DYNAMIC REVENUE MAXIMIZATION ON A NETWORK

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ABSTRACT. This paper studies the allocation of several heterogeneous objects to buyers with multidimensional private information. Motivated primarily by airline-pricing problems, we impose certain substitution and complementarity assumptions on the buyers’ preferences over bundles of objects and represent the allocation problem as a directed graph, or a network. We give sufficient conditions for Bayesian implementation of the efficient or revenue-maximizing allocation problems in a static environment where agents can shill-bid. We also study a dynamic revenue-maximization problem where a monopolist needs to sell all the objects before a certain deadline to short-lived consumers that arrive over time. We show the optimal allocation rule is a cut-off rule and this rule can be implemented by a post-price mechanism in the case where agents are not allowed to shill-bid. We then give sufficient conditions for post-price being optimal when agents can shill-bid. The cut-offs (or prices) for each object are deterministic and evolve over time, depending on not only the supply of this object, but also the supplies of complementary and substitute objects.

1. Introduction

In many real-world monopoly-pricing problems, sellers have different products to sell within a certain amount of time. These products are often heterogeneous and linked to each other. This link comes from buyers’ preferences over bundles of these products. That is, buyers can treat products in a bundle as complements and see different bundles as substitutes. For instance, in the airline industry, each airline company owns a flight network. It has many different flights going on the same time, most of which are interconnected. Moreover, consumers have different departure cities and destinations, and they can choose different routes to satisfy their demand. For instance, a customer who wants travel from New York to Barcelona has many choices: she can buy either a direct flight or indirect connecting flights. Another such example is IKEA, where a customer who wants to decorate a kitchen can either choose an already designed kitchen or buy pieces separately.

These examples suggest that a seller should take both the links among objects and the time till the deadline in to account while determining his pricing policy. Moreover, on the demand side, each consumer has multi-dimensional private information (the subsets of desired objects and valuations)
which creates novel incentive problems. As an example to these problems recently United Airlines sue cheap-airfare website called Skiplagged.\footnote{http://techcrunch.com/2014/12/30/united-and-orbitz-sue-hidden-cities-flight-search-engine-skiplagged/} The Skiplagged website finds a longer flights that include a stop in your destination en route to another destination. Existing research usually focuses on one aspect of this problem: auctions of heterogeneous objects are static; on the other hand, the literature on dynamic pricing mostly deals with homogeneous goods (Board and Skrzypacz, 2013) or heterogeneous goods but buyers private information is one-dimensional (Gershkov and Moldovanu, 2012). In these papers, we can not observe the effect of the heterogeneity to the revenue management problem.

In this paper, we provide a succinct framework to analyze revenue management problems in the presence of the heterogeneity described above. In particular, we model this heterogeneity as a directed graph where the edges of the graph represent different objects and the adjacency matrix of the graph describes the complementarities among the objects. The model captures the essential features of many real-life situations. For instance, in the airline pricing problem mentioned before, for a given directed graph, each consumer’s demand is represented by three parameters: her departure and destination cities and the valuation attached to her trip.

We study both static and dynamic versions of this model. In the static case, we first consider the efficient-allocation problem. In this problem it is natural to assume that buyers can shill-bid\footnote{We assumed short-lived consumers as a first approximation of the problem. From the literature we know that problem of long-lived consumers is much more complicated.} (Imagine a reservation system for airline). Shill-bid allows for submitting bids using different identifiers. We use an example to show that the VCG mechanism is not dominant-strategy incentive compatible. This example also helps us to observe the effect of network structure on allocation rule. Moreover, this particular incentive to shill bid can be very costly to the seller. Then we give necessary and sufficient conditions on the distributions of buyers’ valuations, under which the buyers do not have the incentive to shill-bid (dominant strategy incentive compatibility) in the optimal mechanism.

As a next step we consider a dynamic monopoly-pricing problem where the monopolist needs to sell a limited number of heterogeneous objects before a certain deadline and short-lived consumers arrive over time.\footnote{http://techcrunch.com/2014/12/30/united-and-orbitz-sue-hidden-cities-flight-search-engine-skiplagged/} In this case the optimal allocation rule is a cut-off rule: in each period and for each object, if a consumer arrives and demands that object, then she receives that object only if her valuation is above a certain cut-off, which is also the transfer she needs to pay. The cut-offs for each object are deterministic and evolve over time, depending on not only the supply of this object,
but also the supplies of complementary objects. Under certain assumptions the optimal shill-bid proof mechanism can be implemented by post price mechanisms in which prices are determined from the cut-offs. Specifically, a good’s price will increase whenever a substitute of that good sold and it’s price will decrease whenever its complement sold. To put it differently, the heterogeneity among the objects in our dynamic model allows for rich and intuitive price dynamics, which can be taken as a theoretical basis for further empirical investigations of the pricing patterns in various industries. Indeed, in the airline industry, there is strong empirical evidence (McAfee and te Velde, 2006, Lazarev, 2012) that ticket prices vary frequently in a non-monotone fashion. Our results give at least a partial explanation of this phenomenon: the ticket price of each flight varies with the supplies of all connecting flights; this link through network increase the volatility of prices.

The rest of the paper is organized as follows: In section 2 we survey the literature. In Section 3 we introduce the static model, formulate and solve the efficient allocation and revenue maximization problems. Section 4 studies the dynamic model. Section 5 discusses several extensions. The Appendix contains all the proofs that are not given in the main text.

2. Literature Review

Revenue management (RM) problems have been studied extensively in Operation Research (OR) and Economics. The former is older and extensive, recently economists have also started to show interest in the topic and build on the models in the OR literature. In OR, RM models have some common features: (i) assuming myopic and non strategic buyers (lack of private information), (ii) using dynamic models, (iii) focus on computational performance (applicability) of the proposed solution. Talluri and van Ryzin (2006) have an excellent book that surveys the OR literature on RM. In the OR literature, our paper is closest to Talluri and van Ryzin (1998), which shows the sub-optimality of bid pricing in the network RM models. Our paper introduces private information and shill-bidding behavior to the framework of Talluri and van Ryzin (1998).

The economics literature incorporates private information into the problem and use mechanism design techniques to analyze them.\(^3\) Pai and Vohra (2013) and Board and Skrzypacz (2013) analyze the strategic-buyer case. In these models buyers are strategic in the sense that they can lie about their arrival time to the market. Both of these models are dynamic and consider the case of identical goods. Gershkov and Moldovanu (2009) consider the case of heterogeneous goods, but the goods

\[^3\]On the topic of dynamic mechanism design, Bergemann and Said (2011) and Vohra (2012) are excellent surveys. The most recent papers on the dynamic revenue management are Pai and Vohra (2013a), Board and Skrzypacz (2013), Dizdar, Gershkov and Moldovanu (2011), Gershkov and Moldovanu (2009), and Dilme and Li (2012). Our paper shares some similarities with each of these papers, but it does not fully overlap with any of them.
are ordered according to a common quality parameter. Our model also features heterogeneous goods, but the ordering is different from the one in Gershkov and Moldovanu (2012).\textsuperscript{4} Different than the former papers, Dilme and Li (2012) look at the case in which the designer cannot commit to a mechanism a priori.

In static models, combinatorial auctions are the closest to our setup.\textsuperscript{5} Similar to our paper, Abhishek and Hajek (2010) and Ledyard (2007) look at the optimal (revenue maximizing) combinatorial auction in the case of single-minded buyers. However, both of the papers are silent about shill-bidding. Yokoo et. al (2004) show that the VCG mechanism is vulnerable to shill-bidding and give sufficient conditions when it is not. Compared to Yokoo et al (2004), our focus and conditions for sufficiency are different.

\section{3. The Static Problem}

A seller sells \(K\) types of products which is represented by a supply vector \((c_1, \ldots, c_K) \in \mathbb{N}^K\). Each buyer has unit demand for a specific type of product. Specifically, a buyer of type \((k, v) \in \{1, \ldots, K\} \times \mathbb{R}_+\) only wants to buy one unit of product \(k\) and his private valuation associated with product \(k\) is \(v\). Each buyer’s type and valuation of the product are his private information.

\textbf{Objects.} Products in this model can be described as a directed graph, \(G(N, E)\), where \(N\) denotes the set of nodes and \(E\) denotes the set of edges.\textsuperscript{6} This graph \(G\) can be represented by a \(N \times N\) matrix with entries denoting the edges. For instance having 1 at entry \((1, 2)\) denotes that there exist an edge from node 1 to node 2, this edge has origin 1 and destination 2. If entry \((1, 2)\) is zero this means there is no edge from 1 to 2. A path \(p\), is sequence of distinct edges (without cycles) which has an origin \(o\) and a destination \(d\). For instance, \(\{(1, 2), (2, 3), (3, 5)\}\) denotes a path from 1 to 5. Every path represents a specific type of product. Let \(\mathcal{P}_G\) denotes set of all paths of graph \(G\) and \(\tilde{\mathcal{P}}_G\) be the set of of all subsets of \(\mathcal{P}_G\). Following the matrix notation we can denote the seller’s supply vector by matrix \(C\) where each entry denotes supply of each type of product. For instance, having 5 on entry \((1, 2)\) means that seller has 5 units of product of type \((1, 2)\).

\textbf{Buyers.} Each buyer’s private information is represented by a 2-dimensional vector \((v, \theta)\), where \(\theta\) denotes the acceptance set (i.e., the set of bundles that satisfy the buyer’s demand), and \(v\) is her valuation. Thus, we denote the space of private information as \(\tilde{\mathcal{P}}_G \times V\). We assume that the total demand for each type of product is random. Specifically, for each \(\theta\), the number of corresponding

\textsuperscript{4}We also suggest some similar extension at Section 5.
\textsuperscript{5}For combinatorial auctions, Cramton (2006) is an excellent reference.
\textsuperscript{6}Similar to Talluri and van Ryzin (1998).
buyers $L_\theta$ is a discrete random variable that takes values from $\{m_\theta, m_\theta + 1, \ldots, m_\theta\} \subset \mathbb{N}_+$ with a strictly positive probability mass function $l_\theta$. Let $L = \prod_{i \in \tilde{P}_G} L_i$ denote the (random) demand in the problem. The valuation $v$ of each type $(v, \theta)$ takes values from $[\underline{v}_\theta, \bar{v}_\theta]$ and is distributed according to a cumulative distribution $G_\theta$ with strictly positive density $g_\theta$. Each buyer is risk neutral and knows the supply of each good; however, she does not observe the exact number of buyers for each type of the product.

Buyer $i$ gets utility $v$ if and only if she obtains a set of objects which is an element of her acceptance set. Otherwise she will receive zero utility. Each $\theta$ is derived from an origin $(o)$ and destination $(d)$ pair. Let $r(o, d)$ be the set of all possible paths from $o$ to $d$ given the graph $G$. We denote $\theta^{i}(o, d)^7$ the the collection of subsets that contain $r(o, d)$. The payoff of buyer $i$ with private information $(v, \theta)$ when she is assigned to bundle $x^i$ is

$$u^i(x^i; v, \theta) = \begin{cases} v & \text{if } x^i \text{ is an element of } \theta^i \\ 0 & \text{otherwise} \end{cases}$$

Remark 1. This model is equivalent to a combinatorial auction with single-minded buyers. Single-mindedness means that buyers only care for a specific bundle of goods and will get utility if and only if their assigned bundles contain that specific bundle. In our setup single-mindless can be interpreted as only caring about the set $\theta^i$. 8

Allocation and Payment. By the revelation principle, we focus on direct mechanisms.9 A mechanism $(A, T)$ consists of an allocation rule $A$ and a transfer rule $T$. The (random) allocation rule $A$ is defined as

$$A : V \times \tilde{P}_G \rightarrow \Delta \left( \tilde{P}_G \right).$$

The transfer rule $T$ is given by

$$T : V \times \tilde{P}_G \rightarrow \mathbb{R}.$$

Note that it is without loss of generality to consider deterministic transfer rules. In fact we will show that the optimal mechanism is deterministic.

7Since there is a one to one mapping between acceptance set and $(o, d)$ pair we omit the subscript while mentioning $\theta$.

8Our setup is also related to the problems with dichotomous domain considered by Mishra and Roy (2013). A dichotomous domain is a domain in which each buyer’s type is described by two parameters: an acceptance set and a valuation. However, our domain does not satisfy the richness assumption in Mishra and Roy (2013) due to the fact that $\theta$ can not be a singleton.

**Solution Concept.** We mostly focus on Bayesian incentive compatible (BIC) mechanisms. We will also show that, different from Myerson (1981), in this model Bayesian and dominant strategy incentive compatibility are not equivalent in general; however, under our assumptions they will become equivalent.

Let \( P(v', \theta' | v, \theta) \) be the highest probability that type \((v, \theta)\) gets an element in \(\theta\) by reporting \((v', \theta')\). Formally \( P(\cdot | \cdot) \) defined as follows

\[
P(v, \theta | v, \theta) = \begin{cases} 
\alpha & \text{if } \exists p \text{ (path) s.t. } p \in \theta \text{ and } p \subseteq A(v, a) \\
0 & \text{otherwise}
\end{cases}
\]

where \(\alpha\) is the maximum probability of getting such a path.

Given a mechanism \(\langle A, T \rangle\), the payoff of a type \((v, \theta)\) buyer when she reports type \((v', \theta')\) is

\[vP(v', \theta' | v, \theta) - T(v', \theta')\]

The incentive compatibility (IC) constraints are: for all \((v, \theta)\)

\[vP(v, \theta | v, \theta) - T(v, \theta) \geq vP(v'_i, \theta'_i | v, \theta) - T(v'_i, \theta'_i),\]

for all \((v', \theta')\).

The shill-bidding constraints (SB) are (with slight abuse of notation):

\[vP(v, \theta | v, \theta) - T(v, \theta) \geq vP \left( \bigcup_{j \in D} (v_j, \theta_j) | v, \theta \right) - \sum_{i \in D} T(v_j, \theta_j)\]

for every set of identifiers \(D\), for each \(j \in D\) and for every \((v_j, \theta_j)\)\(^{10}\).

Individual Rationality (IR) constraints are: for all \((v, \theta)\),

\[vP(v, \theta | v, \theta) - T(v, \theta) \geq 0\]

Note that our shill-bidding constraints are stronger than the IC constraints, since they allow buyers to submit multiple bids and each bid can be arbitrary. Since the number of buyers is stochastic, the seller can not detect a deviation by just looking the number of bids. Let \(U(v, \theta) \equiv vP(v, \theta | v, \theta) - T(v, \theta)\). We define the feasibility constraint (F) for the allocation function as follows: for each component \((m, n)\), \(C_{(m, n)} \leq X_{(m, n)}\) where

\[C = \sum_k \sum_{i \in N} a_i(v, \theta).\]

\(^{10}\)In this model shill-bidding is costless while submitting multiple identifiers agents do not incur any internal cost.
Then we can write the revenue maximization problem of the seller as follows\textsuperscript{11}

\[
\max_{a,T} \sum_{L(Z)} \sum_{\theta \in Z} \sum_{i=1}^{N_\theta} \int g_{\theta i}(v, \theta) dv_\theta \\
\text{s.t} IC, IR, SB, F.
\]

We can write the program for the efficient allocation problem as follows

\[
\max_{a,T} \sum_{L(Z)} \sum_{\theta \in Z} \sum_{i=1}^{N_\theta} \int g_{\theta i} dv_\theta \\
\text{s.t} : IC, IR, SB, F.
\]

Next we define the following partial order ($\succeq$) over the set of types:\textsuperscript{12} \((v', \theta') \succeq (v, \theta)\) iff \(v' \geq v\) and \((o, d)\) pair of \(\theta\) is a member of some path in \(r(o', d')\) of \(\theta'\). We assume the following condition based on this order:

**Assumption 1.**

\[
(v', \theta') \succeq (v, \theta) \Rightarrow \frac{g_{\theta}(v')}{1 - G_{\theta'}(v')} \leq \frac{g_{\theta}(v)}{1 - G_{\theta}(v)}
\]

This assumption implies that, for any fixed \(\theta\), the conditional hazard rate is non-decreasing in \(v\). Also, for any given \(\theta\) and \(\theta'\) if \((o, d)\) pair of \(\theta\) is a member of some path in \(r(o', d')\) of \(\theta'\), then valuations are ordered in the hazard rate order.\textsuperscript{13} The hazard rate order implies that \(G_{\theta'} \geq_{\text{FOSD}} G_{\theta}\). Intuitively, for any two buyers, if one buyer wants a longer path (in the order we defined) than the other, then the former should have a higher value in the sense of first order stochastic dominance than the latter. This assumption is plausible in many environments, including the airline-pricing models.

3.1. **Efficiency.** We first show there is no efficient, dominant strategy incentive compatible (DSIC), and individually rational (IR) mechanism. It is well-known that the VCG (Vickrey-Clarke-Groves) mechanism is the unique mechanism satisfying all these requirements in standard models without shill-bidding. In the following example, we show that the VCG mechanism does not work due to shill-bidding.\textsuperscript{14}

\textsuperscript{11}Note that \(A\) and \(T\) are interim allocation and transfer rules, their respective ex-post versions are defined as \(a\) and \(t\).

\textsuperscript{12}Similar to Abhishek and Hajek (2010), Pai and Vohra (2013a), Dizdar, Moldovanu and Gershkov (2011).

\textsuperscript{13}For more on stochastic orders, see Shaked and Shanthikumar (2007).

\textsuperscript{14}Alternatively, we can look the case collusion among the buyers but in our setup which is not feasible. ( For instance in airline pricing examples)
Consider an airline company that operates two flights. Flight 1 is from city A to city B, flight 2 from city B to city C. For simplicity, assume that there is one seat available on each flight. There are three types of buyers: AB, BC, and AC. Suppose that there are two AC buyers with valuations 10 and 9, respectively. Suppose that there is only one AB buyer and one BC buyer; these two buyers have the same valuation 3.

\[
\begin{array}{ccc}
\text{AB} & \text{BC} & \text{AC} \\
3 & 3 & 10 \\
& & 9 \\
\end{array}
\]

Suppose that a VCG mechanism is adopted. If each buyer reports her value truthfully, then AC buyer with valuation 10 wins and pays 9, while all other buyers do not pay anything. Now consider the following deviation by the AC buyer with valuation 9, she can shill-bid as an AB buyer and a BC buyer, both bidding 9 for each good. As a result, she wins both AB and BC and only pays 6 in total, and her payoff is 3 rather than 0. This profitable deviation comes from the fact that there are no reserve prices in a VCG mechanism (or the lowest bids to accept are not defined correctly).

Let us focus on the above example and try to find conditions for BIC. After shill-bidding there are three possibilities for an AC buyer, either she only wins AB, or she only wins BC, or she wins both AB and BC. In the last case where the buyer wins both objects, if conditional on winning she pays more than when she sticks to bidding as an AC type, then shill-bidding is not profitable.\(^{15}\) This gives the following inequality:

\[
P(\theta_1 + \theta_2, AC \mid v, AC) [v - T(\theta_1 + \theta_2, AC \mid v, AC)] \geq \\
P(\theta_1, AB \mid v, AC) P(\theta_2, BC \mid v, BC) [v - T(\theta_1, AB \mid v, AC) - T(\theta_2, BC \mid v, AC)].
\]

If \(T(AC, \theta_1 + \theta_2) < T(\theta_1, AB \mid v, AC) - T(\theta_2, BC \mid v, AC)\), then the above inequality holds, since we have

\[
P(\theta_1 + \theta_2, AC \mid v, AC) \geq P(\theta_1, AB \mid v, AC) P(\theta_2, BC \mid v, BC).
\]

Suppose that the valuations of AB and BC buyers are distributed uniformly on \([0, 1]\) and AC buyers valuations are distributed uniformly on \([0, 2]\), and suppose that there are \(N_{AB}, N_{BC}, N_{AC}\) buyers present for each type.\(^{16}\) Let \(k, m\) and \(n\) be the supplies of AB, BC and AC, respectively.

\(^{15}\) Yokoo et al. (2004) look for sufficient conditions for DSIC.

\(^{16}\) Here we choose the uniform distribution to use the fact that the order statistic of (regular) uniform distribution follows beta distribution and it is relatively easier to find its expectation using beta functions. For normal and exponential distributions we can get similar results (see David and Nagaraja, 2004).
For ease of exposition, assume \( k = m = n = 1 \). Then conditional on winning the expected transfer of \( AC \) buyer is:
\[
E \left[ \min \left\{ V_{AC}^{(1,N_{AC}-1)}, V_{AB}^{(1,N_{AB})} + V_{BC}^{(1,N_{BC})} \mid P(v, \theta_{AC} \mid v, \theta_{AC}) = 1 \right\} \right],
\]
which is less than \( E \left[ \min V_{AC}^{1,N_{AC}-1} \mid P(v, \theta_{AC} \mid v, \theta_{AC}) = 1 \right] \).

Conditional on winning, sum the expected transfer of \( AB \) and \( BC \) types is given by,
\[
E \left[ \max \left\{ V_{AC}^{(1,N_{AC})} - \theta_1, V_{AB}^{(1,N_{AB}-1)} \mid P(v, \theta_{AB} \mid v, \theta_{AB}) = 1 \right\} \right]
+E \left[ \max \left\{ V_{AC}^{(1,N_{AC})} - \theta_1, V_{BC}^{(1,N_{BC}-1)} \mid P(v, \theta_{BC} \mid v, \theta_{BC}) = 1 \right\} \right],
\]
which is larger than
\[
E \left[ V_{AB}^{1,N_{AB}} \mid P(v, \theta_{AB} \mid v, \theta_{AB}) = 1 \right] + E \left[ V_{BC}^{1,N_{BC}} \mid P(v, \theta_{BC} \mid v, \theta_{BC}) = 1 \right].
\]

Our goal is to show
\[
E \left[ V_{AC}^{1} \mid P(v, \theta_{AC} \mid v, \theta_{AC}) = 1 \right] \leq E \left[ V_{AB}^{1,N_{AB}} \mid P(v, \theta_{AB} \mid v, \theta_{AB}) = 1 \right]
+E \left[ V_{BC}^{1,N_{BC}} \mid P(v, \theta_{BC} \mid v, \theta_{BC}) = 1 \right],
\]
which, by the assumption of uniform distribution, reduces to
\[
\frac{N_{AC}(\theta_1 + \theta_2)}{1 + N_{AC}} < \frac{N_{AB}\theta_1}{1 + N_{AB}} + \frac{N_{BC}\theta_2}{1 + N_{BC}}.
\]

We want the above condition to hold for every \( \theta_1, \theta_2 \in [0, 1] \). This is guaranteed when \( N_{AC} \leq N_{AB} \) and \( N_{AC} \leq N_{AB} \). That is, for an \( AC \) type buyer, if there is a profitable deviation by shill-bidding, then there is always a better deviation by bidding directly in the \( AC \) market. By this argument and the fact that the VCG pricing rule is weakly DSIC, we conclude that buyers will not shill-bid. In other words, this condition means that if the sub-markets are competitive then it is not profitable for buyers to enter these markets. Note that in this case our condition does not involve supplies; however, this is not true in general (see Appendix). Another observation is the following: in dynamic settings, \( i \) and \( k \) are decreasing over time; as a result, shill-bidding is less likely to be profitable as time passes. Note that this assumption is more reasonable in

\[17\]In appendix we give a proof for general supplies.
settings where the number of buyers are much larger than the capacity (such as airline or clothing industry).

Efficiency Problem: Solution. Under the above assumption, the efficient allocation can be implemented by VCG transfers. The main question then is how to compute the efficient allocation. Here we give an example where we use a simple algorithm to compute the efficient allocation. Again, suppose there are only three nodes (A, B, and C) and three edges (AB, BC, and AC) with arbitrary finite supplies.

An algorithm for computing the efficient allocation:

Suppose the supplies of AC, AB and BC are $k_1$, $k_2$, and $k_3$, respectively.

Step 1) Take the first $k_1$ highest valuations of AC and assign each of them an AC object.

Step 2) Case 1: Look at the difference $k_2 - k_3 > 0$ then take the $k_2 - k_3$ highest of AB and assign them AB objects. Define the vector $\{AB + BC\}$ in the following way by adding $(k_2 - k_3 + i)$-th highest of AB to $i$th highest of BC $\{AB^{k_2-k_3+i} + BC^i\}_{i=1}$. Then consider the following set

$$\{AB^{k_2-i} + BC^{k_3-i}\}_{i=1}^{k_3-1} \bigcup \{AC^{j}\}_{j=k_1+1}^{k_3}$$

and then choose the highest $k_3$ from this set and assign their respective objects.

Case 2: If $k_2 - k_3 < 0$ then take the $k_3 - k_2$ highest of BC and assign them BC objects.

Define the vector $\{AB + BC\}$ in the following way by adding $k_3 - k_2 + ith$ highest of AB to $ith$ the highest of BC $\{AB^{k_2-i} + BC^{k_3-i}\}_{i=1}^{k_3-1}$. Then consider the following set

$$\{AB^{k_2-i} + BC^{k_3-i}\}_{i=1}^{k_2-1} \bigcup \{AC^{j}\}_{j=k_1+1}^{k_3}$$

and then choose the highest $k_2$ from this set and assign their respective objects.

Case 3: If $k_2 = k_3$, then define the vector of $\{AB + BC\}$ in the following way $ith$ highest of AB to $ith$ highest of BC $\{AB^i + BC^n\}_{i=1}$. Then consider the following set

$$\{AB^{k_2-i} + BC^{k_2-i}\}_{i=1}^{k_2} \bigcup \{AC^{j}\}_{j=k_1+1}^{k_2}$$

and then choose the highest $k_2$ from this set and assign their respective objects.

Two immediate implications of the above algorithm are: (i) if there is at least one AC product available, then it is not optimal to assign an AC type buyer one AB and one BC products (ii) for each product the marginal value of an additional capacity is decreasing, keeping the other capacities fixed.
Following a similar algorithm, we can find the efficient allocation for any graph $G$, and arbitrary supplies by creating its conflict graph and finding the Maximum Weight Independent sets. In general it is well known that this problem is NP-hard and difficult to approximate. Therefore, in this paper we will focus on simple graphs.

3.2. **Revenue Maximization.** We will solve the revenue maximization mechanism in the following way. First, we divide each IC constraint into two parts. The first deals with lies about the valuations

$$vP(v, \theta | v, \theta) - T(v, \theta) \geq vP(v', \theta | v, \theta) - T(v', \theta)$$

for all $v$ and $v'$. The second deals with lies about the destination

$$vP(v, \theta | v, \theta) - T(v, \theta) \geq vP(v, \theta' | v, o, d) - T(v, \theta')$$

for all $(o', d')$ and last constraint is the shill-bidding

$$vP(v, \theta | v, \theta) - T(v, \theta) \geq vP \left( \bigcup_{j \in D} (v_j, \theta_j) \right) | v, \theta \right) - \sum_{i \in D} T(v_j, \theta_j)$$

for any set $D$ of the edges.

We focus on exact allocation rules (without loss of generality). Exact allocation rules are the rules such that for each buyer it assigns a “minimal” bundle from their acceptance set or the empty set. An allocation rule $A$, is exact iff for all $(v, \theta)$

$$A(v, \theta) = p \in \Delta (v, \theta)$

Given the fact that we focus on exact allocation rules, the only possible misreport is $\theta' \supseteq \theta$. Then this makes above constraints sufficient for global incentive compatibility. In fact, adding

$$vP(v, \theta | v, \theta) - T(v, \theta) \geq vP(v, \theta' | v, o, d) - T(v, \theta')$$

to

$$vP(v, \theta' | v, \theta') - T(v, \theta') \geq vP(v', \theta' | v, \theta') - T(v', \theta')$$

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18To be more specific, in our setup we are trying to find a Maximum Weight Independent set in a matroid, we can always write a greedy algorithm to solve this problem. This method is applied by Lehmann, O’Callaghan, and Shoham (2002). Mu’alem and Nisan (2008) improve the existence results in the case with one dimensional private information using linear programming relaxation. For more on combinatorial optimization over matroids, Chapter 40 of Schrijver (2003) is an excellent reference.

19Our definition of simple graph is little bit restrictive than the graph theoretic definition. Beside, not having loops and multiple edges we also assume there is always at most two path connecting any two vertices.
and using exactness of the allocation rule, we have
\[ vP(v, \theta | v, \theta) - T(v, \theta) \geq vP(v', \theta' | v, \theta) - T(v', \theta'). \]

We then follow the classic mechanism design approach: first we ignore the constraints of truthfully reporting the destination and shill-bidding proofness, then we impose a condition on the distribution function in order to satisfy those constraints. Finally we show that the optimal mechanism will satisfy all the constraints.

By envelope theorem (following standard methods), for every \( \theta \), we write the equilibrium utility of each \( (v, \theta) \) type as follows
\[
U(v, \theta) = U(v, \theta) + \int_0^v P(v, \theta) dv
\]

Next using the IR constraint we could set \( U(v, \theta) = 0 \). This allows us to write the seller’s relaxed problem as follows (the proof is in the appendix)
\[
\max_P \sum_{\theta \in Z} L(Z) \sum_{i=1}^{N_\theta} \int \left( v - \frac{1 - G_\theta(v)}{g_\theta(v)} \right) P(v, \theta) g_\theta(v) dv_
\]
\[ s.t \quad v \geq v' \Rightarrow P(v, \theta) \geq P(v', \theta) \]

Finally, define the conditional virtual valuation \( \psi_\theta \) as
\[
\psi_\theta(v) = v - \frac{1 - G_\theta(v)}{g_\theta(v)}.
\]

Under assumption 1, given \( \theta \) the conditional virtual valuation is increasing in \( v \). Also, for a fixed \( v \), and for every \( \theta, \theta' \) such that \( (\theta_1, \theta_2, \ldots, \theta_j) \) pair of \( \theta \) is a member of some path in \( r(\theta_1, \theta_2, \ldots, \theta_j) \) of \( \theta' \), we have \( \psi_\theta(v) \geq \psi_{\theta'}(v) \). Then this condition guarantees that buyers will not misreport their destinations. We then obtain a solution to the relaxed problem by maximizing the above expression point-wise. We will also give a necessary and sufficient conditions to make buyers’ shill-bidding unprofitable for the solution of the relaxed problem.

**Proposition 1.** Under assumption 1, a mechanism (revenue maximizing mechanism which uses the transfer rule \( T(v, \theta) = \inf \{v | P(v, \theta) = 1\} \)) is dominant shill-bid proof iff for every \( \theta \) and for every set \( D \) such that the edges\( (\theta_1, \ldots, \theta_j) \) can form a path in \( \theta \), \[ \sum_{\theta_i \in D} \psi_{\theta_i}^{-1}(0) \geq \psi_\theta. \]
Proof. Suppose the mechanism is shill-bid proof but condition does not hold. Then there exist a case beside acceptance set $\theta$ all other acceptance sets buyers have virtual valuation below the $(\psi_\theta(v) \leq 0)$ the reserve price and at $\theta$ there exist $j$ such that $\psi^j_\theta(v) = \bar{v}_\theta$. Then choose an buyer $i$ ($i \neq j$), which has acceptance set $\theta$ and valuation such that $v > \sum_{\theta_i \in D} \psi^{-1}_{\theta_i}(0)$ then by shill-bidding buyer $i$ can get the good and pay $\sum_{\theta_i \in D} \psi^{-1}_{\theta_i}(0)$. This contradicts that mechanism is shill-bid proof.

Suppose the condition does hold but mechanism is not shill-bid proof. Then there exist an buyer $i$ and $\theta$ such that $v - \sum_{i \in D} T(v_j, \theta_j) > 0$. But this contradicts the assumption since $\sum_{i \in D} T(v_j, \theta_j) \geq \sum_{\theta_i \in D} \psi^{-1}_{\theta_i}(0) \geq v_\theta$. \hfill $\square$

Remark 2. There are conditions in the literature to deter buyers from shill-bidding (See Ausubel and Milgrom, 2002, Lehmann et al., 2006, and Sher, 2012). All these conditions are substitutes conditions; however, our domain does not satisfy these conditions: some goods are complements and some are substitutes.\textsuperscript{20}

Remark 3. If Assumption 1 is not satisfied, AB and BC customers are willing to imitate to AC customers and win both objects and disregard the extra object.

The previous remark suggests that if a firm can control how it creates the products before implementing the mechanism, then it should split the products into many nearly identical markets so that the mechanism is immune to shill-bidding.

Remark 4. Suppose the assumptions we imposed are not satisfied, then one of the IC constraints must be binding at the optimal solution. Given that our optimization problem is well behaved we can use Lagrangian relaxation approach. We create new virtual values as adding dual Lagrangian variables of binding constraints to the conditional virtual values. If we knew that shill-bidding constraints bind, then in the optimal mechanism buyers will have the bigger $\theta$ pairs in the sense of our order $\succeq$ will be subsidized. This approach was suggested in Pai and Vohra (2013a) without calculating the duals but fully executed by Pai and Vohra (2013b) in a different context.

Remark 5. Myerson’s optimal auction (under the assumptions that buyers are ex-ante symmetric and virtual valuations are increasing) is efficient in the sense that when it assigns the object it assigns to the buyer with the highest valuation.\textsuperscript{21} In our case the optimal mechanism is not efficient

\textsuperscript{20}Sher (2012) shows that in the case of mixture goods, finding optimal shill-bidding strategy (cost minimization problem) for the buyer is equivalent to finding the efficient allocation in the combinatorial auction (the winner determination problem).

\textsuperscript{21}In the literature this notion is called as quasi-efficiency.
in the above sense. To see this, note that the optimal mechanism assigns two products, \((i, k)\) and \((k, j)\), to a buyer of type \((i, j)\) because \(\psi_{(i,j)}(v) > \psi_{(i,k)}(v') + \psi_{(k,j)}(v'')\), but it is possible that \(v < v' + v''\), where inefficiency occurs. It is also interesting to note that the resulting mechanism (under the assumptions) is weakly dominant strategy incentive compatible. This result is due to the existence of reserve prices. We know that in our setup any mechanism that implements the efficient allocation is not dominant strategy incentive compatible from Yokoo et. al (2004).

**Example 1.** Suppose there is only 3 types \(AB, BC,\) and \(AC\). Assume that the valuation of each type of buyer is distributed uniformly over \([0, 1]\). That makes \(\psi_{o,d}(v) = 2v - 1\) for each \((o, d)\) and then all of our assumptions above are satisfied. Let supply vector be \((1, 1, 1)\) and suppose two buyers are available for each type. Then we can describe the optimal mechanism as follows: Let \(\psi_{o,d}(v^i)\) denotes the \(i\)th highest virtual value of given \((o, d)\) pair. Then

1. \(v^1_{AC}\) gets the object iff 
   \[\psi_{AC}(v^1_{AC}) \geq 0;\]
2. \(v^2_{AC}\) gets the object iff 
   \[\psi_{AC}(v^2) \geq \max\{\psi_{AB}(v^1_{AB}), 0\} + \max\{\psi_{BC}(v^1_{BC}), 0\};\]
3. \(v^1_{AB}\) gets the object iff 
   \[\psi(v^1_{AB}) \geq 0\text{ and } \psi_{AB}(v^1_{AB}) + \max\{\psi_{BC}(v^1_{BC}), 0\} \geq \max\{\psi_{AC}(v^2_{AC}), 0\};\]
4. \(v^1_{BC}\) gets the object iff 
   \[\psi(v^1) \geq 0\text{ and } \psi_{BC}(v^1_{BC}) + \max\{\psi_{AB}(v^1_{AB}), 0\} \geq \max\{\psi_{AC}(v^2_{AC}), 0\}.\]

4. **The Dynamic Problem without Shill-Bidding**

In this section, we consider the dynamic version of the basic model. Suppose that the monopolist wants to sell \(K\) types of products with supply vector \((c_1, \ldots, c_K) \in \mathbb{R}_+^K\) over \(T \geq 2\) periods. Time is discrete, labeled as \(t = 1, 2, \ldots, T\). We assume that buyers arrive over time and are impatient, i.e., they need to be served upon arrival.

**Buyer Arrival Process and Distributions.** We further assume that in each period \(t\), there will be only one buyer in the market.\(^{22}\) First, we consider the case where the monopolist observes which type of buyers arrive in every period.\(^{23}\) A type \(\theta\) buyer’s valuation \(v\) takes values from

\(^{22}\)We can handle the multiple buyers of same type case, our proves in the appendix handle multiple buyers case. Alternatively, one can think time is continuos then our assumption holds trivially.

\(^{23}\)We elaborate on shill-bidding next section.
[\nu_\theta, \bar{v}_\theta] and is distributed according to a cumulative distribution \( G_\theta \) with strictly positive density function \( g_\theta \). We assume the demand process is stationary.

**Histories.** Let \( z_t \) be the reports received by the mechanism in period \( t \). Analogously to static model, let \( C_t \) denote the available supply in period \( t \). Define \( \omega_t \) as all the reports up to period \( t \) and allocations up to period \( t - 1 \), and \( \overline{z}_t \) as all reports and allocation decisions up to and including period \( t \). Let \( H_t \) be set of all possible histories in period \( t \), and let \( \overline{H}_t \) be the set of all possible end of period histories.

**Buyers.** Similar to the static case, each buyer is described by a pair \((v, \theta)\). At the beginning of period \( t \), each buyer, up on arrival, reports her type; if she is not assigned an object at time \( t \), she will leave the market. Each buyer only knows her type and the available capacities in that period. She does not observe the number of buyers in the market and the past history.

**Allocation Rule.** Define \( a_t(v, \theta) \) as the period \( t \) allocation rule

\[
a_t : \left( V \times \mathcal{P}_G \right) \leq t \times H_t \rightarrow \Delta \left( \mathcal{P}_G \right).
\]

Note that here we consider ex-post allocation rules as oppose to interim allocation rules. The dynamic version of the feasibility constraint (F) becomes:

\[
C \geq \sum_t \sum_k \sum_{i \in N} a_{i,t}(v, \theta).
\]

**Sellers Problem.** We can write seller’s revenue maximization problem as follows:

\[
\max_{(a_t)_{t=1}^T} \sum_{t=1}^T \sum_{Z_t} \sum_{\theta \in Z_t} \sum_{i=1}^{\theta_t} \int g_{\theta t_i}(v, \theta) d\nu_\theta
\]

s.t IC, IR, F.

This revenue maximization problem can be written recursively as

\[
V_t(R, C) = \max_{a_t} \left( \sum_{i \in N} a(v, \theta) \psi_\theta(v_i) + \mathbb{E}_{R'V_{t+1}(R', C')} \right)
\]

s.t : \( K = \sum_{i \in N} a(v, \theta) \)

\( C' = C - K \)

\( R' \) i.i.d
where \( R \) denotes the realization of the buyers value and \( V_t(R, C) \) is the seller’s value function in period \( t \) given the current state variables.

**Knapsack Problems.** Before analyzing this problem, first note that our problem can be viewed as a generalized version of the stochastic knapsack problem. Knapsack problem can be described as follows: Given a set of objects, each with a mass and a value, agent tries to maximize the sum of the values given capacity constraints. Our problem is a dynamic version of this problem with stochastic values but fixed masses (supplies). Each edge can be seen as a knapsack with given capacity and the buyers can be interpret as an object that covers one capacity. Different than the classical knapsack problem, objects can be decomposed into pieces and can be put to multiple knapsacks. For instance, an AC ticket can be decomposed in to AB and BC tickets and then put in to the corresponding knapsacks.

In the economics literature there are papers that study this problem, such as Pai and Vohra (2013a), and Dizdar, Gershkov and Moldovanu (2011). Similar to Pai and Vohra (2013a) we will focus on simple index rules. An index function, \( I : T \to \mathbb{R} \), is a mapping from the type space to the reals. we say a solution is an index rule, if the solution of the knapsack problem uses only this index and the local information at each period.\(^{24}\)

### 4.1. Dynamic Efficiency

To implement the efficient allocation, we can use the dynamic VCG mechanism. Each buyer \( i \) reports her type \( v \) (one dimensional private information) to the designer in the period \( t \) when she arrives. The payment rule will become \( p_i(h_t) = v_i - V^*(h_t) - V^*(h_{t-i}) \), where \( h_{t-i} \) denotes the history up to period \( t \) without the buyer \( i \). Moreover, under this payment rule buyers will always report their arrival times truthfully.\(^{25}\)

### 4.2. Dynamic Revenue Maximization

Under our assumptions, the result in the dynamic setting can be summarized by the following propositions.

**Proposition 2.** Given supply matrix \( C \), the optimal allocation rule awards a unit to the highest value buyer in any given period if her value passes a deterministic cut-off. The cutoffs are determined by \( \Delta_i V_t(R, C_{AC}, C_{AB}, C_{BC}) \), where

\[
\Delta_{AC} V_t(R, C_{AC}, C_{AB}, C_{BC}) = V_t(R, C_{AC}, C_{AB}, C_{BC}) - V_t(R, C_{AC} - 1, C_{AB}, C_{BC}),
\]

others cut-offs are defined analogously.

---

\(^{24}\)In our case the virtual valuation is the index function. Alternatively we can call it as a Markov decision rule.

Proof. See the appendix. □

Next proposition states the properties of these cut-offs.

**Proposition 3.**

1) \( \Delta^t_i(C_{AC}, C_{AB}, C_{BC}) \) is (weakly) decreasing in each term (strictly for some) for each \( i \in \{AC, AB, BC\} \).

2) For \( i \in \{AB, BC\} \), \( \Delta^t_i(C_{AC}, C_{AB}, C_{BC}) \) is weakly decreasing in \( C_{AC} \).

3) \( \Delta^t_{AC}(C_{AC}, C_{AB}, C_{BC}) \) is (weakly) decreasing with \( C_{AB} \) and \( C_{BC} \).

4) For \( i, j \in \{AB, BC\} \) and \( i \neq j \), \( \Delta^t_i(C_{AC}, C_{AB}, C_{BC}) \) is (weakly) increasing with \( C_j \).

5) For each \( i \in \{AC, AB, BC\} \), \( \Delta^t_i(C_{AC}, C_{AB}, C_{BC}) \) is (weakly) decreasing on \( t \).

Proof. See the appendix. □

This result says that over time if no object is assigned, every cut-off decreases due to the deadline effect. When a sale of a given type of object occurs, it has two effects: the first is a direct effect as now objects of that given type have fewer supply; the second is an indirect effect on other types of objects. The latter effect comes from the network structure. For instance, we show that after a sale of an \( AC \) type object, the cut-offs of both \( AB \) and \( BC \) types increase, due to the fact that buyers who want to obtain these objects now face more competition. Interestingly, when a sale of an \( AB \) type object occurs, the cut-off of \( BC \) type object weakly decreases, as now \( BC \) type buyers will face less competition from \( AC \) type buyers. In other words, we show that the network effect leads to interesting patterns for the evolution of the cut-offs. This case can be implemented by post-price mechanisms in which prices are equal to the cut-offs. Cut-offs are in terms of virtual valuations to derive price cut-offs we invert the virtual value function then corresponding valuation becomes the prices.

### 5. The Dynamic Problem with Shill-Bidding

In this section we give sufficient conditions for above mechanism to be still optimal under shill-bidding. Consider the possibility that a buyer may misreport her type \( \theta \).

**Proposition 4.** Under Assumption 1, for every \( t \), type \( AB \) or \( BC \) buyers incentive constraints are satisfied in the optimal mechanism without shill-bidding.

Proof. There are two cases to check i) \( C_{AC} > 0 \), and ii) \( C_{AC} = 0 \). In the first case, since we know that when a buyer reports \( AC \) type and there is supply of \( AC \), he will be assigned an \( AC \) ticket. Therefore in this case \( AB \) or \( BC \) type buyers will not lie. For the second case, observe that for every
t, $\Delta_{AC} V_t(0, C_{AB}, C_{BC}) \geq \max(\Delta_{AB} V_t(0, C_{AB}, C_{BC}), \Delta_{BC} V_t(0, C_{AB}, C_{BC}))$. Previous observation coupled with Assumption 1\textsuperscript{26} imply that neither $AB$ nor $BC$ type will lie on the $\theta$ dimension.

Assume that even if $AC$ submits two bid: one $AB$, one $BC$ seller do not notice that $AC$ type is lying.\textsuperscript{27} The following condition guarantee that $AC$ will never shill-bid.

**Assumption 2.** Assume $\psi^{-1}_{AB}(0) + \psi^{-1}_{BC}(0) \geq \psi^{-1}_{AC}(0)$ and $\inf_{(\psi^{-1}_{AC}(0), \bar{v}_{AC})} \psi'_{AC}(x) \geq \sup_{(\psi^{-1}_{i}(0), \bar{v}_i)} \psi'_{i}(x)$ for $i \in AB, BC$.

This assumption says that cut-offs in virtual valuation agrees with cut-offs in valuations. One way to guarantee this is assuming $AC$ type’s virtual valuation curve is always steeper then the $AB$ and $BC$’s virtual valuation curve. Assumption 1 and Assumption 2 are always satisfied if each type comes from the same distribution with super-additive virtual value function. Uniform and exponential families also satisfies the assumptions.

**Proposition 5.** Under assumption 1 and 2\textsuperscript{28} optimal mechanism can be implemented by post-price.

**Proof.** There are two cases to check $i) C_{AC} > 0$, and $ii) C_{AC} = 0$. In both cases, $\Delta_{AC} V_t(C_{AC}, C_{AB}, C_{BC}) \leq \Delta_{AB} V_t(C_{AC}, C_{AB}, C_{BC}) + \Delta_{BC} V_t(0, C_{AB}, C_{BC})$ then this fact coupled with the above assumption implies $AC$ will never shill-bid.

If Assumption 2 holds, then in the optimal mechanism buyers’ shill-bidding constraints do not bind. Therefore the seller can still implement the optimal mechanism by post price. A natural question is what distributions satisfy Assumption 2. On the other hand, Assumption 2 is strong, and it does not always hold. A sub-optimal solution that is shill-bid proof, the seller can always first determine the cut-offs of $AB$ and $BC$ then the set the price of $AC$ to be bounded below by the sum of two previous cut-offs. When it is violated, some shill-bidding constraints bind in the optimal mechanism. As another approach we can use the Lagrangian relaxation method: if we know the corresponding duals then we can define new index as the old index plus the dual variables. Papers that use this method include Deb and Pai (2014), Mierendorff (2013), Pai and Vohra (2013b). Besides Pai and Vohra (2013b), these papers are silent on how these duals are computed. Since finding the duals requires solving the general problem, as a result, the optimal mechanism will not be a simple index rule that uses only local information.

\textsuperscript{26}Observe that without Assumption 1, the following is still possible, $\Delta_{AC} V_t(0, C_{AB}, C_{BC}) \geq \Delta_{AB} V_t(0, C_{AB}, C_{BC})$ but $\psi^{-1}_{AC}(\Delta_{AC}) < \psi^{-1}_{AB}(\Delta_{AB})$ then $AB$ still might want to lie.

\textsuperscript{27}This contradicts our assumption that there can be only one customer in each point of time.

\textsuperscript{28}Notice that assumption 2 is a sufficient condition to deter shill-bidding. A weaker condition is: If $\Delta_{AB} + \Delta_{BC} \geq \Delta_{AC}$ then $\psi^{-1}_{AB}(\Delta_{AB}) + \psi^{-1}_{BC}(\Delta_{BC}) \geq \psi^{-1}_{AC}(\Delta_{AC})$.  


6. Extensions

6.1. The Discounted Case. The baseline model can be extended to cover this possibility as follows. Suppose that each buyer type is described by a favorite object, an acceptable set of objects, and a valuation. Following our airline pricing example, consider an AC type buyer, if she gets a direct flight (her favorite object), she will get utility $v$ (her valuation); if she gets any indirect flight (one of the acceptable objects), then she will get $\delta v$ (her discounted valuation), where $\delta \in (0, 1)$. If each buyer’s favorite object and the discount factor $\delta \in (0, 1)$ are public knowledge (we explicitly assume that the favorite object is the shortest path connecting the origin and the destination), then we can interpret the favorite object as the shortest path for the buyer. On the other hand, if each buyer’s discount factor is her private knowledge, then there are additional IC constraints, i.e., each buyer should report her discount factor truthfully. We leave this possibility for future research. Finally, we note that if all buyers share a common discount factor $\delta \in (0, 1)$, then as indicated in the next lemma, our original algorithm still works.

**Lemma 1.** The algorithm for efficient allocation in the static case still works under common discounting.

*Proof.* See the Appendix.

6.2. A Continuous-Time Model with Three Types of Buyers. Here we consider a continuous-time formulation of the problem. Suppose that time is continuous and buyers’ arrivals follow a Poisson process with rate $\lambda_\theta$ for each type $\theta \in \{AB, BC, AC\}$. This implies that at each point in time there will be only one type of buyers in the market. 29

6.3. Patient Buyers. If we extend our model to the case where buyers can stay in the market, the problem becomes more complicated. One easy case to analyze is that the seller and all buyers do not discount the future, yet the time horizon is still finite. Then the dynamic problem is equivalent to the static problem. The seller will not allocate any good prior to the deadline $T$, and at the deadline $T$ he will implement the static mechanism. In the case where the seller discounts the future and buyers report their arrival times, if we impose the assumption that those buyers who arrive earlier have higher valuations according to our order $\succ$ over types, then by Proposition 3 of Vohra and Pai (2013a) we can accommodate the strategic reporting of the arrival times. (Note that

29Akan and Ata (2009) consider a different model where the customers arrive as a flow and show that in this setup bid-pricing is $\epsilon$-optimal. Clearly, under bid-pricing shill bidding does not have any bite. Alternatively, shill bidding can be interpret as another justification for bid-pricing. Bid-pricing refers to additive pricing of bundle of goods
deadline is public in our model) We conjecture the buyers patience will not destroy the network effects.

6.4. **Multiple Sellers.** Another natural question is what happens if there are multiple sellers. In this case each seller, while determining the optimal mechanism, should take into account both his and the competitors networks. Thus competition may generate different price dynamics. Martínez-de-Albéniz and Talluri (2011) answers this question (without network constraints) and finds that the seller who makes the first sale will finish his supply of goods then the other seller starts selling his products.

6.5. **Multi-unit Demand.** Suppose the agents can demand multiple goods, then we can not guarantee the concavity of the value function (see Dizar, Moldovanu, and Gershkov, 2011) We need to make additional assumptions similar to Dizdar et al. (2011) to guarantee the concavity.

6.6. **Network Formation.** Our model can be used to analyze how airline-companies creates their flight network. Major questions here is: Why the airline companies owns a star-network? Basic trade-off is by creating new routes companies can satisfy more demand but these new routes introduces extra costs and may reduce the competition in the existing routes. Determining the optimal network is left for future research.

7. **Appendix**

**Lemma 2.** Sufficient conditions for the VCG mechanism to be Bayesian incentive compatible.

**Proof.** Let us recall that the p.d.f. of the \( i \)th order statistic for uniform distribution over interval \([0, 1]\) is:

\[
f_{i,n}(y) = \frac{n!}{(i-1)! (n-1)!} y^{i-1} (1-y)^{n-i} \quad \forall \ y \in [0, 1].
\]

Then conditional on winning the expected transfer becomes:

\[
\frac{\int_0^\theta y^i (1-y)^{n-i} dy}{\int_0^\theta y^{i-1} (1-y)^{n-i} dy} = \frac{B(\theta; i+1, n-i+1)}{B(\theta; i, n-i+1)},
\]

where \( B \) denotes the incomplete beta function and \( i \) denotes the \( i \)th order statistic. If we assume an AC type buyer’s value is uniform on \([0, 1]\), then if \( \theta_1 + \theta_2 > 1 \) then we truncated it and assume it is fixed at 1. Note that when, \( \theta_{AC} = 1 \), buyer almost sure get the good. We want joint conditions on supply and number \((i, n)\) of the buyers to following condition holds:
\[
\frac{B(\theta_1; i_1 + 1, n_1 - i_1 + 1)}{B(\theta_1; i_1, n_1 - i_1 + 1)} + \frac{B(\theta_2; i_2 + 1, n_2 - i_2 + 1)}{B(\theta_2; i_2, n_2 - i_2 + 1)} > \frac{B(h(\theta_1, \theta_2); i + 1, n - i + 1)}{B(h(\theta_1, \theta_2); i, n - i + 1)}
\]

where \( h(\theta_1, \theta_2) = \begin{cases} 
\theta_1 + \theta_2 & \text{if } \theta_1 + \theta_2 \leq 1 \\
1 & \text{if } \theta_1 + \theta_2 > 1
\end{cases} \)

We know that for every \((\theta_k, i)\) pair \(\frac{B(\theta; i + 1, n - i + 1)}{B(\theta; i, n - i + 1)}\) is increasing \(n\). Then, this suggest that we can increasing the number of buyers in the market get the desired result. Also, for every pair \((\theta, n)\), \(\frac{B(\theta; i + 1, n - i + 1)}{B(\theta; i, n - i + 1)}\) is decreasing on supply \((i)\), this implies that to get the desired result we can increase the supply of AC. Using these two arguments we can always find bounds on \((\theta_k, i)\) numerically. \(\square\)

7.1. **Multiple buyers in dynamic model.** Now we maintain the assumption that there is only one type of buyer in every period, however, number of buyers can be arbitrary. Specifically, for each \(\theta\), the number of corresponding buyers \(L_{\theta}\) is a discrete random variable that takes value from \(\{m_\theta, m_\theta + 1, \ldots, m_\theta\}\) with a strictly positive probability mass function. At the beginning of each period \(t\), the corresponding demand is denoted by \(L_{\theta,t}\).

**Lemma 3.** The optimal mechanism can be characterized by cut-offs.

**Proof.** Assume by way of contradiction that, we have the supply matrix \(C\) and we assign unit \(j\) to when value \(y_j\) is equal to \(x_j\) but not when \(y'_j > x_j\).

Then by optimality

\[
\psi(y^j) + V_t(R, C_{AC} - 1, C_{AB}, C_{BC}) \geq \delta V_{t+1}(R', C_{AC} - 1, C_{AB}, C_{BC})
\]

\[
\psi(y') + V_t(R, C_{AC} - 1, C_{AB}, C_{BC}) \leq \delta V_{t+1}(R', C_{AC} - 1, C_{AB}, C_{BC})
\]

then we add both equations we have \(\psi(y') - \psi(y^j) \leq 0\). This is a contradiction. \(\square\)

An immediate corollary of this result is that the unit \(j\) is allocated if and only if it passes the corresponding cut-off.

---

\(\square\)The function is not convex in \(n\), but it still increases quite fast. To see the intuition suppose for now \(\theta = 1\) then \(\frac{B(\theta; i + 1, n - i + 1)}{B(\theta; i, n - i + 1)} = \frac{1}{n+1}\) if \(i = n\), it is easy to see that we can make \(\frac{n}{n+1}\) close to 1 arbitrarily. Unfortunately in the case \(0 < \theta < 1\), we don’t have clean characterization but we can get close to \(\theta\) arbitrarily, this implies we can get numerical bounds.
Lemma 4. Suppose $g : X^3 \to \mathbb{R}$ satisfies decreasing differences (for each argument separately). Let $f : X^3 \to \mathbb{R}$ be defined by

$$f(x_1, x_2, x_3) = \max_{a=0,1,\ldots,m} \left\{ \sum_{i=1}^{a} p_i + g(x_1 - a, x_2, x_3) \right\}$$

for any given $p_1 \geq p_2 \ldots p_m \geq 0$ and non-negative integer $m \leq x$. Then $f$ satisfies decreasing differences for each argument as well.\(^{31}\)

Proof. Let’s start changing the variables $y = x_1 - a$ then we can write $f(x_1, x_2, x_3) = \hat{f}(x_1, x_2, x_3) + rx_1$ where

$$\hat{f}(x_1, x_2, x_3) = \max_{x-m \leq y \leq x} \left\{ -\sum_{i=1}^{y} p_i + g(y, x_2, x_3) \right\}$$

Let $y^* = \argmax_{y \geq 0} \{-yp + g(y, x_2, x_3)\}$. Since $g(y, x_2, x_3)$ satisfies decreasing differences $-\sum_{i=1}^{y} p_i + g(y, x_2, x_3)$ satisfies as well, moreover is non-decreasing for values $y \leq y^*$ and non-increasing for values $y \geq y^*$. Therefore, for a given $m$ and $p$

$$\hat{f}(x_1, x_2, x_3) = \begin{cases} 
-\sum_{i=1}^{x} p_i + g(y, x_2, x_3) & x \leq y^* \\
-\sum_{i=1}^{y} p_i + g(y^*, x_2, x_3) & y^* \leq x \leq y^* + m \\
-\sum_{i=1}^{x-m} p_i + g(x-m, x_2, x_3) & x \geq y^* + m
\end{cases}$$

Case 1: $x < y^*$ then using the fact $g$ satisfies decreasing differences

$$\hat{f}(x_1 + 1, x_2, x_3) - \hat{f}(x_1, x_2, x_3) = -p_1 + g(x_1 + 1, x_2, x_3) - g(x_1, x_2, x_3) \leq -p_1 + g(x_1, x_2, x_3) - g(x_1 - 1, x_2, x_3) = \hat{f}(x_1, x_2, x_3) - \hat{f}(x_1 - 1, x_2, x_3).$$

Case 2: $y^* \leq x < y^* + m$, in this range the difference is always zero. Therefore it satisfies the condition trivially.

Case 3: $x \geq y^* + m$ then again using the fact that $g$ satisfies decreasing differences,

$$\hat{f}(x_1 + 1, x_2, x_3) - \hat{f}(x_1, x_2, x_3) = -p_1 + g(x_1 + 1 - m, x_2, x_3) - g(x_1 - m, x_2, x_3) \leq -p_1 + g(x_1 - m, x_2, x_3) - g(x_1 - 1 - m, x_2, x_3) = \hat{f}(x_1, x_2, x_3) - \hat{f}(x_1 - 1, x_2, x_3).$$

we conclude that \( \hat{f}(x_1, x_2, x_3) \) satisfies decreasing differences and we know that \( x \geq 0 \) therefore \( f(x_1, x_2, x_3) \) satisfies it.

\[ \hat{f}(x_1, x_2, x_3) \text{satisfies decreasing differences and we know that } x \geq 0 \text{ therefore } f(x_1, x_2, x_3) \text{satisfies it.} \]

\[ \square \]

Let's define difference as follows:

\[ \Delta_t^{AC}(C_{AC}, C_{AB}, C_{BC}, R) = V_{t+1}(R', C_{AC}, C_{AB}, C_{BC}) - V_{t+1}(R', C_{AC} - 1, C_{AB}, C_{BC}) \]

**Lemma 5.** In the optimal mechanism, the assignment starts from AC, then it will assign the connection flight.

**Proof.** We need to show

\[ V_t(C_{AC} - 1, C_{AB}, C_{BC}) \geq V_t(C_{AC}, C_{AB} - 1, C_{BC} - 1) \]

for all \( t \). It is obvious when \( t = T \). The proof follows by backward induction. To see that

\[ H(C_{AC}, C_{AB}, C_{BC}, R) = \max_{0 \leq a \leq \min\{l_j, x\}} \left\{ \sum_{i=1}^{a} \psi_i + V_{t+1}(C_{AC} - a, C_{AB}, C_{BC}) \right\} \]

and define

\[ D(C_{AC}, C_{AB}, C_{BC}, R) = \max_{0 \leq a \leq \min\{l_j, x\}} \left\{ \sum_{i=1}^{a} \psi_i + V_{t+1}(C_{AB}, C_{BC} - a, C_{BC} - a) \right\} \]

where \( R \) denotes the realization of the buyers, then by induction hypothesis for each realization \( H \geq D \), then since expectation is a linear operator, it is never optimal to do that.

\[ \square \]

**Lemma 6.** Comparative statics:

1) \( \Delta_t^i(C_{AC}, C_{AB}, C_{BC}) \) is (weakly) decreasing in each term (strictly for some) for each \( i \in \{AC, AB, BC\} \).

2) For \( i \in \{AB, BC\} \), \( \Delta_t^i(C_{AC}, C_{AB}, C_{BC}) \) is weakly decreasing in \( C_{AC} \).

3) \( \Delta_t^{AC}(C_{AC}, C_{AB}, C_{BC}) \) is (weakly) decreasing with \( C_{AB} \) and \( C_{BC} \).

4) For \( i, j \in \{AB, BC\} \) and \( i \neq j \), \( \Delta_t^i(C_{AC}, C_{AB}, C_{BC}) \) is (weakly) increasing with \( C_j \).

5) For each \( i \in \{AC, AB, BC\} \) \( \Delta_t^i(C_{AC}, C_{AB}, C_{BC}) \) is (weakly) decreasing on \( t \).

**Proof.** 1) We need to show when \( C_i \) increases, \( \Delta_t^i(C_{AC}, C_{AB}, C_{BC}, \cdot) \) decreases. We proceed by backward induction. First, \( V_{T+1}(C_{AC}, C_{AB}, C_{BC}) = 0 \) for all \( C_{AC}, C_{AB}, C_{BC} \), so at time \( T \) it holds
trivially. Next assume $V_{t+1}$ satisfies the property and consider period $t$. Note that the Bellman equation\textsuperscript{32} of the problem can be written as

$$V_t(C_{AC}, C_{AB}, C_{BC}) = \mathbb{E} \left[ \max_{0 \leq a \leq \min\{N, x\}} \left\{ \sum_{i=1}^{a} \psi_i + V_{t+1}(C_{AC} - a, C_{AB}, C_{BC}) \right\} \right].$$

Since the term inside the expectation operator in the right-hand-side of the Bellman equation is the same as our previous lemma, for each realization of buyers,

$$H(C_{AC}, C_{AB}, C_{BC}, R) = \max_{0 \leq a \leq \min\{N, x\}} \left\{ \sum_{i=1}^{a} \psi_i + V_{t+1}(C_{AC} - a, C_{AB}, C_{BC}) \right\}$$

satisfies decreasing differences in $C_{AC}$. Since expectation is a linear operator, $V = \mathbb{E}[H]$ satisfies decreasing differences.

2) We need to show $\Delta^t_{AB}(C_{AC} - 1, C_{AB}, C_{BC}) \geq \Delta^t_{AB}(C_{AC}, C_{AB}, C_{BC})$. We proceed by backward induction. At time $T$ it holds trivially. Suppose it holds for time $t + 1$, and we now show it holds for time $t$.

Similar to previous argument, define for each realization,

$$H(C_{AC}, C_{AB}, C_{BC}, R) = \max_{0 \leq a \leq \min\{N, x\}} \left\{ \sum_{i=1}^{a} \psi_i + V_{t+1}(C_{AC} - a, C_{AB}, C_{BC}) \right\}$$

and

$$H(C_{AC} - 1, C_{AB}, C_{BC}, R) = \max_{0 \leq a \leq \min\{N, x\}} \left\{ \sum_{i=1}^{a} \psi_i + V_{t+1}(C_{AC} - 1, C_{AB} - a, C_{BC}) \right\}.$$

Then we know that for each realization of values,

$$\Delta H (C_{AC} - 1, C_{AB}, C_{BC}, R) - \Delta H (C_{AC}, C_{AB}, C_{BC}, R) \geq 0.$$

To see that $\Delta H (C_{AC}, C_{AB}, C_{BC}, R)$ is bounded by $\max \{\psi_i, \Delta V_{t+1}(C_{AC}, C_{AB}, C_{BC})\}$ at each $C_2$, and $\Delta H (C_{AC} - 1, C_{AB}, C_{BC}, R)$ is bounded by $\max \{\psi_j, \Delta V_{t+1}(C_{AC} - a, C_{AB}, C_{BC})\}$ where by induction hypothesis and the fact that $j < i$, the desired inequality holds. Since for each realization this holds, by the linearity of expectations, it holds for $V_t$.

3) Here we need to show $\Delta^t_{AC}(C_{AC}, C_{AB} - 1, C_{BC}) \geq \Delta^t_{AC}(C_{AC}, C_{AB}, C_{BC})$. The proof uses the same argument as in case 2 by backward induction.

\textsuperscript{32}Note that in our model two forms are equivalent either putting realization of the buyers as a state variable or the the form $\mathbb{E}(\max)$. For a proof see Talluri and van Ryzin (2005) Appendix D.
Here we need to show $\Delta_{t}^{\text{AB}}(C_{AC}, C_{AB}, C_{BC} - 1) \geq \Delta_{t}^{\text{AB}}(C_{AC}, C_{AB}, C_{BC})$. The proof again follows the same argument as in case 2 by backward induction.

We have

$$\Delta_{1}^{\text{AC}}(C_{AC}, C_{AB}, C_{BC}) = \mathbb{E} \left[ \sum_{i=1}^{\pi_{1}} \psi_{i} + V_{t-1}(C_{AC} - \pi_{1}, C_{AB}, C_{BC}) \right] - \mathbb{E} \left[ \sum_{i=1}^{\pi_{3}} \psi_{i} + V_{t-1}(C_{AC} - 1 - \pi_{2}, C_{AB}, C_{BC}) \right] \geq \mathbb{E} \left[ \sum_{i=1}^{\pi_{2}} \psi_{i} + V_{t-1}(C_{AC} - \pi_{2}, C_{AB}, C_{BC}) \right] - \mathbb{E} \left[ \sum_{i=1}^{\pi_{3}} \psi_{i} + V_{t-1}(C_{AC} - 1 - \pi_{2}, C_{AB}, C_{BC}) \right] \geq \mathbb{E} \left[ \sum_{i=1}^{\pi_{2}} \psi_{i} + V_{t-1}(C_{AC} - \pi_{2}, C_{AB}, C_{BC}) \right] - \mathbb{E} \left[ \sum_{i=1}^{\pi_{3}} \psi_{i} + V_{t-1}(C_{AC} - 1 - \pi_{2}, C_{AB}, C_{BC}) \right] \geq \mathbb{E} \left[ \Delta V_{t-1}(C_{AC} - \pi_{1}, C_{AB}, C_{BC}) \right] \geq \Delta V_{j-1}(C_{AC}, C_{AB}, C_{BC})$$

where $\pi_{1} \equiv \min \left\{ N_{j}, (x - y_{j-1}^{*})^{+} \right\}$, $\pi_{2} \equiv \min \left\{ N_{j}, (x - y_{j-2}^{*})^{+} \right\}$, $\pi_{3} \equiv \min \left\{ N_{j}, (x - 1 - y_{j-2}^{*})^{+} \right\}$ and $y_{j-1}^{*}$ denotes the optimal choice at period $j - 1$. The second equality follows from $\pi_{2} - \pi_{3} \geq 0$ and the last equality comes from the fact that $\Delta V_{t-1}$ is decreasing in its first argument.

**Lemma 7.** The algorithm for efficient allocation in the static case still works under common discounting.

**Proof.** The only modification we need from the given algorithm is:

$$\left\{ AB^{k_{2}-i} + BC^{k_{3}-1} \right\}_{i=1}^{k_{2}-1} \bigcup \left\{ AC^{i} \right\}_{j=k_{1}+1}^{k_{3}}$$

We should multiply each AC with $\delta$. \hfill \square

**References**


