2_{\emptyset}^{[n-1]}}} \in \mu^R \cdot \Delta^R$ for every $R \in 2^{[n-1]} \setminus 2_{\emptyset}^{[n-1]}$ and $\sum_{R \in 2^{[n-1] \setminus 2_{\emptyset}^{[n-1]}}}^\mu \cdot \mu^{(n)} \in \{t_j, T_j\}$ for every $j \in [n-1]$. For every $R \in 2^{[n-1]} \setminus 2_{\emptyset}^{[n-1]}$, if $\mu^R = 0$, then we define $\mu_j^R \triangleq 0$ and $\mu_j^{R_j[n]} \triangleq 0$ for every $j \in [n-1]$; otherwise, we define $\mu_j^R \triangleq \frac{\mu^R}{\mu^R \cdot \Delta^R}$ and $\mu_j^{R_j[n]} \triangleq \mu_j^{R_j[n]} \cdot \Delta^R$ for every $j \in [n-1]$. We further define $\mu_j^{(n)} \triangleq 0$ for every $R \in 2^{[n-1] \setminus 2_{\emptyset}^{[n-1]}} \setminus \{n\}$, $\mu_j^{(n)} \triangleq \mu_j^{(n)}$ and $\mu_j^{(n)} \triangleq 0$ for every $j \in [n-1]$. If $\mu^R = 0$ then by definition $(\mu_j^R)_{j \in [n]} \equiv 0 \cdot \Delta^R = \mu^R \cdot \Delta^R$. and similarly $(\mu_j^{R_j[n]})_{j \in [n]} \equiv 0 \cdot \Delta^R = \mu^{R_j[n]} \cdot \Delta^{R_j[n]}$; otherwise, $\mu_j^{R_j[n]} \equiv \Delta^{R_j[n]}$. For every $R \in 2^{[n-1]} \setminus 2_{\emptyset}^{[n-1]}$, if $\mu^R = 0$ then by definition $(\mu_j^R)_{j \in [n]} \equiv 0 \cdot \Delta^R = \mu^R \cdot \Delta^R$. and similarly $(\mu_j^{R_j[n]})_{j \in [n]} \equiv \mu^{R_j[n]} \cdot \Delta^{R_j[n]}$. Finally, $\sum_{R \in 2^{[n-1]} \setminus 2_{\emptyset}^{[n-1]}}^\mu \cdot \mu^{(n)} = \mu^{(n)} \in [t_n, T_n]$ where $\mu^{(n)} = m_C(\{n\}) \leq T_C(\{n\}) = T_n$ by normality of $C'$ and the proof is complete.

Proof of Lemma 11. Let $C = \left( \left( \mu^R_{R \in 2^{[n-1] \setminus 2_{\emptyset}^{[n-1]}}} \right)_{j=1}^n \right)$ be a normal distribution constraint. We prove the claim by induction on $n \in \mathbb{N}$.

(Outer induction) Basis: For $n = 1$, we have by definition that $m_C(\{1\}) = \mu^{(1)} = M_C(\{1\})$, and so $t_1 = t_C(\{1\}) \leq M_C(\{1\}) = \mu^{(1)} = m_C(\{1\}) \leq T_C(\{1\}) = T_1$. Therefore, setting $\mu^{(1)} \triangleq \mu^{(1)}$ completes the proof of the (outer) induction base.
Lemma 13(c) \[ \sum \]

Lemma 14(c), \( C^\prime \) as defined inLemma 14 is normal, and by the (outer) induction hypothesis for \( n - 1 \), \( C^\prime \) is satisfiable. By Lemma 14(d), \( C \) is satisfiable as well.

(Inner induction) Step: Assume that \( \{R \in 2^{[n]}_{\neq \emptyset} \mid \{n\} \subseteq 2^{[n]}_{\neq \emptyset} \& \mu^R > 0 \& Q_C^R > 0\} = 0 \) and that the claim holds whenever this set is of smaller cardinality. Therefore, there exists \( R \in 2^{[n]}_{\neq \emptyset} \) s.t. \( \{n\} \subseteq R \), \( \mu^R > 0 \) and \( Q_C^R = 0 \). Let \( \mu = \min\{Q_C^R, \mu^R\} > 0 \), and define \( C^\prime \) w.r.t. \( R \) and \( \mu \) as in Lemma 13. By Lemma 13(c), \( C^\prime \) is normal. If \( \mu = \mu^R \), then \( \mu^R = 0 \); otherwise, \( \mu = Q_C^R \) and by Lemma 13(c), \( Q_C^R = Q_R^C - \mu = 0 \). Either way, and by definition of \( C^\prime \) and as by Lemma 13(c) \( Q_C^R = 0 \) whenever \( Q_C^R = 0 \), we have that \( \{R' \in 2^{[n]}_{\neq \emptyset} \mid \{n\} \subseteq R' \& \mu^R > 0 \& Q_C^{R'} > 0\} = 0 \) and so, by the (inner) induction hypothesis, \( C^\prime \) is satisfiable. By Lemma 13(d), \( C \) is satisfiable as well and the proof is complete.

\[ \square \]

A.1.4 Existence

Lemma 15 \( h_G = \max_{S \in 2^{[n]}_{\neq \emptyset}} E_G(S) \). Let \( G = ((f_j)_{j=1}^n; (\mu^R)_{R \in 2^{[n]}_{\neq \emptyset}}) \) be a resource selection game.

(a) \( \max_{S \in 2^{[n]}_{\neq \emptyset}} E_G(S) \in \mathbb{R} \) is well defined.

(b) If \( f_1, \ldots, f_n \) are continuous, then \( h_G = \max_{S \in 2^{[n]}_{\neq \emptyset}} E_G(S) \).

In both parts, the value undefined is treated as \(-\infty\) for comparisons by the Max operator.

Proof. To show Part a, note that by Lemma 2(a), \( E_G(\{1\}) = \text{Equalize}_{f_1}(\mu^{(1)}) = f_1(\mu^{(1)}) \in \mathbb{R} \); therefore, \( E_G(\{1\}) \in \mathbb{R} \), and so \( \max_{S \in 2^{[n]}_{\neq \emptyset}} E_G(S) \in \mathbb{R} \), as required.

Define \( A = \arg \max_{S \in 2^{[n]}_{\neq \emptyset}} E_G(S) \). Before proving Part b, we first show that for every \( S \in A \) and \( M' \in M_G(S) \), if \( (f_j)_{j \in S \setminus M'} \) are continuous, then also \( S \setminus M' \in A \). Let, therefore, \( S \in A \) and let \( M' \in M_G(S) \) s.t. \( (f_j)_{j \in S \setminus M'} \) are continuous. By definition of \( M' \), we have both that \( M' \subseteq S \) (see Lemma 4) and that \( \text{Equalize}_{f_k:k \in M'}(\mu) \neq E_G(S) \) for every \( \mu \leq \sum_{R \in 2^{[n]}_{\neq \emptyset}} \mu^R \).

By Part a, \( E_G(S) \in \mathbb{R} \), and so by definition there exist \( (\mu_j)_{j \in S} \in \mathbb{R}^{|S|} \) s.t. \( \sum_{j \in S} \mu_j = \sum_{R \in 2^{[n]}_{\neq \emptyset}} \mu^R \) and \( f_j(\mu_j) = E_G(S) \) for every \( j \in S \). Therefore, \( \text{Equalize}_{f_k:k \in S \setminus M'}(\sum_{j \in S \setminus M'} \mu_j) = E_G(S) \), and also \( \text{Equalize}_{f_k:k \in M'}(\mu) = E_G(S) \), where \( \mu = \sum_{j \in M'} \mu_j \).

As \( M' \in M_G(S) \), we thus have that \( \mu > \sum_{R \in 2^{[n]}_{\neq \emptyset} \setminus S \setminus M'} \mu^R \). Therefore, \( \sum_{R \in 2^{[n]}_{\neq \emptyset} \setminus S \setminus M'} \mu^R = \sum_{R \in 2^{[n]}_{\neq \emptyset} \setminus S \setminus M'} \mu^R - \mu = \sum_{j \in S} \mu_j - \mu = \sum_{j \in S \setminus M'} \mu_j = \sum_{j \in S \setminus M'} \mu_j \). Recall that \( \text{Equalize}_{f_k:k \in S \setminus M'}(\sum_{j \in S \setminus M'} \mu_j) = E_G(S) \in \mathbb{R} \); therefore, by continuity of \( (f_k)_{k \in S \setminus M'} \) and by Lemma 3(b), we obtain that also \( \text{Equalize}_{f_k:k \in S \setminus M'}(\sum_{R \in 2^{[n]}_{\neq \emptyset} \setminus S \setminus M'} \mu^R) \in \mathbb{R} \). Hence, by Lemma 1 we conclude that \( E_G(S \setminus M') = \text{Equalize}_{f_k:k \in S \setminus M'}(\sum_{R \in 2^{[n]}_{\neq \emptyset} \setminus S \setminus M'} \mu^R) \geq \text{Equalize}_{f_k:k \in S \setminus M'}(\sum_{j \in S \setminus M'} \mu_j) = E_G(S) \), and so indeed \( S \setminus M' \in A \), as required.
We conclude by proving Part b. By definition $h_G \leq \max_{S \in 2^{[n]} \setminus \emptyset} E_G(S)$; we therefore have to show that $h_G \geq \max_{S \in 2^{[n]} \setminus \emptyset} E_G(S)$. Let $S \in A$. We sequentially define a series $(S_i)_{i=0}^k$, for $k \in \mathbb{N}$ to be determined, as follows:

- $S_0 \define S$.
- If $M_G(S_i) = \emptyset$, then we set $k \define i$ and conclude. Otherwise, choose $M_i \in M_G(S_i)$ arbitrarily, and set $S_{i+1} = S_i \setminus M_i$.

We now show by induction that $S_i \in A$ and $|S_i| \leq |S| - i$ for every $i$ for which $S_i$ is defined.

- Base: By definition, $S_0 = S \in A$ and $|S_0| \leq |S| - 0 = |S| - 0$, as required.

- Step: Let $i > 0$ for which $S_i$ is defined. By the induction hypothesis, $S_{i-1} \in A$; therefore, as shown above and by continuity of $(f_j)_{j \in S_{i-1} \setminus M_{i-1}}$, we have that $S_i = S_{i-1} \setminus M_{i-1} \in A$.

Furthermore, as by definition $M_{i-1} \neq \emptyset$, we have by the induction hypothesis that $|S_i| = |S_{i-1}| - |M_{i-1}| \leq |S_{i-1}| - 1 \leq |S| - (i-1) - 1 = |S| - i$, as required.

We conclude that the process constructing $(S_i)_i$ indeed stops (i.e., $k$ is well defined), and with $k < |S|$. By definition, $M_G(S_k) = \emptyset$, and as $S_k \in A$, by Part a we have $E_G(S_k) = \max_{S \in 2^{[n]} \setminus \emptyset} E_G(S') \in \mathbb{R}$; therefore, $S_k \in D_G$. Therefore, $h_G \geq E_G(S_k) = \max_{S' \in 2^{[n]} \setminus \emptyset} E_G(S')$, as required.

We note that it can be shown that, in the context of Lemma 15(b), for every $S \in A$ s.t. $M_G(S) \neq \emptyset$, in fact $\bigcup M_G(S) \subseteq M_G(S)$ and $M_G(S \setminus \bigcup M_G(S)) = \emptyset$. While this may be used to avoid the inductive construction concluding the proof of this lemma, the need to prove these facts would result in a considerably longer total length for the proof.

**Lemma 16.** Let $G = ((f_j)_{j=1}^n; (\mu^R)_{R \in 2^{[n]}})$ be a resource selection game s.t. $f_1, \ldots, f_n$ are continuous. For every $S \in \arg \max_{S' \in D_G} E_G(S')$, there exists a strategy profile $s$ in the $|S|$-resource selection game $G' \define ((f_j)_{j \in S'}; (\mu^R)_{R \in 2^{[n]}})$, s.t. $h^*_j = h_G$ for every $j \in S$.

**Proof.** For every $j \in S$, let $t_j \define \min f_j^{-1}(h_G)$ and $T_j \define \max f_j^{-1}(h_G)$ if sup $f_j^{-1}(h_G) \neq \infty$ and $T_j \define \sum_{R \in 2^{[n]} \setminus \emptyset} \mu^R$ otherwise ($t_j$ and $T_j$ are well defined by continuity of $f_j$ and since $E_G(S) = h_G$); regardless of how we define $T_j$, we have that both $f_j(T_j) = h_G$ and $T_j \geq t_j$ (when sup $f_j^{-1}(h_G) = \infty$, this is since $E_G(S) = h_G$ and since $f_j$ is nondecreasing).

We now show that $C \define (\mu^R)_{R \in 2^{[n]} \setminus \emptyset}, ([t_j, T_j])_{j \in S}$ is a normal distribution constraint. (See Appendix A.1.3; we slightly abuse notation by treating $S$ in the context of $C$ as $[|S|]$, using an arbitrary isomorphism.) Let $S' \in 2^{|S|}$ \setminus \emptyset. As $M(S) = \emptyset$, there exists $\mu \leq \sum_{R \in 2^{[n]} \setminus \emptyset} |2^{S'} \setminus S'| \mu^R$ s.t. Equalize$_{f_k \in K}(\mu) = E_G(S) = h_G$; therefore, $I_C(S') = \sum_{j \in S'} t_j \leq \mu \leq \sum_{R \in 2^{S'} \setminus \emptyset} \sum_{j \in S'} |2^{S'} \setminus S'| \mu^R = M_C(S')$. Assume by way of contradiction that $\sum_{R \in 2^{S'} \setminus \emptyset} \mu^R > \sum_{j \in S'} T_j$. As $f_j(T_j) = h_G$ for every $j \in S'$, we have Equalize$_{f_k \in K}(\sum_{j \in S'} T_j) = h_G \in \mathbb{R}$. By continuity of $(f_j)_{j \in S'}$ and by Lemma 3(b), we thus have that $E_G(S') \in \mathbb{R}$; therefore, there exist $\mu_j \in S'$ s.t. $\sum_{j \in S'} \mu_j = \sum_{R \in 2^{S'} \setminus \emptyset} \mu^R$ and $f_j(\mu_j) = E_G(S')$ for every $j \in S'$. As $\sum_{j \in S'} \mu_j = \sum_{R \in 2^{S'} \setminus \emptyset} \mu^R > \sum_{j \in S'} T_j$, there exists $j \in S'$ s.t. $\mu_j > T_j$; as $T_j < \mu_j \leq \sum_{R \in 2^{S'} \setminus \emptyset} \mu^R \leq \sum_{R \in 2^{[n]} \setminus \emptyset} \mu^R$, we have that $T_j = \max f_j^{-1}(h_G)$, and so $E_G(S') = f_j(\mu_j) > h_G = \max_{S' \in 2^{[n]} \setminus \emptyset} E_G(S')$ (where the last equality is by Lemma 15(b)) — a contradiction. Therefore, $m_C(S') = \sum_{R \in 2^{S'} \setminus \emptyset} \mu^R \leq \sum_{j \in S'} T_j = T_C(S')$. 

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As $C$ is normal, by Lemma 11 it is satisfiable, and so there exist $(s_j(R))_{j \in S \neq \emptyset}^{R e \in 2^S}$ s.t. $s(R) \in \mu R \cdot \Delta R$ for every $R \in 2^S \neq \emptyset$ and $\sum_{j \in S \neq \emptyset} s_j(R) \in [t_j, T_j]$ for every $j \in S$. By the former, $s$ is a strategy profile in $G'$, and by the latter, for every $j \in S$ we have that $\mu_j \in [t_j, T_j]$, and so by definition of $t_j$ and $T_j$ and since $f_j$ is nondecreasing, $h_j^s = f_j(\mu_j^s) = h_G$ and the proof is complete.

□

Lemma 17 ($E_G(P_G) = h_G$). Let $G = (\{f_j\}_{j=1}^n; (\mu^R)_{R \in 2^S \neq \emptyset})$ be a resource selection game. If $f_1, \ldots, f_n$ are continuous, then $P_G \in \arg \max_{S \in D_G} E_G(S)$.

Proof. Define $A \triangleq \arg \max_{S \in D_G} E_G(S)$. By Lemma 4, $A \neq \emptyset$. Therefore, by definition of $P_G$, it is enough to show that $S' \cup S'' \in A$ for every $S', S'' \in A$. Let, therefore, $S', S'' \in A$; it is enough to show that $S' \cup S'' \in D_G$ and that $E_G(S' \cup S'') = h_G$.

By Lemma 16, there exists a strategy profile $s'$ in the game $((f_j)_{j \in S'}; (\mu^R)_{R \in 2^S \neq \emptyset})$ s.t. $h_j^s = h_G$ for every $j \in S'$. Therefore, by Lemma 1, we obtain $E_G(S' \cup S'') \geq (\mu^R)_{R \in 2^S \neq \emptyset} h_G = h_G$. Thus, by Lemma 15(b), $E_G(S' \cup S'') = h_G$. It therefore remains to show that $S' \cup S'' \in D_G$

We have to show that for every $S \in 2^{S' \cup S''}$, there exists $\mu \in \sum_{R \in 2^S \neq \emptyset} \mu R$ s.t. $E_{f, k} \in s(C) = E_G(S' \cup S'')$: let therefore $S \in 2^{S' \cup S''}$. Define $\mu \triangleq \sum_{j \in S} R_j$. As $f_j(\mu_j) = h_G$ for every $j \in S' \cup S''$, we have $E_{f, k} \in s(C) = h_G = E_G(S' \cup S'')$. By definition of $s$, we have that $\mu = \sum_{j \in S} \mu_j \leq \sum_{R \in 2^S \neq \emptyset} \mu R \leq \sum_{R \in 2^S \neq \emptyset} \mu R$, and the proof is complete.

□

Proof of Lemma 7. Part a follows directly from Lemmas 16 and 17. We therefore prove Part b. Let $G' \triangleq G - P_G$ and assume by way of contradiction that $h_{G'} \geq h_G$: recall that by definition $P_G' \subseteq \{n\} \setminus P_G$ and so $P_G$ and $P_G'$ are disjoint. As by Lemma 4, $P_G' \neq \emptyset$, we aim to obtain a contradiction by showing that $S' \cup S'' \subseteq D_G$.

By Lemma 17, $P_G' \in \arg \max_{S \in D_G} E_G(S)$; therefore, we have by definition that $h_G = E_G(P_G) = E_{f, k} \in s(C) = \sum_{R \in 2^S \neq \emptyset} \mu R$. By Lemma 4, $h_G \in \mathbb{R}$ and so there exist $(\mu_j)_{j \in P_G} \in \mathbb{R}_s \sum_{j \in P_G} \mu_j = \sum_{R \in 2^S \neq \emptyset} \mu R$ and $f_j(\mu_j) = h_G$ for every $j \in P_G$.

Similarly, by Lemma 17, $P_G' \in \arg \max_{S \in D_G} E_G(S)$; therefore, and by definition of $G'$, we have that $h_{G'} = E_{G'}(P_{G'}) = \sum_{R \in 2^S \neq \emptyset} \mu R$. By Lemma 4, $h_{G'} \in \mathbb{R}$ and so there exist $(\mu_j')_{j \in P_{G'}} \in \mathbb{R}_s \sum_{j \in P_{G'}} \mu_j' = \sum_{R \in 2^S \neq \emptyset} \mu R$ and $f_j(\mu_j') = h_{G'} \geq h_G$ for every $j \in P_{G'}$. Let $j \in P_{G'}$. As $f_j$ is nondecreasing, by Lemma 2(a) and by definition of $h_G$,
we have $f_j(0) \leq f_j(\mu(j)) = E_G(\{j\}) \leq h_G$. By continuity of $f_j$ and by the intermediate value theorem, there thus exists $\mu_j \in [0, \mu_j']$ s.t. $f_j(\mu_j) = h_G$.

As $f_j(\mu_j) = h_G$ for every $j \in P_G \cup P_G'$, by definition Equalize $f_k\in P_G \cup P_G'(\sum_{j \in P_G \cup P_G'} \mu_j) = h_G$. As $\sum_{j \in P_G \cup P_G'} \mu_j = \sum_{j \in P_G} \mu_j + \sum_{j \in P_G'} \mu_j \leq \sum_{j \in P_G} \mu_j + \sum_{j \in P_G'} \mu_j' = \sum_{R \subseteq 2^{P_G \cup P_G'}} R^R + \sum_{R \subseteq 2^{P_G \cup P_G'}} R^R$, we have by Lemma 3(b) that $E_G(P \cup P_G') \in \mathbb{R}$ and therefore, by Lemma 1, $E_G(P_G \cup P_G') \geq h_G$. Therefore, by definition of $P_G$, in order to show that $P_G \subseteq P_G$ and complete the proof, it is enough to show that $M_G(P_G \cup P_G') = \emptyset$.

Let $S \in 2^{P_G \cup P_G'}$. By Lemma 17, $M_G(P_G) = \emptyset$ and so, if $S \cap P_G \neq \emptyset$, then there exists $\mu'' \leq \sum_{R \subseteq 2^{P_G \cup P_G'}} R^R(\sum_{j \in P_G} \mu_j) \leq \sum_{R \subseteq 2^{P_G \cup P_G'}} R^R$ s.t. Equalize $f_k\in S \cap P_G'(\mu'') = E_G(P_G) = h_G$. If $S \cap P_G = \emptyset$, then set $\mu'' = 0$.

Similarly, by Lemma 17, $M_G(P_G') = \emptyset$ and so, if $S \cap P_G' \neq \emptyset$, then there exists $\mu' \leq \sum_{R \subseteq 2^{P_G \cup P_G'}} R^R(\sum_{j \in P_G'} \mu_j) \leq \sum_{R \subseteq 2^{P_G \cup P_G'}} R^R$ s.t. Equalize $f_k\in S \cap P_G'(\mu') = E_G(P_G') = h_G'$. As $f_j(\mu_j) = h_G$ for every $j \in P_G'$, we also have in this case that Equalize $f_k\in S \cap P_G'(\sum_{j \in S \cap P_G'} \mu_j) = h_G'$. By both of these and by Lemma 1, Equalize $f_k\in S \cap P_G'(\min\{\mu', \sum_{j \in S \cap P_G'} \mu_j\}) = \min\{h_G', h_G\} = h_G$ in this case. If $S \cap P_G' = \emptyset$, then set $\mu'' = 0$.

Define $\mu \equiv \mu'' + \min\{\mu', \sum_{j \in S \cap P_G} \mu_j\}$. By definition of $\mu$ (and by Lemma 2(b) if neither $S \cap P_G = \emptyset$ nor $S \cap P_G' = \emptyset$), we have that Equalize $f_k\in S(\mu) = h_G$; it is therefore enough to show that $\mu \leq \sum_{R \subseteq 2^{P_G \cup P_G'}} R^R(\sum_{j \in S \cap P_G} \mu_j) \leq \mu'' \leq \sum_{R \subseteq 2^{P_G \cup P_G'}} R^R(\sum_{j \in S \cap P_G} \mu_j) \leq \sum_{R \subseteq 2^{P_G \cup P_G'}} R^R$, as required. □

A.2 Proof of the Theorems and Corollary from Section 3.2

We defer the proof of Theorem 1 until after that of Theorem 3.

Proof of Theorem 2. We prove the theorem by full induction on $n$.

Let $n \in \mathbb{N}$ and assume that the theorem holds for all smaller natural values of $n$. Let $G = (G_{ij})^n_{i=1}$ be an $n$-resource selection game and let $s, s'$ be Nash equilibria in $G$. By Lemma 6(a,b), $h_j^* = h_j^s = h_j^s'$, for every $j \in P_G$. If $P_G = \{n\}$, then the proof is complete. Otherwise, let $s'' = \sum_{R \subseteq \emptyset} n_{-P_G} \rightarrow \{0\}^{n_{-P_G}}$ be the functions defined by $f_j^*(R') \equiv \sum_{R \in R(R',G-P_G)} s_j(R)$ and $s_j^*(R') \equiv \sum_{R \in R(R,G-P_G)} s_j(R)$ for every $j \in [n] \setminus P_G$. By Lemma 6(a,d), $s_j^*, s_j^{*'}$ are both Nash equilibria in $G - P_G$, and so, by the induction hypothesis (since $P_G \neq \emptyset$ by Lemma 4), we obtain that $h_j^{*'} = h_j^{*''}$ for every $j \in [n] \setminus P_G$. Therefore, by Lemma 6(a,d), we have $h_j^* = h_j^{s''} = h_j^{s'''} = h_j^s$ for every $j \in [n] \setminus P_G$ as well, and so $h_j^* = h_j^{s''}$ for every $j \in [n]$, as required. □

Proof of Corollary 1. We start by proving Part a. Let $s, s'$ be Nash equilibria in $G$, and let $R \in 2^{\emptyset \neq \emptyset}$. By definition of Nash equilibrium and by Theorem 2, we have for every $k \in \text{supp}(s(R))$ and $k' \in \text{supp}(s'(R))$, that $h_k^* = \min_{j \in R} h_j^* = \min_{j \in R} h_j^s = h_k^{s'}$, as required.

We move on to prove Part b: Let $j \in [n]$. Let $S = \{k \in [n] \mid h_k^* = h_j^*\}$; by Theorem 2, $S = \{k \in [n] \mid h_k^* = h_j^s\}$ as well. Let $R \equiv \{R \in 2^{[n]} \setminus \text{supp}(s(R)) \subseteq S\}$; by Theorem 2 and Part a, $R = \{R \in 2^{[n]} \setminus \text{supp}(s'(R)) \subseteq S\}$ as well. Assume w.l.o.g. that $h_j^*$ is not a plateau height of any of $S \setminus \{j\}$; we therefore have to show that $\mu_k^* = \mu_k^s$ for every $k \in S$. 36
For every $k \in S \setminus \{j\}$, as $h^*_j$ is not a plateau height of $f_k$, there exists a unique value $\mu_k \in \mathbb{R}_{\geq 0}$ s.t. $f_k(\mu_k) = h^*_j$. Therefore, and as by definition of $S$ and by Theorem 2 we have that $f_k(\mu^*_j) = h^*_k = h^*_j = h^*_j = f_k(\mu^*_j)$ for every $k \in S \setminus \{j\}$, we have that $\mu^*_k = \mu_k = \mu^*_k$ for every $k \in S \setminus \{j\}$. By Part a, we have that $\sum_{k \in S} \mu^*_k = \sum_{R \in R} \mu^*_R = \sum_{k \in S} \mu^*_k$, and so $\mu^*_j = \sum_{R \in R} \mu^*_R - \sum_{k \in S(\setminus \{j\})} \mu^*_k = \sum_{R \in R} \mu^*_R - \sum_{k \in S(\setminus \{j\})} \mu^*_k = \mu^*_j$. Therefore, $\mu^*_k = \mu^*_k$ for every $k \in S$ and the proof is complete. 

Proof of Theorem 3. We begin by proving Part a by full induction on $n$. Let $n \in \mathbb{N}$ and assume that the claim holds for all smaller natural values of $n$. Let $G = (\{f_j\}_{j=1}^n; (\mu_R)_{R \in 2^{[n] \setminus \emptyset}})$ be an $n$-resource selection game and let $s$ be Nash equilibrium in $G$. For every $R \in 2^{[n] \setminus \emptyset}$ with $\mu^*_R \neq 0$, let $h_R \triangleq h^*_j$ for every $j \in \text{supp}(s(R))$. Let $s'$ be a consumption profile s.t. $h^*_k < h_R$ for every $R \in 2^{[n] \setminus \emptyset}$ and $k \in \text{supp}(s'(R))$ s.t. $s'_k(R) > s_k(R)$. We must show that $s' = s$.

We begin by showing that $s'(R) = s(R)$ for every $R \in 2^{[n] \setminus \emptyset}$, for $P^*$ as defined in Lemma 6. Assume by way of contradiction that $s'(R) \neq s(R)$ for some $R \in 2^{[n] \setminus \emptyset}$. Let $S = \{j \in P^* | h^*_j < h_R\} \subseteq P^*$. As $s'(R) \neq s(R)$, there exists $k \in R$ s.t. $s'_k(R) > s_k(R)$ and so $k \in \text{supp}(s'(R))$. Therefore, by definition of $s'$ and by Lemma 6(b), we have that $h^*_k < h^*_k < h^*_k$ and so $k \in S$; in particular, $S \neq \emptyset$.

For every $j \in S$, by definition of $S$ and by Lemma 6(b), we have that $f_j(\mu^*_j) = h^*_j < h^*_j = f_j(\mu^*_j)$; therefore, as $f_j$ is nondecreasing we have that $\mu^*_j < \mu^*_j$ for every such $j$. Therefore, as $S \neq \emptyset$, $\sum_{j \in S} \mu^*_j < \sum_{j \in S} \mu^*_j$. By definition of consumption profile and by Lemma 6(c), $\sum_{j \in P^*} \mu^*_j \geq \sum_{R \in 2^{[n] \setminus \emptyset}} h^*_R = \sum_{j \in P^*} \mu^*_j$. By both of these, $\sum_{j \in P^*} \mu^*_j < h^*_R$. It follows that there exists $j \in P^* \setminus S$ s.t. $\mu^*_j > \mu^*_j$; hence, there exists $R' \in 2^{[n] \setminus \emptyset}$ such that $s'_j(R') > s_j(R')$ (and so $j \in \text{supp}(s'(R'))$), however, by definition of $S$ and as $f_j$ is nondecreasing, we have that

$$h^*_j = f_j(\mu^*_j) \geq f_j(\mu^*_j) = h^*_j = h^*_j > h^*_R \tag{2}$$

(as $s'_j(R') > 0$, $h^*_R$ is well defined), even though $s'_j(R') > s_j(R')$ — a contradiction. Therefore, $s'(R) = s(R)$ for every $R \in 2^{[n] \setminus \emptyset}$, by definition of $P^*$ and by Lemma 6(c), we thus obtain that $s'_j \equiv s_j$ for every $j \in P^*$.

If $P^* = [n]$, then the proof is complete. Otherwise, define $s'' : 2^{[n] \setminus \emptyset} \rightarrow \mathbb{R}_{\geq 0}^{[n] \setminus P^*}$ by $s''_j(R') \triangleq \sum_{R \in R(\mathcal{G},G,P^*)} s_j(R)$ for every $j \in [n] \setminus P^*$. By Lemma 6(d), $s''$ is a Nash equilibrium in $G - P^*$, and $h^*_j = h^*_j$ for every $j \in [n] \setminus P^*$. For every $R' \in 2^{[n] \setminus P^*}$ with $\mu^*_R \neq 0$, let $h^*_R \triangleq h^*_R$ for every $j \in \text{supp}(s''(R))$, by definition of $h^*_R$. For every $R' \in 2^{[n] \setminus P^*}$ and $R \in R(\mathcal{G},G,P^*)$ s.t. $\mu^*_R \neq 0$.

Similarly, define $s''' : 2^{[n] \setminus \emptyset} \rightarrow \mathbb{R}_{\geq 0}^{[n] \setminus P^*}$ by $s'''_j(R') \triangleq \sum_{R \in R(\mathcal{G},G,P^*)} s_j(R')$ for every $j \in [n] \setminus P^*$. As $s'''_j \equiv s_j$ for every $j \in P^*$, we have that, similarly to the proof of Lemma 6(d), $s'''$ is a strategy profile in $G - P^*$ and $h^*_j = h^*_j$ for every $j \in [n] \setminus P^*$. By definition, we have that $h^*_j = h^*_j < h^*_j = h^*_j$ for every $R' \in 2^{[n] \setminus P^*}$ and $k \in \text{supp}(s'''(R'))$ s.t. $s'''_j(R') > s_j(R')$, where $R \in R(\mathcal{G},G,P^*)$ s.t. $k \in \text{supp}(s'(R'))$ and $s_j(R') > s_k(R)$ (there exists such $R$ by definition of $R'$). By the induction hypothesis (since $P^* \neq \emptyset$ by definition), $s''' = s''$, and so $s' = s$ and the proof of Part a is complete.

The proof of Part b is very similar; the main difference is that in Eq. (2) we would have, by $h^*_j$ not being a plateau height of $f_j$, that $h^*_j < f_j(\mu^*_j) < h^*_j = h^*_j < h^*_R$. The remaining trivial differences between Parts a and b are left to the reader. 

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Proof of Theorem 1. We prove by full induction on \( n \) that in every \( n \)-resource selection game \( G = ((f_j)_{j=1}^n; (\mu^R)_{R \in 2^n_\emptyset}) \) s.t. \( f_1, \ldots, f_n \) are continuous, there exists a Nash equilibrium \( s \) s.t. \( \max_{j \in [n]} h^*_j \leq h_G \); from this claim, the existence of Nash equilibrium \textit{a fortiori} follows (while the existence of Nash equilibrium also follows from a theorem of Schmeidler (1973), we constructively reprove it here via hydraulic analysis rather than a nonconstructive fixed-point theorem).

The theorem then follows by Theorem 3. Let \( n \in \mathbb{N} \) and assume that the claim holds for all smaller natural values of \( n \). Let \( G = ((f_j)_{j=1}^n; (\mu^R)_{R \in 2^n_\emptyset}) \) be an \( n \)-resource selection game.

By Lemma 7(a), there exists a strategy profile \( s'' \) in the \([P_G]\)-resource selection game \( G'' \triangleq ((f_j)_{j \in P_G}; (\mu^R)_{R \in 2^n_\emptyset}) \) s.t. \( h^*_j = h_G \) for every \( j \in P_G \). By definition of Nash equilibrium, \( s'' \) is a Nash equilibrium in \( G'' \). If \( P_G = [n] \), then \( s \triangleq s'' \) is a Nash equilibrium as required, and the proof of the induction step is complete. Assume, therefore, that \( P_G \subseteq [n] \); hence, and since \( P_G \neq \emptyset \) by Lemma 4, by the induction hypothesis there exists a Nash equilibrium \( s' \) in the \([n] \setminus P_G\)-resource selection game \( G' \triangleq G - P_G \), s.t. \( \max_{j \in [n] \setminus P_G} h^*_j \leq h_{G'} \).

We construct a strategy profile \( s \) in \( G \) as follows: \( s(R) \triangleq s''(R) \) for every \( R \in 2^n_{\emptyset} \), and for every \( R' \in 2^n_{\emptyset} \setminus P_G \), we pick \( (s(R))_{R \subseteq R'(G')} \) arbitrarily among the tuples satisfying \( s(R) \in \mu^R \cdot \Delta_{R'} \) for every \( R \in \mathcal{R}(R', G') \) and \( \sum_{R \in \mathcal{R}(R', G')} s(R') = s''(R') \). This is a well-defined strategy profile in \( G \) since \( R' = R \setminus P_G \subseteq R \) for every \( R \in \mathcal{R}(R', G') \) and \( R' \in 2^n_{\emptyset} \setminus P_G \), and by definition of the player mass in \( G' \) and \( G'' \). By definition of \( s \), we have that \( h^*_j = h^*_j = h_G \) for every \( j \in P_G \) and \( h^*_j = h^*_j \) for every \( j \in [n] \setminus P_G \). Therefore, by definition of \( s'' \) we have that \( h^*_j = h^*_j \leq h_G < h_{G'} \) for every \( j \in P_G \), and by definition of \( s' \) and by Lemma 7(b), we have that \( h^*_j = h^*_j \leq h_G < h_{G'} \) for every \( j \in [n] \setminus P_G \). Therefore, we have that \( h^*_j \leq h_G \) for every \( j \in [n] \).

We complete the proof by showing that \( s \) is a Nash equilibrium in \( G \). For every \( R \in 2^n_{\emptyset} \), \( k \in \text{supp}(s(R)) \subseteq R \) and \( j \in R \), we have by definition of \( s \), \( s'' \) that \( h^*_k = h^*_k = h_G = h^*_j = h^*_j \). Let \( R \in 2^n_{\emptyset} \setminus 2^n_{\emptyset} \), \( k \in \text{supp}(s(R)) \) and \( j \in R \). By definition of \( s \), we have that \( k \in \text{supp}(s(R) \setminus P_G) \) \( \subseteq 2^n_{\emptyset} \setminus P_G \). If \( j \in [n] \setminus P_G \), then \( j \in R \setminus P_G \) and by definition of \( s \), \( s' \) we have that \( h^*_k = h^*_k = h^*_j \leq h^*_j \); otherwise, i.e., if \( j \in P_G \), then by Lemma 7(b) and by definition of \( s \), \( s' \), \( s'' \) we have that \( h^*_k = h^*_k \leq h_G < h_G = h^*_j = h^*_j \). Either way, \( h^*_k \leq h^*_j \) and the proof is complete.

\[ \square \]

A.3 Proof of Theorem 5 from Section 5

Proof of Theorem 5. As in the main text, we prove only one part, leaving the proof of the other (trivial) part to the reader. Assume that no fractional perfect marriage exists.

By Theorem 1, there exists a (strong) Nash equilibrium \( s \) in \( G \). As no fractional perfect marriage exists, by Lemma 8 we have that not all loads in \( s \) are 1. As by definition of \( G \) the average of all loads in \( s \) is 1, we have that the highest load in \( s \) is greater than 1. By Lemma 6(b), we therefore have \( h_G > 1 \). Therefore, by definition there exists a set of pistons \( S \in 2^n_{\emptyset} \) s.t. \( E_G(S) > 1 \). As \( f_j = \text{id} \) for all \( j \in S \), we have, as in Remark 3 and by definition of \( \mu^R \), that \( 1 < E_G(S) = \text{Equalize}_{f_j; j \in S} \left( \sum_{R \subseteq 2^n_{\emptyset}} \mu^R \cdot |S| \right) = \left( \sum_{R \subseteq 2^n_{\emptyset}} \mu^R \right) \cdot |S| = \sum_{R \subseteq 2^n_{\emptyset}} \mu^R \cdot |S| \). As by definition, \( R \subseteq 2^n_{\emptyset} \subseteq S \), we have that \( \left| \{ i \in [n] \mid R^i \in 2^n_{\emptyset} \} \right| > |S| \geq \left| R \subseteq 2^n_{\emptyset} \right| \), i.e., that \( |I| > |R^i| \) for \( I \triangleq \{ i \in [n] \mid R^i \in 2^n_{\emptyset} \} \), as required.

\[ \square \]
A.4 Proof of Proposition 1 from Section 6, and Auxiliary Results

**Definition 13.** Let $G = ((f_j)_{j=1}^n; (\mu^R)_{R \subseteq 2^n \setminus \emptyset})$ be a resource selection game s.t. $f_1, \ldots, f_n$ are continuous. We denote by $h_j(G) \in \mathbb{R}$ the value $h_j^*$ for every Nash equilibrium $s$ in $G$.

**Remark 9.** $h_j(G)$ is well defined by Theorems 1 and 2.

**Lemma 18.** Let $G = ((f_j)_{j=1}^n; (\mu^R)_{R \subseteq 2^n \setminus \emptyset})$ be a resource selection game s.t. $f_1, \ldots, f_n$ are continuous, and let $S \subseteq [n]$. For every $j \in [n] \setminus S$, $h_j(G - S) \geq h_j(G)$.

**Remark 10.** By Lemma 6(a,d), taking $S \triangleq P_G$ in Lemma 18 yields an equality.

*Proof of Lemma 18.* We prove the lemma by full induction on $n$.

(Outer induction) Step: Let $n \in \mathbb{N}$ and assume that the lemma holds for all smaller values of $n$. We prove the (outer) induction step by full induction on $n - |S|$.

(Inner induction) Base: If $S = [n]$, then the claim vacuously holds.

(Inner induction) Step: Let $S \subset [n]$ and assume that the (outer) induction step holds for all $S$ of larger cardinality. We consider two cases.

If $S \supseteq P_G$, then by Lemma 5(b), by the (outer) induction hypothesis (since $P_G \neq \emptyset$ by Lemma 4), and by Remark 10, we have $h_j(G - S) = h_j((G - P_G) - (S \setminus P_G)) \geq h_j(G - P_G) = h_j(G)$, as required.

Otherwise, i.e., if $P' \triangleq P_G \setminus S \neq \emptyset$, we claim that $h_{G - S} \geq h_G$. By Lemma 17, we have that $P_G \in \arg \max_{S' \subseteq D_G} E_G(S')$; in particular, $P_G \in D_G$. Since $E_G(P_G) = h_G \in \mathbb{R}$ by Lemmas 4 and 17, we therefore have that $M(P_G) = \emptyset$. Therefore, $P' \notin M(P_G)$, and so there exists $\mu \leq \sum_{R \subseteq 2^n \setminus \emptyset}^{P'} 2^{(S \setminus P') / \mu R} \mu^R$ s.t. $\text{Equalize}_{f_k, k \in P'}(\mu) = E_G(P_G)$. We note that $2^{P_G \setminus 2^n \setminus \emptyset} \setminus 2^{(S \setminus P') / \mu R} = \bigcup_{R \in 2^n \setminus \emptyset}^{P'} \mathcal{R}(R', G - S)$, where the union is of disjoint sets; therefore, $\mu \leq \sum_{R \subseteq 2^n \setminus \emptyset}^{P'} 2^{(S \setminus P') / \mu R} \mu^R \leq \sum_{R' \subseteq 2^n \setminus \emptyset}^{P'} \mathcal{R}(R', G - S) \mu^R$. By Lemmas 1 and 3(b), we therefore have that $h_G = E_G(P_G) = \text{Equalize}_{f_k, k \in P'}(\mu) \leq \text{Equalize}_{f_k, k \in P'}(\bigcup_{R' \subseteq 2^n \setminus \emptyset}^{P'} \mathcal{R}(R', G - S) \mu^R) = E_G(S) \leq h_{G - S}$, where the last inequality is by Lemma 15(b).

For every $j \in P_G - S$, by Lemma 6(a,b), we have $h_j(G - S) = h_{G - S} \geq h_G \geq h_j(G)$. For every $j \in [n] \setminus (S \cup P_G - S)$, by Remark 10, by Lemma 5(b) and by the (inner) induction hypothesis (since $P_G - S \neq \emptyset$ by Lemma 4), we have $h_j(G - S) = h_j((G - S) - P_G - S) = h_j(G - (S \cup P_G - S)) \geq h_j(G)$ and the proof is complete.

*Proof of Proposition 1.* We prove that $h_1, \ldots, h_n$ are nondecreasing by full induction on $n$. Let $n \in \mathbb{N}$ and assume that the claim holds for all smaller values of $n$. Let $G = ((f_j)_{j=1}^n; (\mu^R)_{R \subseteq 2^n \setminus \emptyset})$ be an $n$-resource selection game s.t. $f_1, \ldots, f_n$ are continuous, let $R \subseteq 2^n \setminus \emptyset$, let $\mu^R > \mu^R$ and let $G'$ be the game obtained from $G$ by increasing the mass of player type $R$ from $\mu^R$ to $\mu^R$.

By Lemma 15(b) and by Lemma 1, $h_G' = \max_{S \subseteq 2^n \setminus \emptyset}^{P_G} E_G'(S) \geq \max_{S \subseteq 2^n \setminus \emptyset}^{P_G} E_G = h_G$. Therefore, by Lemma 6(a,b), $h_j(G') = h_{G'} \geq h_G = h_j(G)$ for every $j \in P_G'$; it therefore remains to show that $h_j(G') \geq h_j(G)$ for every $j \in [n] \setminus P_G'$ as well. Before we show this, we claim that $h_j(G' - P_G') \geq h_j(G - P_G')$ for every $j \in [n] \setminus P_G'$; to show this, we consider two cases. If $R \subseteq P_G'$, then by definition $G' - P_G' = G - P_G'$, and so $h_j(G' - P_G') = h_j(G - P_G')$ for every $j \in [n] \setminus P_G'$. Otherwise, i.e., if $R \cap P_G' \neq \emptyset$, then by definition $G' - P_G'$ is the game obtained from $G - P_G'$ by increasing the mass of player type $R \setminus P_G'$ by $\mu^R - \mu^R > 0$. Therefore, by the induction hypothesis (since $P_G' \neq \emptyset$ by Lemma 4), we therefore have that $h_j(G' - P_G') \geq h_j(G - P_G')$ for every $j \in [n] \setminus P_G'$ in this case as well. Finally, by Remark 10 and Lemma 18, we therefore have for every $j \in [n] \setminus P_G'$ that $h_j(G') = h_j(G' - P_G') \geq h_j(G - P_G') \geq h_j(G)$, and the proof by induction is complete.

We move on to prove continuity of $h_1, \ldots, h_n$; for simplicity, we show continuity only for the case in which $f_1, \ldots, f_n$ are strictly increasing. W.l.o.g. we will show that $h_1$ is continuous;
let $\varepsilon > 0$. For every $j \in [n]$, let $\mu_j(G) = f_j^{-1}(h_j(G))$ — this is the value $\mu_j^s$ for every Nash equilibrium $s$ in $G$. By continuity of $f_1$, there exists $\delta > 0$ s.t. $|f_1(\mu) - f_1(\mu_1(G))| < \varepsilon$ for every $\mu \in (\mu_1(G) - \delta, \mu_1(G) + \delta)$. Let $\mu^R \in (\mu^R - \delta, \mu^R + \delta)$ and denote by $G'$ the game obtained from $G$ by changing the mass of player type $R$ from $\mu^R$ to $\mu^R$. For every $j \in [n]$, let $\mu_j(G') = f_j^{-1}(h_j(G'))$. We first consider the case in which $\mu'_R \geq \mu^R$. In this case, as shown above, for every $j \in [n]$ we have that $h_j(G') \geq h_j(G)$; as $f_j$ is increasing, therefore $\mu_j(G') \geq \mu_j(G)$ for every such $j$. As $\sum_{j\in [n]} \mu_j(G') = \sum_{R\in 2^{[n]} - \{R\}} \mu^R + \mu^R = \sum_{j\in [n]} \mu_j(G) + \mu^R - \mu^R < \sum_{j\in [n]} \mu_j(G) + \delta$, we therefore have that $\mu_j(G) \leq \mu_j(G') < \mu_j(G) + \delta$ for every $j$. In particular, $\mu_1(G) \leq \mu_1(G') < \mu_1(G) + \delta$, and so $0 \leq h_1(G') - h_1(G) = f_1(\mu_1(G')) - f_1(\mu_1(G)) < \varepsilon$, as required. The case in which $\mu'_R < \mu^R$ is analogous.

The proof of Part b is virtually identical to that of the continuity of $h_1, \ldots, h_n$, noticing that we can choose $\delta = \frac{\varepsilon}{K}$, where $K$ is the Lipschitz constant of $f_1$. □