STRATEGIC EXIT WITH RANDOM OBSERVATIONS

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ABSTRACT. In the standard optimal stopping problems, actions are artificially restricted to the moments of observations of costs or benefits. In the standard experimentation and learning models based on two-armed Poisson bandits, it is possible to take an action between two sequential observations. The latter models do not recognize the fact that timing of decisions depends not only on the rate of arrival of observations, but also on the dynamics of costs or benefits. We combine together these two strands of literature and consider bandits of “evolving shade of grey” instead of two-armed bandits who are either “white knights” or “black villains.” Stopping decisions in a model with Poisson bandits of “evolving shade of gray” are qualitatively different from those in optimal stopping or Poisson bandit models. We consider a case of two firms operating a technology which may experience costly breakdowns. The cost of breakdowns follows a jump-diffusion process. Breakdowns occur at random times, which follow a Poisson process independent of the cost process. The arrival rate of breakdowns may be high or low, but it is initially unknown. The firms differ by the rates of arrival, recovery rates and costs of breakdowns. We solve for the optimal exit strategy of the players.

Keywords: optimal stopping, jump-diffusion process, Cox processes, Poisson arrival, technology adoption

JEL: C73, C61, D81

1. INTRODUCTION

1.1. Objectives and the main result. Timing is a key feature of many economic decisions such as financing an innovation, marketing a new product, or abandoning a piece of equipment prone to breakdowns. There are two major strands of literature that deal with timing of actions in a risky environment. The real options literature emphasizes the fact that performing an irreversible action (for example entry into a new market) when payoffs are stochastic involves sacrificing the option to perform this action in the future. The optimal timing rule is formulated in terms of a stopping time, i.e., the first time the underlying stochastic variable enters a certain state space region, called action or stopping region. In such models, information either arrives continuously or at equally distanced time moments. However, in reality, news about such events as new drugs discoveries, stock market crashes, terrorism acts, or environmental catastrophes arrive at random times. Therefore, the first question we
ask in the paper is: “given that observations arrive at random times, is it always optimal to undertake an irreversible action only when one observes a realization of the state?” If the answer to the first question is “no,” then when is it optimal to act?

The insights of the literature on experimentation and learning indicate that if the information about, say, quality of a new technology, arrives at random times, it may be optimal to take an action such as adoption of the risky technology between two consecutive observations. In models based on two-armed Poisson bandits, it is typically assumed that the “safe” arm of a bandit generates a known payoff, and the “risky” arm generates an unknown payoff. This unknown payoff may, for example, characterize the quality of the new technology or a venture capital (VC) project. The new technology may be “good” or “bad” in terms of frequency of breakdowns. Similarly, the VC project may be “good” or “bad” in terms of frequency of breakthroughs. Thus, in these models, the quality of a project, equipment, or technology is intrinsically “good” or “bad,” and an experimenter tries to assess the failure or success rates of a project. While waiting for an observation to arrive, the DM updates her beliefs about the rate of success or failure and takes an action when the posterior belief reaches a certain cut-off. In particular, if the “good” technology never breaks down, the first breakdown will induce the experimenter to abandon the risky technology. Similarly, if the “bad” project can never be successful, the first breakthrough results in the investment into the VC project.

The settings of the above models ignore such features as dynamics of costs of breakdowns or profits of breakthroughs. For example, nuclear stations may have easily repairable breakdowns, or breakdowns on the scale of Chernobyl. When making decisions about optimal abandonment of a project with costly breakdowns, one takes into account not only the frequency of failures, but also the cost incurred after each breakdown, and whether the manufacturer successfully (say, a number of cars recalled decreases) or unsuccessfully (say, a new version of software fails no matter what) improves the equipment. Similarly, when a breakthrough happens, the profitability of the project may depend on specific market conditions (e.g., new shale oil extraction technology may be more or less profitable depending on oil prices). We believe that in order to account for such features, instead of considering two-armed bandits who are either “white knights” or “black villains,” it is necessary to consider bandits of “evolving shade of grey.” To this end, we consider two kinds of models: breakdowns with stochastic costs and breakthroughs with stochastic benefits. In the model with breakdowns, stochastic costs are incurred at random times. In the model with breakthroughs, stochastic benefits are observed at random times. In either of the models, the arrival rates of failures or successes can be high or low, and they are initially unknown. We ask the following questions: (i) If a breakdown or breakthrough is conclusive, is it optimal to act immediately at the time of the first observation? (ii) If not, when is it optimal to act? (iii) When is it optimal to act in case of inconclusive failures or successes?

We show that even if the first breakdown is conclusive, immediate action is not always optimal after the first failure. We demonstrate that if at the time of the news
arrival it is not optimal to act, then it is optimal to fix a calendar time \( T(x) \) that depends on the observed realization of the shock, and exercise the option at time \( T(x) \) unless the new piece of information arrives at \( \tau \leq T(x) \).

In models with strategic learning, players impose information externalities on their counterparts as long as they keep experimenting. In our model, there are two sorts of information externalities: when one of the firms incurs a breakdown, both firms not only update their posterior beliefs about the rate of arrival of failures, but also they learn about the state of the underlying cost process. That is why it may be optimal to stay in the business even in the case of conclusive breakdowns after the first failure is observed.

1.2. **Empirical facts.** Here are some examples that demonstrate that not only the occurrence of an event, but also its size matters, and actions not necessarily take place right after an observation of a random event.

1. Before the 9/11 terrorist attacks, the terrorism risk was included as an unnamed peril into commercial insurance contracts in the U.S. Such events as the first bombing of the World Trade Center in New York in 1993, or the 1998 bomb attacks on the U.S. embassy in Nairobi, Kenya, which caused significant insurance losses (see [39] for details) did not change the attitude of the U.S. insurers or international reinsurers to terrorism coverage. The situation changed dramatically after 9/11 attacks, which resulted in unprecedented losses. Private reinsurers, who covered the majority of these losses exited the market, and a few months after the attacks, the insurers excluded terrorism from their policies in most states (see [39] for details).

2. Other examples of how insurers who suffered large losses from a disaster are reluctant to continue offering coverage against this risk are hurricane insurance market in Florida and earthquake insurance market in California. Even though hurricanes in Florida are not rare events, the especially large losses during 2004 and 2005 hurricane seasons caused a failure of private insurance market. In California, private insurance companies decided to stop covering the residential property after the Northridge earthquake of 1994. Moreover, the earthquake happened in January 1994, but private insurance companies decided to quit the residential property market only in 1995. Hence, a one year delay was observed.

3. Following the shortage of reinsurance after such catastrophic events as Hurricane Andrew in 1992 and Northridge earthquake in 1994, there emerged a special financial contracts in order to complement reinsurance in covering large losses. One of such contracts are catastrophe (CAT) bonds. All CATs are structured to pay on triggers; in particular, there are CAT bonds that pay on insurer-specific catastrophe losses, or insurance-industry catastrophe loss indices (see, e.g., [22] for details).

4. Before the Fukushima 2011 disaster, 442 nuclear power reactors in 30 countries produces 14 per cent of all world’s electricity. This number dropped to 11 per cent in 2012 as 15 reactors, mainly in Germany and Japan, exited service.
5. Some research suggests that energy-efficient technologies appear not to be adopted by consumers and businesses to the degree that would seem justified, even on a purely financial basis (see Gerarden et al. [30] for details).

1.3. **Literature review.** The insights of the real options literature have significantly influenced capital budgeting decisions of corporations in the recent decades. In the majority of models of optimal exercise of American options and real options (real investment problems), it is presumed that the underlying process is observed either continuously (see, e.g., [24, 11, 17, 18]) or at equally distanced discrete time moments, when nothing can happen between any two consecutive observations (see, e.g., [16, 15, 18]). In reality, in many instances, the information flow is neither continuous, nor discrete. In fact, financing decisions are affected not only by uncertainty of the underlying payoff process, but also by uncertainty about the time of arrival of the next piece of news. Such news may be, for example, a breakthrough innovation, an evidence of efficiency of a new drug, a discovery of a new gas or oil site. All such pieces of information arrive at random times and they may radically change a financing decision of a related project. The adoption of new technologies crucially hinges upon an assessment of the risks they might entail. In this case, the critical events that convey new information are breakdowns of new equipment or technology which also happen at random times and also influence decisions about which technology to use.

The simplest way to model the uncertainty in the time of arrival of the next piece of news is to use a Poisson process, and assume that the option exercise (investment) is possible only at times of arrival of a piece of news. By now, there is a number of papers, which solve the corresponding problem in several situations such as portfolio optimization problem and option exercise [29, 32, 21, 49]. However, in these papers, no justification of optimality of actions restricted to the times of observations is provided.

At the same time, there is extensive literature on learning and experimentation that uses the multi-armed bandit framework. Bandit problems are used to study the trade-off between exploration and exploitation. Bandit models were successfully used in various settings in economics, for example, learning and matching in labor markets, monopolist pricing with unknown demand, choice between R&D projects, or financing of innovations (see, e.g., [4, 5, 6, 7, 8, 9, 35, 50, 51, 53] and references therein). The situation becomes even more interesting if several DM’s participate in experimentation because in this case information externalities are present, and, potentially, free riding problems may arise. See, for example, [10, 25, 33, 34, 36, 37].

In models based on two-armed bandits, the decision maker (DM) has to decide on optimal allocation of her time between the safe action (arm of the bandit) and the risky action (arm). The safe action generates a payoff given by a known distribution (in many instances, this payoff is deterministic), and the risky action generates an unknown payoff. In continuous time models, the latter payoff follows a certain continuous time stochastic process whose parameters are not known. For example, Bolton and Harris [10] model the unknown payoff as a Brownian motion with unknown drift.
and known variance in a model of strategic experimentation. Decamps et al. [26, 27] study timing a fixed size investment into a risky project with the payoff generated by a Brownian motion with unknown drift and known variance. Keller et al. [34], Keller and Rady [36, 37] use a Poisson process with unknown rate of arrival to model the risky arm. Decamps and Mariotti [25] study a duopoly model of investment where a signal about the quality of the project is modeled as a Poisson process. In two-armed bandit problems, it is typically assumed that the expected payoff generated by the risky arm is higher (respectively, lower) than the expected payoff generated by the safe arm, if the risky arm is, in some sense, “good” (respectively, “bad”). The optimal stopping rule in such problems is of a cut-off type: as long as the posterior belief that the payoff distribution of the risky arm is “good” has not reached the cut-off, the DM continues experimentation with the risky arm. When the posterior belief reaches the cut-off, the DM switches to the safe arm. In particular, in learning models with conclusive breakthroughs or breakdowns, this means that the first observation of a success or a failure implies immediate action (for example, adoption or discarding of a risky technology).

1.4. Other results and the structure of the paper. The rest of the paper is organized as follows. In Section 2, we consider exit in a duopoly in a model with conclusive breakdowns after the first costly breakdown had taken place. In this case, the players have learned the rate of arrival of breakdowns and only have to decide on timing exit. We start with the case of a single decision-maker who determines the optimal timing of abandoning a project with costly breakdowns after the other firm had exited the market; and then determine the optimal exit strategy when two firms are still active. We characterize conditions that ensure simultaneous exit and conditions, under which the firms exit sequentially.

In Section 3, we derive optimal exit strategies in a model, where rates of arrival of breakdowns depend on the last observed state and on time. In particular, if arrival rates are only time dependent, this corresponds to the case of inconclusive breakdowns in the strategic learning literature, when the players have to update their posterior beliefs after each new observation. Dependence on time and state together corresponds to the case with inconclusive failures where the prior may depend on the observed initial state.

2. Strategic exit; Poisson rates of arrival are known

In this Section, we consider the case of timing of a strategic abandonment of a project which generates a stream of deterministic profits, and may be either “good” or “bad” depending on the frequency of randomly incurred lump-sum costs. The rate of arrival of costs may be either 0 or $\lambda > 0$, so that costs can be incurred only if the project is “bad.” We start with the case, when at least one lump-sum cost was incurred, so the project is conclusively “bad.” As an example, consider two insurance
companies subject to shocks of the same type. The companies contemplate exiting regions that have recently become subject to climate disasters which impose high costs on insurance companies at random. Each active insurance company imposes an informational externality on the counter party, because the other firm observes realizations of the cost process more frequently than when being on its own. The question is whether the companies should exit the region immediately after a costly disaster occurs or wait for a while and try to exit at some optimal calendar time unless a new disaster happens and new cost is due. If the latter is the case, will the companies exit simultaneously or sequentially?

Time is continuous. For simplicity, assume that an active insurance company \( j \) gets the stream of instantaneous deterministic profits \( R_j > 0 \). The riskless rate is \( r > 0 \).

We model the instantaneous cost incurred by firm \( j \in \{1, 2\} \) at time \( t \) as \( A_j e^{X_t} \), where \( A_j > 0 \), and \( X = \{X_t\}_{t \geq 0} \) is a jump-diffusion process with i.i.d. increments (Lévy process) on \( \mathbb{R} \). On the filtered measurable space generated by \( X \), we chose a probability measure \( Q \) so that the filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t < \infty}, Q)\) satisfies the usual properties. See [52] for the definitions. If \( \mathbb{E}^Q[e^{X_1}] < \infty \), then the Lévy exponent \( \Psi := \Psi^Q \) of \( X \) under \( Q \) is definable from \( \mathbb{E}^Q[e^{\beta X_t}] = e^{t \Psi(\beta)} \) for \( \beta \in [0, 1] \). Assume that the supremum process \( \bar{X}_t = \sup_{0 \leq s \leq t} X_s \) is non-trivial.

Assume that the cost of company \( j \) is incurred (and publicly observed) at random times, which follow the Poisson process \( N_j \) with rate \( \lambda_j \), independent of \( X \) and the Poisson process \( N_i \) (\( i \neq j, i, j \in \{1, 2\} \)). Between the jumps in \( N^1 \) or \( N^2 \), no new piece of information arrives. We assume that the disaster never strikes both firms simultaneously (in the future work, we plan to analyze how this assumption can be relaxed).

The time of the last observation is normalized to 0. Let \( \tau_j \sim \text{Exp} \lambda_j \) be the random time of arrival of the next disaster for firm \( j \). (After the next disaster is observed, the clock is adjusted and \( \tau_j \) is reassigned the value 0).

In the case of exit at time \( t_j \), company \( j \) gets value \( G_j(X_t) \). For simplicity, we assume that the exit value is constant, and it is a fraction of the value of the perpetual stream \( R_j \). Thus, \( G_j(X_t) = \alpha_j R_j / r \), where \( \alpha_j \in [0, 1), j \in \{1, 2\} \). Although the nature of shocks is the same, hence, the process for the size of an individual disaster is the same, the locations where the firms operate are different, hence, the hazard rates, risk premiums, the costs and recovery rates are different: \( \lambda_j, R_j, C_j(X_t) = A_j e^{X_t} \) and \( \alpha_j, j = 1, 2 \).

**Lemma 2.1.** The expected present value \( C_j(x) \) of the cost is finite if and only if \( \Psi(1) < +\infty \) and

\[
(2.1) \quad r + \lambda_j - \Psi(1) > 0.
\]

If \( (2.1) \) holds, \( C_j(x) = \lambda_j A_j e^{x}/(r + \lambda_j - \Psi(1)) \).

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1Namely, \( \mathcal{F}_0 \) contains all the \( P \)-null sets of \( \mathcal{F} \), and the filtration \((\mathcal{F}_t)_{0 \leq t < \infty}\) is right continuous.
Proof. We have
\[ E^Q_{x,0} [e^{-r\tau X_t}] \equiv E^Q [E^{-r\tau X_t} | X_0 = x] \]
\[ = \int_0^{+\infty} \lambda_j A_j e^{-(r+\lambda_j)t} E^Q_{x,0} [e^{X_t}] \, dt = \int_0^{+\infty} \lambda e^{-(r+\lambda_j)t} e^{\Psi(1)+x} \, dt. \]
(The more rigorous notation would be \( E^Q_{x,0} \) instead of \( E^Q_{x,0} \), where \( P \) is the probability measure associated with \( \tau \). We use \( E^Q_{x,0} \) to simplify the notation.) The integral converges if and only if (2.1) holds. \( \square \)

One can also consider the costs \( C_j(X_t) \) which are decreasing functions of the shock, and which have sufficiently large limits at infinity. The techniques of the paper is applicable to this more general situation as well. The Bellman equations for the exits thresholds will involve functions \( C_j(T, x) = E^x [C_j(X_T)] \). The condition that we need is that \( C_j(T, x) \) are increasing in both arguments, and \( C_j(T, x) \) tends to a sufficient large limit (non-necessarily infinite) as \( T \to +\infty \).

Remark 2.2. This setting presumes that the company can always bear the cost, which is not quite realistic. However, the methodology of the paper is applicable to more realistic situations. It is possible to model the size of a disaster as a function of \( X \) that is bounded from above or increases very slow. Then, assuming that the company has sufficiently deep pockets, the question of the default does not arise; to be more accurate, the probability of default can be disregarded as being too low.

The cost of a disaster for the insurance company \( j \) is bounded if the cost above a certain level \( H_j \) is covered by reinsurers. In such setting, it would be necessary to assume that \( R_j \) is the profit flow net of reinsurance costs and that, after each disaster, the cost of reinsurance jumps up (hence, \( R_j \) jumps down), and, possibly, \( H_j \) jumps up. The natural assumption would be that, when a disaster of a too large scale happens, the reinsurer will abandon the client, and the company will remain on its own. In this case, it would be natural to introduce the default level for the insurance company. One can start with the company that does not use a reinsurer but defaults when a certain level \( H_j \) for the costs is reached. Notice, however, that if the insurance company is large, and can withstand significant costs, such complications (which are possible to analyze) do not lead to a sizable change in the optimal exit rule. Therefore, the qualitative conclusions of this section and the next one will not change.

2.1. Lévy processes, random walks and Wiener-Hopf factorization.

2.1.1. Lévy processes: main definitions. We need several basic definitions and facts from [52, 18]. Let \( X \) be the Lévy process on \( \mathbb{R} \) with the Lévy density \( F(dx) \), Lévy exponent \( \Psi \) and infinitesimal generator \( L \), under a probability measure \( Q \) on the filtered space generated by \( X \). If \( \int_{\mathbb{R}\setminus0} F(dx) < \infty \), which means that the jump component of the process is a compound Poisson, or, more generally, if \( \int_{\mathbb{R}\setminus0} \min\{|x|, 1\} F(dx) < \infty \) (that is, the jump component is of finite variation), then there exist \( \sigma^2 \geq 0 \) and \( b \in \mathbb{R} \)
(variance and drift of the Brownian motion component of \(X\)) such that, for all \(\beta\) on
the imaginary line \(i\mathbb{R}\),

\[
\Psi(\beta) = \frac{\sigma^2}{2} \beta^2 + b\beta + \int_{\mathbb{R}<0} (e^{\beta x} - 1) F(dx)
\]

(this is a special case of the Lévy-Khintchine formula). If \(\mathbb{E}^Q[e^{X_1}] < \infty\), equivalently, \(\Psi(1)\) is well-defined and finite, then (2.2) is valid for all \(\beta\) in the strip \(\text{Re} \beta \in [0,1]\).

If there are no jumps, we obtain the Lévy exponent of the Brownian motion with
 drifted. A popular example of jump-diffusion processes is the double-exponential jump-
diffusion model (DEJD) with the Lévy density

\[
F(dx) = \left( c_+ \lambda^+ e^{-\lambda^+ x} \mathbb{1}_{(0,\infty)}(x) + c_- (-\lambda^-) e^{-\lambda^- x} \mathbb{1}_{(-\infty,0)}(x) \right) dx,
\]

where \(c_+ > 0\), and \(\lambda^- < 0 < \lambda^+\), and the Lévy exponent

\[
(2.4) \quad \Psi(\beta) = \frac{\sigma^2}{2} \beta^2 + b\beta + \frac{c_+ \beta}{\lambda^+ - \beta} + \frac{c_- \beta}{\beta - \lambda^-}.
\]

The condition \(\Psi(1) < \infty\) is equivalent to \(\lambda^+ > 1\).

Note that, if \(\Psi(1) < \infty\), then, for any \(\beta\) in the strip \(\text{Re} \beta \in [0,1]\), \(Le^{\beta x}\) is well-defined, and \(Le^{\beta x} = \Psi(\beta)e^{\beta x}\). Furthermore, if \(q > 0\), \(q - \Psi(1) > 0\), and \(g\) is a measurable function satisfying a bound

\[
|g(x)| \leq C(1 + e^x), \quad x \in \mathbb{R},
\]

where \(C\) is a constant independent of \(x\), then the (discounted) expected present value of the stream \(g(X_t)\) under \(Q\)

\[
\mathcal{E}_q g(x) := E^Q_x \left[ \int_0^\infty q e^{-qt} g(X_t) dt \right] := E^Q_x \left[ \int_0^\infty q e^{-qt} g(X_t) dt \mid X_0 = x \right]
\]
is finite. In particular, for any \(\beta\) in the strip \(\text{Re} \beta \in [0,1]\), and \(g(x) = e^{\beta x}\),

\[
\mathcal{E}_q g(x) = E^Q_x \left[ \int_0^\infty q e^{-qt} e^{\beta X_t} dt \right] = \int_0^\infty q e^{-qt} E^Q_x \left[ e^{\beta X_t} \right] dt
\]

\[
= \int_0^\infty q e^{-qt+\Psi(\beta)t+x} = \frac{q}{q - \Psi(\beta)} e^x.
\]

The action of the EPV operator \(\mathcal{E}_q\) admits a convenient interpretation in terms of an exponentially distributed random variable \(T_q \sim \text{Exp} q\) of mean \(1/q\), independent of \(X\): \(\mathcal{E}_q g(x) = E^Q_x \left[ g(X_{T_q}) \right] \) (it would be more accurate to write \(= E^{Q \otimes P_q,x} \left[ g(X_{T_q}) \right] \),
where \(P_q\) is the probability measure associated with \(T_q\).) Finally, note that

\[
(2.6) \quad q(q - L)^{-1} = \mathcal{E}_q
\]
as operators in appropriate function spaces. For details, see [18].
2.1.2. **Wiener-Hopf factorization for Lévy processes.** Let $X$ and $T_q$ be the same as above. Introduce the supremum and infimum processes $\overline{X}_t = \sup_{0 \leq s \leq t} X_s$ and $\underline{X}_t = \inf_{0 \leq s \leq t} X_s$ (trajectories of both processes are well-defined, almost surely), and, by analogy with the definition of $E_q$, introduce the expected present value operators under the supremum and infimum processes

$$E^+_q g(x) = \mathbb{E}^Q[g(x + \overline{X}_{T_q})], \quad E^-_q g(x) = \mathbb{E}^Q[g(x + \underline{X}_{T_q})],$$

and the notation $\kappa^+_q(\beta) = \mathbb{E}^Q[e^{\beta \overline{X}_{T_q}}], \quad \kappa^-_q(\beta) = \mathbb{E}^Q[e^{\beta \underline{X}_{T_q}}]$. The Wiener-Hopf factorization formula used in probability states that, for $\beta \in i\mathbb{R}$, where $i = \sqrt{-1}$ is the imaginary unit,

$$E^Q[e^{\beta X_{T_q}}] = E^Q[e^{\beta \overline{X}_{T_q}}] E^Q[e^{\beta \underline{X}_{T_q}}],$$

(2.7)

equivalently,

$$q/(q - \Psi(\beta)) = \kappa^+_q(\beta)\kappa^-_q(\beta).$$

(2.8)

Note that the equality holds for $\beta$ in the region each function in (2.8) admits the analytic continuation into. If $q - \Psi(1) > 0$, then this region contains a strip $\text{Re} \beta \in [0, 1]$.

The equality (2.8) is a special case of the initial form of the Wiener-Hopf factorization used in Complex Analysis for several decades before the probabilistic form had been invented, and for functions $f(\beta)$ much more general than $f(\beta) = q/(q - \Psi(\beta))$ on the LHS of (2.8). The essence of the analytic form of the Wiener-Hopf factorization of a function $f$ on $i\mathbb{R}$ is: under a certain regularity condition, a function on $i\mathbb{R}$ admits the factorization into a product of two functions. One function (in our case, $\kappa^+_q$) admits the analytic continuation into the left half-plane, the other one (in our case, $\kappa^-_q$) admits the analytic continuation into the right half-plane. Moreover, if each of the factors and their reciprocals are polynomially bounded in the corresponding half-planes, then these Wiener-Hopf factors are uniquely defined up to scalar multiples. The last property will be used for the derivation of the Wiener-Hopf factorization for random walks with the transition operator of the form $E_q$.

The operator form of the Wiener-Hopf factorization is

$$E_q = E^+_q E^-_q = E^-_q E^+_q$$

(2.9)

as operators in appropriate function spaces, for instance, in the space of measurable functions admitting the bound (2.5).

**Example 2.3.** a) Let $X$ be the Brownian motion. Under condition $q - \Psi(1) > 0$, the characteristic equation $q - \Psi(\beta) = 0$ has two roots $\beta^-_q < 0 < 1 < \beta^+_q$, and $\kappa^+_q(\beta) = \beta^+_q/(\beta^+_q - \beta)$. The EPV operators are the convolution operators with exponential kernels: $E^+_q = I^+_{\beta^+_q}$, where $I^+_{\beta^+_q}$ act as follows

$$I^+_{\beta^+_q} u(x) = \int^0_{-\infty} (\beta^-_q)^{-\beta^-_q y} u(x + y) dy,$$

(2.10)

$$I^-_{\beta^-_q} u(x) = \int^0_{\infty} (\beta^-_q)^{-\beta^-_q y} u(x + y) dy.$$

(2.11)
b) Let \( X \) be an DEJD, and \( q - \Psi(1) > 0 \). Then the characteristic equation \( q - \Psi(\beta) = 0 \) has 4 roots, \( \beta^-_2 < \lambda^- < \beta^-_1 < 0 < 1 < \beta^+_1 < \lambda^+ < \beta^+_2 \), and (\[18\] eqn. (11.24))

\[
(2.12) \quad \kappa^\pm_q(\beta) = \frac{\beta^\pm_1}{\beta^\pm_1 - \beta} \cdot \frac{\beta^\pm_2}{\beta^\pm_2 - \beta} \cdot \frac{\lambda^\pm - \beta}{\lambda^\pm}.
\]

The EPV operators admit the representations

\[
\mathcal{E}^\pm_q = \sum_{j=1,2} a^\pm_j(q) I^\pm_{\beta_j(q)},
\]

where (\[18\] eqn. (11.29),(11.30))

\[
(2.13) \quad a^+_1 = \frac{\beta^+_2}{\beta^+_2 - \beta^+_1} \cdot \frac{\lambda^+ - \beta^+_1}{\lambda^+}, \quad a^+_2 = \frac{\beta^+_1}{\beta^+_1 - \beta^+_2} \cdot \frac{\lambda^+ - \beta^+_2}{\lambda^+}.
\]

c) The next example is more natural as the model for the size of the next disaster. Namely, consider a spectrally positive process, with the Lévy density supported on \((0, +\infty)\). Then the equation \( q - \Psi(\beta) = 0 \) has the unique negative root \( \beta^-_q \), and

\[
\kappa^-_q(\beta) = \frac{\beta^-_2}{\beta^-_2 - \beta}, \quad \kappa^+_q = \frac{q}{(q - \Psi(\beta)) \kappa^-_q(\beta)}, \quad \mathcal{E}^-_q = I^-_{\beta^-_q}.
\]

In the general case, an explicit formula for \( \mathcal{E}^+_q \) is involved but if the density of the positive jumps is exponential, then \( \kappa^+_q \) and \( \mathcal{E}^+_q \) admit the representations similar to the ones in the DEJD model.

The Wiener-Hopf factorization allows one to solve entry and exit problems. See \[18\] for explicit results in numerous situations. We will use some of these results below, where explicit references will be given.

2.1.3. **Wiener-Hopf factorization for random walks.** Let \( Y_1, Y_2, \ldots, \) be i.i.d. on \( \mathbb{R} \). The corresponding random walk on \( \mathbb{R} \) starting at \( X_0 = x \) is defined as \( X_t = X_0 + Y_1 + Y_2 + \cdots + Y_t, \ t = 0, 1, \ldots. \) (The definition generalizes to the case when \( X_0 \) is a random variable independent of \( Y_1, Y_2, \ldots. \)) The transition operator \( P \) of \( X \) is defined by \( Pu(x) = \mathbb{E}^Q[u(x + Y)] \). Let \( \beta \mapsto M(\beta) := \mathbb{E}^Q[e^{\beta Y_1}] \) be the moment-generating function of \( X \). Then \( Pe^{\beta x} = M(\beta)e^{\beta x} \). Assume that \( \mathbb{E}^Q[e^{\beta Y_1}] < \infty \). Then \( M(\beta) \) is continuous in the strip \( \text{Re} \beta \in [0, 1] \) and analytic in the interior of the strip.

Let \( q \in (0, 1) \), and let \( T_q \) be the geometric random variable on \( \mathbb{Z}_+ \), \( \text{Prob}(T_q = n) = (1-q)q^n, n = 0, 1, \ldots, \) independent of \( X \). Define the normalized expected value \( \bar{E}_q u(x) \) of the stream \( u(X_t) \) by \( \bar{E}_q u(x) = \mathbb{E}\left[u(X_{T_q}) \mid X_0 = x\right] \). We have \((1-q)(1-qP)^{-1} = \mathcal{E}_q \). The supremum and infimum processes for random walks, the EPV operators under supremum and infimum processes and the Wiener-Hopf factors are defined by the same formulas as for Lévy processes; however, since we will use the EPV operators and the Wiener-Hopf factors for Lévy processes to construct similar objects in certain random walk models, we will add the tilde to the notation as we already did in the
case of the EPV operator $\tilde{q}$. The Wiener-Hopf factorization formulas (2.7) and (2.9) remain valid but (2.8) is replaced with
\[(1 - q)/(1 - qM(\beta)) = \tilde{k}_q^+(\beta)\tilde{k}_q^-(\beta).\]

Below, we need (2.14) in the case $P = \mathcal{E}_{r+\lambda} = (r + \lambda)(r + \lambda - L)^{-1}$, where $\mathcal{E}_{r+\lambda}$ is the EPV operator under the Lévy process $X$ with the Lévy exponent $\Psi$ and infinitesimal generator $L$, and $q = \lambda/(\lambda + r)$. (Here, $r > 0$ is the discount rate, and $\lambda > 0$ is the Poisson rate of arrival). Set
\[a(\beta) = 1 - \lambda/(\lambda + r - \Psi(\beta)) = (r - \Psi(\beta))/(r + \lambda - \Psi(\beta))\]
\[= \frac{r}{r + \lambda} \cdot \frac{r - \Psi(\beta)}{r + \lambda - \Psi(\beta)},\]
and, using the Wiener-Hopf identity (2.8), factorize $a(\beta) = a_+(\beta)a_-(\beta)$,
\[(2.15) a_+(\beta) = \kappa_+^+(\beta)^{-1}\kappa_{r+\lambda}^+(\beta), \quad a_-(\beta) = \kappa_-^- (\beta)^{-1}\kappa_{r+\lambda}^-(\beta),\]
where $\kappa_\pm^-(\beta)^{-1}$ are the Wiener-Hopf factors in the Lévy model. These factors and their reciprocals are bounded by polynomials in the corresponding half-planes, therefore, by the uniqueness of the Wiener-Hopf factorization, $1/a_\pm$ are the factors $\tilde{k}_q^\pm$ in (2.14). Note that in order that $\kappa_+^+(1)$ be finite, it is necessary and sufficient to impose the no-bubble condition (??) on the stream $e^{\lambda t}$.

2.2. A guide to the main notation used in the remainder of the paper. An index $\lambda$ is used to distinguish the value functions and exercise regions in the model when the exit is possible only at a moment of observation. The value functions and exercise regions in the model where the exit is allowed at any time, do not have this index. An index $j$ indicates that we consider firm $j$. If a firm is the only one on the market, its value function and optimal exercise region have an additional index 0.

2.3. Exit is possible only at a moment of a disaster.

2.3.1. The problem of a lonely firm. At the first step of our study, we solve the problem of the lonely firm $j$, that is, assuming that firm $k$ exited earlier or at time of the most recent disaster, normalized to 0. Let $\tau_j \sim \text{Exp} \lambda_j$ be the time of the next disaster of firm $j = 1, 2$, and $V_j^{\lambda,0}(x)$ be the value function of firm $j$ at the time 0 of the last disaster and $X_0 = x$, and let $U_j^{\lambda,0}$ be the inaction region of firm $j$.

Let $\tau_j \in \text{Exp} \lambda$ be the moment of the next disaster that befell firm $j$. We can write the Bellman equation for the value of the firm at time 0 and $X_0 = x \in U_j^{\lambda}$:
\[V_j^{\lambda,0}(x) = \mathbb{E}^{Q,x} \left[ R_j \int_0^{\tau_j} e^{-r\tau_j} dt + e^{-r\tau_j} \left(V_j^{\lambda,0}(X_{\tau_j}) - A_j e^{X_{\tau_j}}\right) \right] \]
\[= \int_0^\infty \lambda_j e^{-\lambda_j t} \left[R \int_0^t e^{-r s} ds + e^{-r t} \mathbb{E}^{Q,x} [V_j^{\lambda,0}(X_t)] - A_j e^{-(r - \Psi(1))t + x}\right] dt.\]
(The more rigorous notation would be $\mathbb{E}^{\mathbb{Q} \otimes \mathbb{P}, x}$ instead of $\mathbb{E}^{\mathbb{Q}, x}$, where $\mathbb{P}$ is the probability measure associated with $\tau$. We use $\mathbb{E}^{\mathbb{Q}, x}$ to simplify the notation.) Simplifying,

$$V_{j}^{\lambda, 0}(x) = \frac{R_{j}}{r + \lambda_{j}} + \frac{\lambda_{j}}{\lambda_{j} + r} E_{\lambda_{j} + r}V_{j}^{\lambda, 0}(x) - \frac{A_{j} \lambda_{j} e^{x}}{r + \lambda - \Psi(1)}.$$  

Thus, we have the Bellman equation: for $x \in U_{j}^{\lambda, 0}$,

$$\left(I - \frac{\lambda_{j}}{\lambda_{j} + r} E_{\lambda_{j} + r}\right) V_{j}^{\lambda, 0}(x) = \frac{R_{j}}{r + \lambda_{j}} - \frac{\lambda_{j} A_{j} e^{x}}{\lambda_{j} + r - \Psi(1)}$$

subject to

$$V_{j}^{\lambda, 0}(x) = \alpha_{j} R_{j} / r, \quad x \notin U_{j}^{\lambda, 0}.$$  

Remark 2.4. If we assume that the firm can cover the losses up to $H_{j}$ only, and then defaults, then the Bellman equation must be modified as follows

$$V_{j}^{\lambda, 0}(x) = \frac{R_{j}}{r + \lambda_{j}} + \frac{\lambda_{j}}{\lambda_{j} + r} E_{\lambda_{j} + r} \left(V_{j}^{\lambda, 0}(x) - \min\{e^{x}, H_{j}\}\right).$$

Since the exit is mandatory at $x \geq \ln H_{j}$, an optimal $U_{j}^{\lambda, 0}$ is a subset of $\{x < \ln H_{j}\}$, and the boundary condition becomes

$$V_{j}^{\lambda, 0}(x) = \frac{\alpha_{j} R_{j}}{r} \mathbb{1}_{e^{x} < H_{j}}, \quad x \notin U_{j}^{\lambda, 0}.$$  

Introduce $W_{j}^{\lambda, 0} = V_{j}^{\lambda, 0} - \alpha_{j} R_{j} / r$. Since $\alpha_{j} R_{j} / r$ is constant, the problem of maximization of $V_{j}^{\lambda, 0}$ is equivalent to the problem of maximization of $W_{j}^{\lambda, 0}$. The Bellman equation for $W_{j}^{\lambda, 0}$ is: for $x \in U_{j}^{\lambda, 0}$,

$$\left(I - \frac{\lambda_{j}}{\lambda_{j} + r} E_{\lambda_{j} + r}\right) W_{j}^{\lambda, 0}(x) = \frac{(1 - \alpha_{j}) R_{j}}{\lambda_{j}} - \frac{\lambda_{j} A_{j} e^{x}}{\lambda_{j} + r - \Psi(1)}$$

subject to

$$W_{j}^{\lambda, 0}(x) = 0, \quad x \notin U_{j}^{\lambda, 0}.$$  

Denote by $g_{j}^{\lambda, 0}(x)$ the RHS of (2.21), set $P_{j} = E_{\lambda_{j} + r}$, $q_{j} = \lambda_{j} / (\lambda_{j} + r)$, and denote by $\tilde{X}_{i}^{j}$ the random walk on $\mathbb{R}$ with the transition operator $P_{j}$. Let $\tilde{L}^{j+}_{q_{j}}$ be the EPV operators in this random walk model. We have the problem of the optimal abandonment of the decreasing stream $g_{j}^{\lambda, 0}(\tilde{X}_{i}^{j})$. In [18, Thm. 9.4.2], it was proved that the optimal inaction region is a semi-infinite interval $U_{j}^{\lambda, 0} = (-\infty, h_{j}^{\lambda, 0})$. Let $\tilde{E}^{\pm}_{q_{j}}$ be the EPV operators in the random walk model. The exit threshold $h_{j}^{\lambda, 0}$ is the unique solution of the equation

$$\left(\tilde{L}^{j+}_{q_{j}} g_{j}^{\lambda, 0}\right)(h) = 0,$$

and the value function is given by

$$V_{j}^{\lambda, 0}(x) = \frac{\alpha R}{r} + W_{j}^{\lambda, 0}(x) = \frac{\alpha R}{r} + (1 - q)^{-1} \left(\tilde{L}^{j+}_{q_{j}} \mathbb{1}_{(-\infty, h_{j}^{\lambda, 0})} \tilde{L}^{j-}_{q_{j}} g_{j}^{\lambda, 0}\right)(x).$$
Since \( L_{q_j}^{j+} e^{\beta x} = \tilde{\kappa}_q^+(\beta) e^{\beta x} \), and \( \tilde{\kappa}_q^+(\beta)^{-1} = a_-(\beta) \) is given by (2.15) in terms of the Wiener-Hopf factors of the Lévy model, we have

\[
L_{q_j}^{j+} g_j^{\lambda,0}(x) = \frac{(1 - \alpha_j) R_j}{r + \lambda_j} - e^{x} \frac{A_j \alpha_j \tilde{\kappa}_q^-(\beta)}{r + \lambda_j - \Psi(1)},
\]

therefore, using the Wiener-Hopf factorization formula (2.8), we can rewrite the equation (2.23) for \( h_j^{\lambda,0} \) as

\[
e^{h_j^{\lambda,0}} = \frac{(1 - \alpha_j) R_j}{A_j \lambda_j} \cdot \frac{1}{\kappa_r^-(1) \kappa_{r+\lambda_j}(1)}.
\]

If \( X \) is a spectrally positive process, that is, a Lévy process without negative jumps, then, for any \( q > 0 \), the characteristic equation \( \Psi(\beta) - q = 0 \) has the only negative root \( \beta_q^- \), \( \kappa_q^- \) \( (\beta) = \beta_q^- / (\beta - \beta_q^-) \), and the RHS in (2.26) can be easily calculated. The calculation of the Wiener-Hopf factors is also easy in the case of DEJD model: see [18, Sect. 9.2.1]. In the same model, an explicit formula for the EPV operator \( L_{q_j}^{j+} \) is not difficult to derive as well. The formula is of the form

\[
L_{q_j}^{j+} = c_{j,0} + \sum_{n=1,2} c_{j,n} I_j^{+} \beta_{r,n},
\]

where the coefficients \( c_{j,n}, n = 0, 1, 2 \), can be found from the decomposition of

\[
\tilde{\kappa}_q^+(\beta) = \frac{\kappa_r^+(\beta)}{\kappa_{r+\lambda_j}(\beta)} = \prod_{n=1,2} \frac{\beta_{r,n} - \beta}{\beta_{r,n} + \lambda_j, n} \frac{\beta^+_{r+\lambda_j, n} - \beta}{\beta^+_{r+\lambda_j, n}}
\]

into the sum of simple fractions

\[
\tilde{\kappa}_q^+(\beta) = c_{j,0} + \sum_{n=1,2} c_{j,n} \frac{\beta^+_{r,n}}{\beta^+_{r,n} - \beta}.
\]

It follows that, for \( x < h_j^{\lambda,0} \), the option value is of the form

\[
V_j^{\lambda,0}(x) = \frac{\alpha_j R_j}{r} + \sum_{n=1,2} a_{j,n} e^{\beta_{r,n} (x - h_j^{\lambda,0})},
\]

where the constants \( a_{j,n} \) can be explicitly calculated. The reader can find numerous examples of similar calculations in [18, Chapt. 9].

2.3.2. The problem for two firms. Note that the constructions and results below remain valid if the firms use different discount rates \( r_j \); the only changes in the constructions below are that \( r_j \), \( L_{\lambda_1 + \lambda_2 + r_j} \) and \( q^j = (\lambda_1 + \lambda_2) / (\lambda_1 + \lambda_2 + r_j) \) should be used instead of \( r \), \( L_{\lambda_1 + \lambda_2 + r} \) and \( q = (\lambda_1 + \lambda_2) / (\lambda_1 + \lambda_2 + r) \).

Let \( V_j^\lambda \) and \( U_j^\lambda \) be the value function and inaction region for firm \( j = 1, 2 \). Clearly,

- \( V_j^\lambda \geq V_j^{\lambda,0}, U_j^\lambda \supset U_j^{\lambda,0} \);
- \( V_j^{\lambda,0}(x) = \lambda_j R_j / r, \forall x \in (U_j^\lambda)^c \);
- \( V_j^\lambda(x) = V_j^{\lambda,0}(x), \forall x \in (U_j^k)^c, j = 1, 2, j \neq k \).
If both firms do not exit at $X_0 = x$, that is, $x \in U_j^\lambda \cap U_k^\lambda$, then

$$V_1^\lambda(x) = \mathbb{E}^Q,x \left[ \mathbbm{1}_{\tau_1 < \tau_2} \left( \int_0^{\tau_1} e^{-rt} R_1 dt + e^{-r\tau_1} \left( \mathbb{1}_{U_2^\lambda}(x_{\tau_1}) V_1^\lambda(x_{\tau_1}) + \mathbb{1}_{(U_2^\lambda)^c}(x_{\tau_1}) V_1^{\lambda,0}(x_{\tau_1}) - A_1 e^{x_{\tau_1}} \right) \right) + \mathbb{1}_{\tau_2 < \tau_1} \left( \int_0^{\tau_2} e^{-rt} R_1 dt + e^{-r\tau_2} \left( \mathbb{1}_{U_2^\lambda}(x_{\tau_2}) V_1^\lambda(x_{\tau_2}) + \mathbb{1}_{(U_2^\lambda)^c}(x_{\tau_2}) V_1^{\lambda,0}(x_{\tau_2}) \right) \right) \right]$$

$$= \int_0^{+\infty} dt \lambda_1 e^{-\lambda_1 t} e^{-\lambda_2 t} \left( \int_0^t e^{-rs} R_1 ds \right) + e^{-rt} \mathbb{E}^Q,x \left[ \mathbb{1}_{U_2^\lambda}(x_t) V_1^\lambda(x_t) + \mathbb{1}_{(U_2^\lambda)^c}(x_t) V_1^{\lambda,0}(x_t) - A_1 e^{x_t} \right] + \int_0^{+\infty} dt \lambda_2 e^{-\lambda_2 t} e^{-\lambda_1 t} \left( \int_0^t e^{-rs} R_1 ds \right) + e^{-rt} \mathbb{E}^Q,x \left[ \mathbb{1}_{U_2^\lambda}(x_t) V_1^\lambda(x_t) + \mathbb{1}_{(U_2^\lambda)^c}(x_t) V_1^{\lambda,0}(x_t) \right]$$

$$= \int_0^{+\infty} dt (\lambda_1 + \lambda_2) e^{-(\lambda_1 + \lambda_2) t} R_1 \frac{1 - e^{-rt}}{r} - \frac{\lambda_1 A_1 e^x}{\lambda_1 + \lambda_2 + r - \Psi(1)} + \frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 + r} \mathcal{E}_{\lambda_1 + \lambda_2 + r} (\mathbb{1}_{U_2^\lambda} V_1^\lambda)(x) + \frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 + r} \mathcal{E}_{\lambda_1 + \lambda_2 + r} (\mathbb{1}_{(U_2^\lambda)^c} V_1^{\lambda,0})(x)$$

Calculating the integral above, we derive the Bellman equation: for $x \in U_1 \cap U_2$,

$$(2.28) \quad \left( I - \frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 + r} \mathcal{E}_{\lambda_1 + \lambda_2 + r} \mathbb{1}_{U_2^\lambda} \right) V_1^\lambda(x) = \frac{R_1}{\lambda_1 + \lambda_2 + r} - \frac{\lambda_1 A_1 e^x}{\lambda_1 + \lambda_2 + r - \Psi(1)} + \frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 + r} \mathcal{E}_{\lambda_1 + \lambda_2 + r} (\mathbb{1}_{(U_2^\lambda)^c} V_1^{\lambda,0})(x),$$

subject to

$$(2.29) \quad V_1^\lambda(x) = V_1^{\lambda,0}(x), \quad x \notin (U_1^\lambda \cap U_2^\lambda).$$

(Recall that, for $x \notin U_1^\lambda$, $V_1^\lambda(x) = V_1^{\lambda,0}(x) = \alpha_j R_j / r$.)

Introduce $W_j^\lambda = V_j^\lambda - V_j^{\lambda,0}$. Assuming $V_j^{\lambda,0}$ has been calculated, the problem of maximization of $V_j^\lambda$ is equivalent to the problem of maximization of $W_j^\lambda$. The
Bellman equation for $W_1^\lambda$ is: for $x \in U_1^\lambda \cap U_2^\lambda$,
\begin{equation}
(2.30) \quad \left( I - \frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 + r} \mathcal{E}_{\lambda_1 + \lambda_2 + r} \right) W_1^\lambda(x) = \frac{R_1}{\lambda_1 + \lambda_2 + r} - \frac{\lambda_1 A_1 e^x}{\lambda_1 + \lambda_2 + r - \Psi(1)} + \frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 + r} \mathcal{E}_{\lambda_1 + \lambda_2 + r} V_1^{\lambda,0}(x),
\end{equation}
subject to
\begin{equation}
(2.31) \quad W_1^\lambda(x) = 0, \quad x \neq U_1^\lambda \cap U_2^\lambda.
\end{equation}
Denote by $g_1^\lambda(x)$ the RHS of (2.30), set $q = (\lambda_1 + \lambda_2)/(\lambda_1 + \lambda_2 + r)$, $P = \mathcal{E}_{\lambda_1 + \lambda_2 + r}$, and denote by $\tilde{X}$ the random walk on $\mathbb{R}$ with the transition operator $P$. Since the set $U_2^\lambda$ is fixed by the decision of firm 2, firm 1 solves the problem of the optimal abandonment of the stream $g_1^\lambda(\tilde{X}_t)$ subject to the restriction that the abandonment must be made not later than the process $X$ enters $U_2^\lambda$. Equivalently, firm 1 finds a stopping time $\tau \leq \tau_U$, where $\tau_U$ is the first entrance time into $U$, which maximizes
\begin{equation}
(2.32) \quad W_1^\lambda(x) = \sup_{\tau \leq \tau_U} \mathbb{E}_{\mathbb{P}^x} \left[ \sum_{n=0}^{\tau-1} q^n g_1^\lambda(\tilde{X}_t) \right].
\end{equation}
Since $V_1^{\lambda,0}$ is non-decreasing and $-e^x$ is decreasing and tends to $-\infty$ as $t \to +\infty$, the function $g_1^\lambda$ is decreasing, positive in a neighborhood of $-\infty$, and tends to $-\infty$ as $x \to +\infty$. Therefore, given the exit set $U_j^\lambda$ of firm $j$, the optimal exit set $U_k^\lambda$ of firm $k$, $k \neq j$, is of the form $[h_j, +\infty)$. Indeed, since the stream is monotone and $X$ is the process with i.i.d. increments, if the exit is optimal at $x$, it must be optimal at any $y > x$. For the rigorous proof, see [13, Thm. 9.4.2].

If $h_2^\lambda = +\infty$ (no restriction on an optimal exit rule), the problem (2.32) is solved in [18, Thm. 9.4.2] (see also [17]); in the restricted case, the result and the proof admit the straightforward modification. Let $\tilde{\mathcal{E}}_{q}$ be the minus Wiener-Hopf operator in the random walk model described above, and set $w_1^\lambda(x) = \tilde{\mathcal{E}}_{q}^{-1} g_1^\lambda(x)$.

**Theorem 2.5.** Assume that firm 2 exits when $\tilde{X}_t$ reaches $(U_2^\lambda)^c = [h_2^\lambda, +\infty)$ for the first time. Then
(a) the equation
\begin{equation}
(2.33) \quad w_1^\lambda(h) = 0
\end{equation}
has the only solution, denote it $h_1^\lambda$;
(b) if $h_1^\lambda \geq h_2^\lambda$, the optimal stopping time for the problem (2.32) is $\tau_1^{\lambda,*} = \tau_1^{+ h_2^\lambda}$, equivalently, the optimal threshold is $h_1^{\lambda,*}(h_2^\lambda) = h_2^\lambda$;
(c) if $h_1^\lambda < h_2^\lambda$, the optimal stopping time for the problem (2.32) is $\tau_1^{\lambda,*} = \tau_1^{+ h_1^{\lambda,*}(h_2^\lambda)}$, where $h_1^{\lambda,*}(h_2^\lambda) = h_1^\lambda$. 

The value function is given by

$$V_1^{\lambda, \ast}(h_2; x) = \frac{\alpha_1 R_1}{r} + (1 - q)^{-1} \tilde{F}_q\left[\frac{1}{(\infty, h_1^{\lambda, \ast}(h_2))} \tilde{F}_q\left[\lambda(x)\right]\right] = x^\lambda.$$  

Lemma 2.6. a) $h_1^{\lambda, \ast}(h_2)$ and $V_1^{\lambda, \ast}(h_2; x)$ are non-decreasing functions of $h_2^\lambda$.  

b) for any $h_2^\lambda$, $h_1^{\lambda, \ast}(h_2^\lambda) \leq h_1^{\lambda, \ast}(+\infty)$, $V_1^{\lambda, \ast}(h_2^\lambda; \cdot) \leq V_1^{\lambda, \ast}(+\infty; \cdot)$.

Proof. a) As $h_2^\lambda$ increases, the restriction on the optimal exercise rule relaxes. Hence, the option value and exit threshold cannot decrease.

b) is immediate from a).

Note that the definitions and results above can be used with the indices 1 and 2 interchanged. Starting from the exit threshold $h_{1,0}^\lambda$, for the lonely firm 1, define, inductively, for $n = 0, 1, 2, \ldots$, $h_{2,n}^\lambda = h_{2,0}^{\lambda, \ast}(h_{1,n}^\lambda)$, $h_{1,n+1}^\lambda = h_{1}^{\lambda, \ast}(h_{2,n}^\lambda)$, and the corresponding value functions $V_{j}^{\lambda, \ast}(h_{k,n}^\lambda; x)$, $j = 1, 2, k \neq j$.

On the strength of Lemma 2.6 all sequences are non-decreasing and bounded from above, and the same is true for the sequences of the value functions $V_{1}^{\lambda, \ast}(h_{2,n}^\lambda; \cdot)$ and $V_{2}^{\lambda, \ast}(h_{1,n}^\lambda; \cdot)$.

Hence, we obtain

Theorem 2.7. For $j = 1, 2$, $k \neq j$, $x \in \mathbb{R}$, the limits $h_j^{\lambda, \ast} = \lim_{n \to +\infty} h_{k,n}^\lambda$, and $V_j^{\lambda, \ast}(x) = \lim_{n \to +\infty} V^*l_{a_j}(h_{k,n}^\lambda, x)$ exist and are bounded by $h_j^{\lambda, \ast}(+\infty)$ and $V_j^{\lambda, \ast}(+\infty, x)$, respectively.

$V_j^{\lambda, \ast}$ is the value function, and $U_j^\lambda = [h_j^{\lambda, \ast}, +\infty)$ is the optimal exit region, $j = 1, 2$.

Note that $h_k^\lambda = +\infty$ means firm $k$, $k \neq j$, precommits itself to never exit and provide the first firm with a valuable source of additional information forever.

2.3.3. The problem for two firms: strategies. Assume that $h_{1,0}^{\lambda, \ast} \leq h_{2,0}^{\lambda, \ast}$. The exit problem of firm 1 is divided into two parts: when (and whether) it is optimal to exit when firm 1 is alive (this is the case $h_1^{\lambda, \ast} < h_{2,0}^{\lambda, \ast}$), and when it is optimal to exit when firm 2 remains alive. It may happen that firm 2 remains alive when firm 2 exits.

Since $h_j^{\lambda, 0} \leq h_{j,0}^{\lambda, \ast}$, the following three cases are possible:

I. $h_1^{\lambda, 0} \leq h_{2}^{\lambda, 0} \leq h_{1,0}^{\lambda, \ast} \leq h_2^{\lambda, \ast}$;

II. $h_1^{\lambda, 0} \leq h_{2,0}^{\lambda, \ast} \leq h_{1,0}^{\lambda, \ast} \leq h_2^{\lambda, \ast}$;

III. $h_{1,0}^{\ast} \leq h_{1,0}^{\lambda, \ast} \leq h_2^{\lambda, 0} \leq h_2^{\lambda, \ast}$.

The monotonicity argument in the proof of Theorem 2.7 implies the following

Lemma 2.8. Let $h_1^{\lambda, 0} \leq h_{2,0}^{\lambda, 0}$ and $V_1^{\lambda, 0} \leq V_{2,0}^{\lambda, 0}$. Then only Cases I and II are possible.  

If, in addition, $h_1^{\lambda, 0} < h_{2,0}^{\lambda, 0}$, then $h_1^{\lambda, \ast} < h_2^{\lambda, \ast}$ as well.

In the descriptions of optimal strategies below, the time of the last disaster is normalized to 0, and it is assumed that at time 0-, both firms remained on the market. The time of the next disaster is denoted $\tau$.

Theorem 2.9. (a) In Cases I and II, if $X_0 \geq h_1^{\lambda, \ast}$, then both firms exit, otherwise, both firms wait until time $\tau$ and make the decision depending on the observed $X_\tau$;
(b) In Case III,
- if \( h_{0,2}^\lambda \leq X_0 \), then both firms exit;
- if \( h_{2,0}^\lambda = h_{1,2}^\lambda \) and \( h_{1,2}^\lambda \leq X_0 \), then both firms exit;
- if \( h_{1,2}^\lambda < h_{0,2}^\lambda \) and \( h_{2,0}^\lambda \leq X_0 < h_{2,0}^\lambda \), then firm 1 exits, and firm 2 waits until time \( \tau \) and makes the decision depending on the observed \( X_\tau \);
- if \( X_0 < h_{1,2}^\lambda \), both firms wait until \( \tau \) and make the decision depending on the observed \( X_\tau \).

To derive sufficient conditions for the case when one of the firms exits but the other one does not, we note that \( h_{0,j}^\lambda < h_{\infty,j}^\lambda < h_{\infty}^\lambda \). For \( h_{j}^\lambda \), we have the equation (2.26), and, similarly to (2.35),
\[
e^{h_{j}^\lambda (+\infty)} = \frac{(1 - \alpha_j)R_j}{A_j\lambda_j} \cdot \frac{1}{\kappa_-(1)\kappa_+^{\lambda_1+\lambda_2}(1)}.
\]
Comparing (2.35) with \( j = 1 \) and (2.26) with \( j = 2 \), we obtain the following sufficient condition that ensures that it is possible that, at the moment of exit of firm 1, firm 2 does not exit:
\[
\frac{(1 - \alpha_1)R_1}{A_1\lambda_1} \cdot \frac{1}{\kappa_-(1)\kappa_+^{\lambda_1+\lambda_2}(1)} \leq \frac{(1 - \alpha_1)R_1}{A_1\lambda_1} \cdot \frac{1}{\kappa_-(1)\kappa_+^{\lambda_1+\lambda_2}(1)}.
\]
It is easily seen that even if both firms are identical with the exception of \( \lambda_j \) (\( \lambda_2 < \lambda_1 \), and the ratio is sizable), the inequality (2.36) can be satisfied. Roughly speaking, if the best source of information coming from the other firm (subject to more frequent shocks) disappears, it may still be optimal to wait because your losses happen not so often, which compensates for the information loss.

3. Exit problem for two firms, Cox processes for arrivals

In this Section, \( \lambda_j(x,t), j = 1, 2, \) are the term structures of hazard rates obtained in the result of updating of prior beliefs, given that the last observation was \( x \), the time of the last observation is labelled 0, and the next piece of information has not arrived by time \( t > 0 \). If we assume that the priors for the hazard rate at different levels of \( x \) are different, and the probability of observation of a disaster in the same state is 0, then the update of the hazard rate in the current state at moments of observation do not matter and can be ignored.

The decision rule of firm \( j \) is given by a function \( T_j : \mathbb{R} \rightarrow [0, +\infty] \), where \( T_j(x) = 0 \) means that firm \( j \) exits at time 0 if a disaster happens at time 0 and \( X_0 = x \), \( T_j(x) = +\infty \) means that firm \( j \) does not exit before the next disaster strikes, and \( T_j(x) \in (0, +\infty) \) means that, if \( X_0 = x \), then firm \( j \) exits at time \( T_j(x) \) unless the disaster happens before or on this date. Set \( U_j = \{ x \mid T_j(x) > 0 \} \); this is the inaction region of firm \( j \) at time 0.

The constructions and results below remain valid if the firms use different discount rates \( r_j \); the only changes in the constructions below are that \( r_j \) should be used instead of \( r \).
3.1. The case of one firm. Suppose that firm $j$ is alone; however, we will keep the index $j$ in order to be able to use the notation and results at the next steps, when the interactions between exit decisions of two firms will be analyzed.

Denote by $V_j(x)$ the value of the firm at time 0 and $X_0 = x$ (naturally, assuming that the firm chooses an optimal exit strategy). If the firm exits, its value is given by $V_j(x) = \alpha_j R_j/r$. If the firm does not exit at $X_0 = x$, consider the value of the firm, which, at time 0, fixes a calendar time $T > 0$ and exits at $T$ if the next piece of information does not arrive by this time. Essentially, we consider potential one-shot deviations of as yet unknown optimal exit rule.

3.1.1. General results. Let $\tau$ be the time of arrival of the next piece of information. For all $t < T \land \tau$, the only information is the one available at time 0. Therefore, at $t < T$, applying the standard infinitesimal accounting techniques, we find that the value of the firm is

$$W_j(V_j; T; t, x) = \frac{\alpha_j R_j}{r} e^{-\int_t^T (r + \lambda_j(x,s)) ds}$$

$$+ \int_t^T e^{-\int_t^{t'} (r + \lambda_j(x,s)) ds} (R_j + \lambda_j(x, t') \mathbb{E}_{Q,x} [V_j(X_{t'}) - A_j e^{X_{t'}}]) dt'.$$

(3.1)

Naturally, for any $x$, $W_j(V_j; 0; 0; x) = \alpha_j R_j/r$, $V_j(x) \geq \alpha_j R_j/r$, and

$$V_j(x) = \sup_{T \geq 0} W_j(V_j; T; 0, x).$$

(3.2)

Set $V_{j0}(x) = \alpha_j R_j/r$, and define, inductively, for $n = 0, 1, 2, \ldots$,

$$V_{j,n+1}(x) = \sup_{T \geq 0} W_j(V_{j,n}; T; 0, x), \quad x \in \mathbb{R}.$$

(3.3)

Theorem 3.1. The following statements hold:

(i) The sequence $\{V_{j,n}\}$ is non-decreasing (point-wise).

(ii) The sequence $\{V_{j,n}\}$ is uniformly bounded from above by $R_j/r$.

(iii) For any $x \in \mathbb{R}$, the limit $V_j(x) = \lim_{n \to \infty} V_{j,n}(x)$ exists, (3.2) holds, and $V_j$ is the value function.

Proof. (i) is evident from (3.1). (ii) It suffices to prove that the LHS of (3.1) is bounded by $R_j/r$ if $V_j$ on the RHS of (3.1) is bounded by $R_j/r$. Replacing $V_j$ with $R_j/r$ and $\alpha_j < 1$ with 1, omitting the negative term $-A_j e^{X_{t'}}$, and multiplying by $r/R_j$, we see that it suffices to prove that

$$I_j(T) := e^{-\int_0^T (r + \lambda_j(x,s)) ds} + \int_0^T e^{-\int_0^{t'} (r + \lambda_j(x,s)) ds} (r + \lambda_j(x, t')) dt'$$

equals 1. But this is just the standard balance equality for the killing process with the rate $r + \lambda_j(x; s)$ (the first term is the fraction of paths remaining alive at time $T$, the integral counts the fraction of killed paths).

(iii) The existence of the limit follows from (i) and (ii). Passing to the limit in (3.3), we obtain that (3.2) holds. Hence, $V_j$ is the value function. □
To derive an equation for an optimal \( T_j = T_j(x) \geq 0 \), calculate the derivative

\[
\partial_T W_j(V_j; T; t, x) = -(r + \lambda_j(x; T)) \frac{\alpha_j R_j}{r} e^{\int_t^T (r + \lambda_j(x; s)) ds} \\
+ e^{\int_t^T (r + \lambda_j(x; s)) ds} \left( R_j + \lambda_j(x, T) \mathbb{E}^{Q,x} \left[ V_j(X_T) - A_j e^{\Psi(T)} \right] \right)
\]

(3.4) \[= e^{-\int_t^T \lambda_j(x; s) ds} \mathcal{U}_j(V_j; T, x).\]

\[
\mathcal{U}_j(V_j; T, x) = -(r + \lambda_j(x; T)) \frac{\alpha_j R_j}{r} + R_j + \lambda_j(x, T) \mathbb{E}^{Q,x} \left[ V_j(X_T) - A_j e^{\Psi(T)} \right]
\]

(3.5) \[= (1 - \alpha_j) R_j + \lambda_j(x, T) \left( \mathbb{E}^{Q,x}[V_j(X_T)] - \frac{\alpha_j R_j}{r} - A_j e^{\Psi(T+x)} \right).\]

Clearly, \( \partial_T W_j(V_j; T; t, x) > 0 \) if \( 0 < x, T > 0 \). \( \mathcal{U}_j(V_j; T; x) > 0 \) if \( 0 < x, T > 0 \).

**Proposition 3.2.** An optimal \( T_j \) is time-consistent: if \( T_j(x) \) maximizes \( W_j(V_j; T; 0; x) \), then, for any \( t \leq T_j(x) \), \( T_j(x) \) is a maximizer of \( W_j(V_j; T; t; x) \).

**Proof.** Set \( f_j(x; t) = \exp \left[ -\int_0^t (r + \lambda_j(x; s)) ds \right] \). For any \( 0 \leq t < T \), we have

\[
W_j(V_j; T; 0; x) = \int_0^T \mathcal{U}_j(V_j; t', x) f_j(x; t') / f_j(x; 0) dt'
\]

\[
= \int_0^t \mathcal{U}(V_j; t', x) \frac{f_j(x; t')}{f_j(x; 0)} dt' + \frac{f_j(x; t)}{f_j(x; 0)} \int_t^T \mathcal{U}(V_j; t', x) \frac{f_j(x; t')}{f_j(x; t)} dt'
\]

\[
= W_j(V_j; t; 0; x) + \frac{f_j(x; t)}{f_j(x; 0)} W_j(V_j; T; t; x).
\]

Since \( W_j(V_j; t; 0; x) \) and \( f_j(x; t) / f_j(x; 0) \) are independent of \( T \), a \( T \) that maximizes \( W_j(V_j; T; 0; x) \) maximizes \( W_j(V_j; T; t; x) \) as well.

Denote by \( T_{j,x}(x) \) and \( T_{j,x}^*(x) \) the minimum and supremum of the set of maximizing \( T_j(x) \).

**Proposition 3.3.** If \( \mathcal{U}_j(V_j; 0; x) > 0 \), then it is non-optimal to exit at time 0 and \( X_0 = x \).

**Proof.** If \( \mathcal{U}_j(V_j; 0; x) > 0 \), function \( W_j(V_j; T; 0; x) \) is increasing in \( T \) in a neighborhood of \( T = 0 \).

**Proposition 3.4.** Let \( \Psi(1) \geq 0 \). Then there exists \( x_+ \) such that if \( X_0 \geq x_+ \), then the exit is optimal at time 0.

**Proof.** Since \( V_j(x) < R_j / r \) for any \( x \), we have

\[
\mathcal{U}_j(V_j; T; x) > (1 - \alpha_j) R_j + \lambda_j R_j / r - A_j e^{\Psi(1) + x}, \quad \forall x \in \mathbb{R}, T \geq 0,
\]

therefore, if \( x \leq \ln[(1 - \alpha_j) R + \lambda_j R_j / r]/A_j \), then \( \mathcal{U}_j(V_j; T; x) < 0 \) for all \( T > 0 \). \( \square \)
Below, we concentrate on the case Ψ(1) > 0; this condition is equivalent to the statement that, on average, the next disaster is greater than the previous one, and we assume that, for each x, the hazard rate λ_j(x, t) is either bounded away from 0 or tends to 0 as t → +∞ not very fast.

**Theorem 3.5.** Assume that the following conditions hold:

(i) \( U_j(V_j; 0; x) > 0 \);
(ii) \( Ψ(1) > 0 \);
(iii) \( λ_j(x, T)e^{TΨ(1)} \rightarrow +∞ \) as \( T \rightarrow +∞ \).

Then it is non-optimal to exit at time 0, and it is non-optimal to wait indefinitely for the next piece of information to arrive: \( 0 < T_j(x) \leq T_j^*(x) \).

**Proof.** Since \( U_j(V_j; 0; x) > 0 \), the function \( W_j(V_j; T; 0; x) \) increases in \( T \) in a neighborhood of 0, hence, it is non-optimal to exit at \( T = 0 \) and \( X_0 = x \). Since \( V_j \leq R_j/r \), and, on the strength of (ii) and (iii), \(-λ_j(x, T)e^{Ψ(1)T+x} \rightarrow -∞ \) as \( T \rightarrow +∞ \), \( W_j(V_j; T; 0; x) \) → \(-∞ \) as well. Hence, function \( T \mapsto W_j(V_j; T; 0; x) \) has a global maximum at some point \( T \in (0, +∞) \), and all such points are on some segment \([a, b] \subset (0, +∞)\). □

Theorem 3.5 gives sufficient conditions for non-optimality of exercise at time 0 and non-optimality of waiting indefinitely for the next piece of information to arrive. However, condition (i) is formulated in terms of the value function that can be found only numerically using the iteration procedure in Theorem 3.1.

**Corollary 3.6.** Let conditions (ii) and (iii) of Theorem 3.5 hold, and let

\[
(3.6) \quad \frac{(1 - α_j)R_j}{A_jλ_j(x, 0)} \geq e^x.
\]

Then the conclusion of Theorem 3.5 hold.

**Proof.** Since \( V_j(x) \geq α_jR_j/r \) for all \( x \), with the strict inequality for \( x \) in a neighborhood of \(-∞\), the inequality \((1 - α_j)R_j \geq A_jλ_j(x, 0)e^x \) implies \( U_j(V_j; 0; x) > 0 \). □

### 3.1.2. Sufficient conditions for the uniqueness of \( T_j \).

**Proposition 3.7.** Assume that the conditions of Theorem 3.5 hold, and, in addition,

\[
(3.7) \quad \text{function } T \mapsto λ_j(x, T) \left( \mathbb{E}_Q^x [V_j(X_T)] - A_je^{Ψ(1)T+x} \right) \text{ is decreasing}
\]

Then a maximizer \( T_j(x) \in (0, +∞) \) is unique, and either \( T_j(x) = 0 \) or \( T_j(x) > 0 \), and \( U_j(V_j; T; x) > 0 \), \( ∀ T \in [0, T_j(x)] \).

The problem with the sufficient condition (3.7) is that it is formulated in terms of the unknown value function. Intuitively, the conditions of Theorem 3.5 and the following two conditions:

\[
(3.8) \quad T \mapsto λ_j(x, T) \text{ is non-increasing;}
\]

\[
(3.9) \quad T \mapsto λ_j(x, T)e^{Ψ(1)} \text{ is increasing}
\]
are expected to imply the condition (3.7). Indeed, if the expected losses at time \( T \) increase with \( T \), it is natural to presume that

\[
T \mapsto \mathbb{E}^{Q,x}[V_j(X_T)] \text{ is non-increasing.}
\]

Then, under condition (3.8),

\[
T \mapsto \lambda_j(x,T)\mathbb{E}^{Q,x}[V_j(X_T)] \text{ is non-increasing.}
\]

Finally, (3.11) and (3.9) yield (3.7). Note that if (3.9) holds, then we may allow for sufficiently slow growth of the function in (3.11); hence, we can relax the (somewhat unnatural) condition of the first order stochastic dominance.

The conditions (3.8)-(3.9) are easy to verify but we were unable to prove that then (3.10), hence, (3.11) hold. Evidently, (3.10) holds if the family of distributions \( \{X_T\}_{T \geq 0} \) is ordered in the sense of the first order stochastic dominance:

**FOSDX property.** Let \( V \) be a bounded non-increasing function. Then, for any \( 0 \leq T \leq T_1 \), and any \( x \in \mathbb{R} \),

\[
\mathbb{E}^{Q,x}[V(X_T)] \geq \mathbb{E}^{Q,x}[V(X_{T_1})].
\]

If \( X \) is a Lévy process, (3.12) implies that the supremum process is trivial. Hence, we may allow for positive jumps and non-negative drift only. However, if (3.9) holds, then (3.7) will hold even if we allow for a small negative drift (and/or a small Brownian motion component). Thus, we have two results: one, that is general but somewhat too restrictive (the size of the next disaster cannot be less than the size of the last one), and a version that allows for some small (albeit temporary) drops in the size of the next disaster.

Let \( V_{j,n}, n = 0,1,\ldots, \) be the sequence of approximations to the value function constructed to prove Theorem 3.1.

**Theorem 3.8.** Let

(a) \( \Psi(1) > 0 \);
(b) FOSDX property hold;
(c) conditions (ii), (iii) of Theorem 3.5, (3.8), and (3.9) be satisfied for all \( x \in \mathbb{R} \).

Then,

(1) for each \( n = 0,1,2,\ldots, \) and each \( x \in \mathbb{R} \), function \( U_j(V_{j,n};T;x) \) is decreasing, hence, the maximizer \( T_{jn}(x) \) of the problem (3.3) is unique and finite;
(2) for each \( x \in \mathbb{R} \), sequences \( \{T_{jn}(x)\} \) and \( \{V_{jn}(x)\} \) are non-decreasing, and have finite limits \( T_j(x) = \lim_{n \to +\infty} T_{jn}(x), V_j(x) = \lim_{n \to +\infty} V_{jn}(x) \);
(3) function \( U_j(V_j;T;x) \) is decreasing, and \( V_j(x) = W_j(V_j;T_j(x);0;x) \);
(4) \( T_j(x) = 0 \) if and only if \( U_j(V_j;0;x) \leq 0 \).

**Proof.** Essentially, we repeat the proof of Theorem 3.5. \( V_{j0} = \lambda_j R_j/r \) is non-increasing, hence, for any \( x \), \( U_j(V_{j0};T;x) \) is decreasing as a function of \( T \), and \( T_{j0}(x) \) is unique and finite. Since \( \mathbb{E}^{Q,x}[V_{j0}(X_T)] \) is non-increasing in \( x \), function \( V_{j1}(x) \) is non-decreasing as well. We conclude that \( U_j(V_{j1};T;x) \) is decreasing as a function of \( T \), and \( T_{j1}(x) \) is unique and finite. Clearly, \( V_{j1}(x) \geq V_{j0}(x) \) for any \( x \), therefore,
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\[ \mathcal{U}_j(V_{j0}; T; x) \leq \mathcal{U}_j(V_{j1}; T; x) \] for all \( T \geq 0 \) and \( x \in \mathbb{R} \), hence, \( T_{j1}(x) \geq T_{j0}(x), \forall x \in \mathbb{R} \). These arguments can be iterated with \( n, n+1 \) in place of 0, 1, for \( n = 0, 1, 2, \ldots \). Being non-decreasing and bounded from above, both sequences have finite limits. \( \square \)

**Theorem 3.9.** Assume that the following conditions hold:

(i) \( X_t = \mu t + Y_t \), where \( \mu \leq 0 \) and the process \( Y_t \) has FOSDX property;

(ii) \( \mu + \Psi \gamma(1) > 0 \);

(iii) the hazard rate \( \lambda_j(x, t) \) is uniformly bounded;

(iv) \( \lambda_j(x, t) \) is decreasing in \( t \), for any \( x \in \mathbb{R} \);

(v) \( \lambda_j(x, t) e^{\Psi(1)} \rightarrow +\infty \), uniformly in \( x \).

Then, for any compact \( K \in \mathbb{R} \), there exists \( \mu_0 < 0 \) such that, for any \( \mu \in (\mu_0, 0) \), and for any \( x \in K \), one of the two possibilities exist:

1. firm \( j \) exits at \( X_0 = x \).
2. solution \( T_j(\mu; x) \) of the equation \( \mathcal{U}_j(V_j(\mu; \cdot); T; x) = 0 \) is unique, and \( V_j(\mu; x) = W_j(V_j(\mu; \cdot); T_j(x); 0; x) \).

**Proof.** Note that the existence of the value function has been proved without using FOSDX property. If \( \mu = 0 \), the property holds, therefore, for any \( x \in \mathbb{R} \), \( \mathcal{U}_j(V_j(\mu; \cdot); T; x) \) is strictly decreasing in \( T \). Moreover, there exists \( T \) such that \( \mathcal{U}_j(V_j(\mu; \cdot); T; x) < 0 \) for all \( x \in K, T \geq \bar{T} \). It is easy to see that \( \mathcal{U}_j(\mu; V_j; T; x) \) depends on \( \mu \) continuously, hence, for any compact \( \mathcal{K} \in \mathbb{R} \times \mathbb{R}_+ \), and sufficiently small (in absolute value) \( \mu_0(\mathcal{K}) < 0 \), \( \mathcal{U}_j(\mu; V_j; T; x) \) is strictly decreasing in \( T \), when \( (x, T) \in \mathcal{K} \). As \( \mu \uparrow 0 \),

\[ \mathcal{U}_j(V_j(\mu; \cdot); T; x) - A_j \lambda_j(x, T) e^{\Psi(1) + x} \rightarrow \mathcal{U}_j(V_j(0; \cdot); T; x) - A_j \lambda_j(x, T) e^{\Psi(1) + x} \]

uniformly in \( (x, T) \) from any compact, and since the difference on the RHS is strictly negative in a neighborhood of \( T_j(0; x) \), it remains negative in the neighborhood of \( T_j(\mu; x) \) if \( -\mu > 0 \) is small enough. \( \square \)

3.1.3. **Hump-shaped hazard rate and non-uniqueness of \( T_j \).** In the real life, one may expect that the hazard rate is non-monotone and may have humps (think about hurricane seasons) and wells between the humps. If the wells are sufficiently deep (\( \lambda(t, 0) \) is close to 0) and wide, then, for \( T \) in a neighborhood of a well, \( \mathcal{U}_j(V_j; T; x) > 0 \), and, in a neighborhood of a hump, \( \mathcal{U}_j(V_j; T; x) < 0 \). Hence, playing with the shape of the hazard rate curve, one can construct \( V_j \) such that function \( T \mapsto W_j(V_j; T; 0; x) \) has several local maxima of varying height; and, continuously deforming the curve, one can obtain several points of the global maximum.

3.2. **The news arrive according to a Cox process; the option exercise is possible at any moment, two firms.** Firm \( k \) precommits to never exit. Define \( \bar{\lambda}(x, t) = \lambda_1(x, t) + \lambda_2(x, t) \). To find the value of firm \( j \), we repeat the same
steps as in the case of the lonely firm replacing (3.1) with
\[
W_j(V_j; T; t, x) = \frac{\alpha_j R_j}{r} e^{-\int_t^T (r + \tilde{\lambda}(x; s))ds} + \int_t^T e^{-\int_s^T (r + \tilde{\lambda}(x; s))ds} ds
\]
Equation (3.2) holds. We construct the sequence \{V_{j,n}\} using \(W_j(V_j; T; t, x)\) defined by (3.13) and repeat the statement and proof of Theorem 3.1 verbatim.

3.4. The news arrive according to a Cox process; the option exercise is possible at any moment, firms are asymmetric. Denote by \(\lambda (x, t) = 2 \cdot \lambda_1 (x, t) = 2 \cdot \lambda_2 (x, t)\), and use \(W_j(V_j; T; t, x)\) defined by (3.13) and \(U_j(V_j; T; t, x)\) defined by (3.15) in all constructions, statements and proofs for the lonely firm. These changes being made, all results remain valid.
(a) \(\Psi(1) > 0\);  
(b) the first instantaneous moment \(m_1 = \Psi'(0) > 0\) (then any sample path of \(X\) tends to \(+\infty\) - see \([52]\));  
(c) \(\lambda(x, t), \ell = 1, 2\), is decreasing in \(t\), for any \(x \in \mathbb{R}\);  
(d) \(\lambda(x, t)e^{\Psi(1)} \to +\infty\), monotonically, \(\ell = 1, 2\), and uniformly in \(x\).

Since \(V_j(x) \downarrow \alpha_jR_j/r\) as \(x \to -\infty\), we conclude that \(\lim_{T \to +\infty} \mathbb{E}^{Q, x}[V_j(X_T)] = \alpha_jR_j/r\), and, therefore, each of the functions: \(U_j(V_j; T; x), U_j^\infty(V_j; T; x)\) and \(U_j^0(V_j; T; x)\), \(j = 1, 2\), have the following asymptotics as \(x \to -\infty\):

\[
U_j(V_j; T; x) \sim U_{j0}(T; x) + \epsilon_j^\infty(x),
\]

where

\[
U_{j0}(T; x) = (1 - \alpha_j)R_j - A_j\lambda_j e^{\Psi(1)T_x},
\]

and function \(\epsilon_j^\infty(x) \to 0\) as \(x \to +\infty\) depends on the type of the function \(U_j\).

The following lemma is immediate.

**Lemma 3.10.**  
(i) There exists \(x_-\) such that, for all \(x \leq x_-\), the functions \(U_j(V_j; T; x), U_j^\infty(V_j; T; x)\) and \(U_j^0(V_j; T; x)\) change the sign only once, and \(T_j(x), T_j^\infty(x), T_j^0(x)\) are well-defined and positive. Each of the latter admit the representation \(\hat{T}_j(x) + o(1)\) as \(x \to -\infty\).

(ii) If

\[
\liminf_{x \to -\infty} \frac{(1 - \alpha_1)R_1}{A_1\lambda_1(x, \hat{T}_1(x))} \frac{A_2\lambda_2(x, \hat{T}_2(x))}{(1 - \alpha_2)R_2} > 1,
\]

then there exists \(x_-\) such that \(\forall x \leq x_-\), \(T_2(x) \leq T_2^\infty(x) < T_1^0(x) \leq T_1(x)\).

**Theorem 3.11.** Let conditions (a)-(d) and \([3.16]\) hold. Then there exists \(x_-\) such that, in any Nash equilibrium, in the subgame for two firms that starts at the moment of the last disaster (time 0) and \(X_0 = x \leq x_-\),

(i) neither of firms exits at time 0;

(ii) if the next disaster happens at \(\tau \leq T_2(x)\), then the game starts anew;

(iii) if the next disaster happens at \(T_2(x) < \tau \leq T_1(x)\), firm 2 exits at time \(T_2(x)\), and firm 1 starts as the lonely firm at the new initial moment \(\tau\), at the level \(X_\tau\) of the shock;

(iv) if the next disaster happens at \(\tau > T_1^0(x)\), firm 2 exits at time \(T_2(x)\), and firm 1 follows at time \(T_1^0(x)\).

**Proof.** On \([0, T_1(x)]\), hence, on \([0, T_2(x)]\), function \(W_1(V_1; t; t; x)\) increases because \(U_1(V_1; t; x) > 0\) for \(t < T_1(x)\). Hence, it is not optimal for firm 1 to exit before firm 2 exits. At the moment of exit of firm 2, firm 1 becomes the lonely firm. Since no new observation has arrived, in order to derive an optimal exit strategy, firm one uses the same hazard rate \(\lambda(x, T)\) and the expected value \(\mathbb{E}^{Q, x} [V_1^0(X_T)]\) as it would had used if it were alone at time 0. Since \(T_1^0(x) > T_2(x)\), function \(W_1^0(V_1^0; t; t; x)\) increases in \(t \in [T_2(x), T_1^0(x)]\), hence, firm 1 will wait until \(T_1^0(x)\) unless the new piece of information arrives.
Clearly, at high levels of $X_0 = x$, both firms exit, and an evident upper bound for the inaction region for firm $j$ is $\sup U_j^0 < +\infty$. Thus, if (a)-(d) and (3.16) hold, it remains to study the exit rules for $x$ in a bounded interval $[x_-, x_+]$. Although the boundaries are unknown in advance, we can formulate reasonable conditions for the existence and uniqueness of a Nash equilibrium, and define the optimal exit strategies.

For simplicity, we assume that the only difference between the two firms is in the hazard rate, and one firm is more lucky than the other in all states and at all times. To be more specific, $\alpha_1 = \alpha_2$, $R_1 = R_2$, $A_1 = A_2$ and

$$\lambda_2(x;t) > \lambda_1(x;t) > 0, \quad \forall x \in \mathbb{R}, t \geq 0,$$

and

$$\lim_{x \to -\infty} \inf_{t \geq 0} \lambda_2(x;t)/\lambda_1(x;t) > 1.$$

Therefore, when both firms are on the market, neither firm has an informational advantage but firm 2 suffers losses more often than firm 1. Therefore, $V_2^0(x) \leq V_1^0(x)$, $V_2(x) \leq V_1(x)$, with the strict inequalities if firm 1 does not exit at $X_0 = x$. It follows that $U_2 \subset U_1$, and the closure of $U_2$ is a proper subset of $U_1$. Evidently, $U_1 \setminus U_2 \subset U_1^0 \subset U_1$, therefore, for $x \in U_1 \setminus U_2$, $T_1(x) = T_1^0(x)$, $V_1(x) = V_1^0(x)$. It remains to consider $x \in [x_-, x_+] \cup U_2$.

As above, the starting object for our constructions is a function $W_j$, whose definition needs an adjustment to account for the fact that the other firm may exit. For $j, k \in \{1, 2\}, j \neq k$, given an exit rule $T_k$ of firm $k$, and a monotone continuous function $V$ satisfying $\alpha_j R_j/r \leq V(x) \leq R_j/r$, introduce function

$$W_j(V; T_k; T; t; x) = \mathbb{I}_{[0, T_k(x)]}(T) \frac{\alpha_j R_j}{r} e^{-f_j^T(r+\tilde{\lambda}(x;s))ds}$$

$$+ \delta_{T=T_k(x)} \mathbb{E}^{Q,x} \left[ V_j^0(X_{T_k(x)}) \right] e^{-f_j^{T_k}(r+\tilde{\lambda}(x;s))ds}$$

$$+ \int_t^T e^{-f_j^T(r+\tilde{\lambda}(x;s))ds}$$

$$\times \left( R_j + \tilde{\lambda}(x, t') \mathbb{E}^{Q,x} \left[ \mathbb{I}_{U_k(X_{t'})} V(X_{t'}) + \mathbb{I}_{U_k^0(X_{t'})} V_j^0(X_{t'}) \right] \right) dt'$$

$$- \int_t^T e^{-f_j^T(r+\tilde{\lambda}(x;s))ds} \lambda_j(x; t') A_j e^{r\Phi(1)+x} dt'.$$

Clearly, the optimal $V_j(\cdot) = V_j(T_k; \cdot)$ satisfies

$$V_j(T_k; x) = \sup_{0 \leq T \leq T_k(x)} W_j(V_j(T_k; \cdot); T_k; T; 0; x).$$
As in the non-restricted case above, on \([0, T_k(x))\),

\[
\partial_t W_j(V; T_k; t, x) = e^{-\int_t^T (r+\lambda_s(x))ds} \left( -(r + \lambda_j(x; T)) \frac{\alpha_j R_j}{\bar{r}} + R_j \right.
\]

\[
+ \bar{\lambda}(x; T) \mathbb{E}_{\mathbb{Q}, x} \left[ \mathbb{1}_{U_k}(X_T)V(X_T) + \mathbb{1}_{U_k'}(X_T)V^0_j(X_T) \right]
- \lambda_j(x, T) A_j e^{T \Psi(1)+x}
\]

\begin{equation}
(3.21)
= e^{-\int_t^T (r+\lambda_s(x))ds} \mathcal{U}_j(V; T, x),
\end{equation}

where

\[
\mathcal{U}_j(V; T, x) = -(r + \bar{\lambda}(x; T)) \frac{\alpha_j R_j}{\bar{r}} + R_j
\]

\[
+ \bar{\lambda}(x, T) \mathbb{E}_{\mathbb{Q}, x} \left[ \mathbb{1}_{U_k}(X_T)V(X_T) + \mathbb{1}_{U_k'}(X_T)V^0_j(X_T) \right]
- \lambda_j(x, T) A_j e^{T \Psi(1)+x}
\]

\begin{equation}
(3.22)
= (1 - \alpha_j) R_j - \lambda_j(x, T) A_j e^{\Psi(1)T+x}
+ \bar{\lambda}(x, T) \left( \mathbb{E}_{\mathbb{Q}, x} \left[ \mathbb{1}_{U_k}(X_T)V(X_T) + \mathbb{1}_{U_k'}(X_T)V^0_j(X_T) \right] - \frac{\alpha_j R_j}{\bar{r}} \right).
\end{equation}

Since \(V_1 \geq V_2, V_1^0 \geq V_2^0\) and \(U_1 > U_2\), we have \(\mathcal{U}_1(V_1; T, x) \geq \mathcal{U}_2(V_2; T, x)\). Therefore, if for all \(x \in U_2, \mathcal{U}_2(V_2; T, x) \geq 0\) for \(T \in [0, T_{2,*}(x))\), and \(T_{2,*}(x)\) is the point of global maximum of \(W_2(V_2; T_1; T; 0; x)\), then it is non-optimal for firm 1 to exit earlier than at \(T_{2,*}(x)\). Therefore, firm 2 may solve the exit problem assuming that firm 1 never exits. We solved this problem already. Thus, \(V_2 = V_2^\infty\), and \(T_{2,*} = T_{2,*}^\infty\). If, as in Section 3.1.2, we assume that function \(V_2\) satisfies the natural condition (3.10), which holds if \(X\) satisfies FOSDX property (3.12), then \(T_{2,*}\) defines the only optimal exit rule for firm 2, and constructions of optimal exit rules for firm 1 below will give all possible Nash equilibria in the exit game. We will leave for the future the study of cases when \(T_{2,*}(x)\) is non-unique; these examples can be constructed if the hazard rates are non-monotone; and then quite non-trivial interactions between the exit decisions of the firms are possible.

If (3.12) holds, then, for any \(x\) and non-increasing \(V, T \mapsto \mathcal{U}_j(V; T; x)\) is decreasing. Hence, for any \(x \in U_2, \mathcal{U}_1(V; 0; x) > 0\), and, therefore, there exists \(c > 0\) such that

\begin{equation}
(3.23)
\mathcal{U}_1(V_1; T; x) > c \quad x \in [x_-, x_+] \cap U_2, \ T \in [0, T_{2,*}(x)]
\end{equation}

Under the standing assumptions (a), (b), (d) at the beginning of Section 3.1, all functions \(W_j, W_j^0, W_j^\infty\) that we may need to consider, are negative on \([T, +\infty) \times [x_-, x_+]\) if \(T\) is sufficiently large. Therefore, all arguments that we make need to be valid only for \((T, x)\) from this compact. Therefore, standard small perturbation considerations imply that if \(X\) is a sum of a process satisfying (3.12) and a sufficiently small drift, then (3.23) still holds (with a smaller \(c > 0\)). Moreover, if \(V_2^\infty\) and \(T_{2,*}^\infty\) have been calculated, and the condition

\begin{equation}
(3.24)
\mathcal{U}_2(V_2; T; x) > 0 \quad x \in [x_-, x_+] \cap U_2, \ T \in [0, T_{2,*}^\infty(x))
\end{equation}
Let \( V(x) \) be the value function without any additional assumption. Assuming that \( T_2 \) is fixed, we can calculate the value function \( V_1 \) of firm 1 as follows. Set \( V_{1,0} = V_1^0 \), and, for \( n = 0, 1, 2, \ldots \), define
\[
V_{1,n+1}(x) = \sup_{T \geq \tau} W_1(V_{1,n}; T; 0; x)
\]

**Theorem 3.12.** Let \( T_2 = T_{2,*}^\infty \) and (3.25) holds. Then

(i) for any \( x \in \mathbb{R} \), the limit \( V_1(x) = \lim_{n \to \infty} V_{1,n}(x) \), \( x \in \mathbb{R} \) exists, and \( V_1 \) is the value function of firm 1;

(ii) if \( T_1^0(x) \) is unique, firm 1 exits at \( T_1(x) := \min\{T_{2,*}^\infty(x), T_1^0(x)\} \) unless \( \tau \leq T_1(x) \);

(iii) if \( T_1^0(x) \) is non-unique and the recovery value \( \alpha_j R_j / r \) is larger than the continuation value \( \sup_{T > T_{2,*}^\infty(x)} W_1(V_1^0; T; T_{2,*}^\infty(x); x) \), then firm 1 exits at \( T_1(x) = \min\{T_{2,*}^\infty(x), T_1^0(x)\} \) unless \( \tau \leq T_1(x) \);

(iv) if \( T_1^0(x) \) is non-unique and the recovery value is less than or equal the continuation value, firm 1 may choose any \( T_1^0(x) \geq T_{2,*}^\infty(x) \) as an optimal exit time and exit at \( T_1^0(x) \) unless \( \tau \) is less than or equal \( T_1^0(x) \).

**Proof.** Under condition (3.25), it is non-optimal for firm 1 to exit while firm 2 remains alive. The same arguments as in the case of the lonely firm, give (i). Statements (ii)-(iv) are evident. \( \square \)

Thus, in principle, even if \( T_{2,*}^\infty \) is unique, multiple Nash equilibria are possible. However, it is unlikely that multiple equilibria exist if \( \lambda_j(x,t) \) are monotone. If they are not, then one can play with the term structures of the hazard rates to ensure that firm 2 exits relatively soon, and then play with the shape of the hazard rate curve \( \lambda_1(x,t) \) on \( (T_2(x), +\infty) \) to ensure that, for some \( x \), at least, \( T_1^0(x) > T_2(x) \) and multiple global maxima of \( W_1(V_1^0; T; 0; x) \) exist.

Assume that \( T_2 = T_2^\infty \) and \( T_1 = \max\{T_2, T_1^0\} \) are unique. Then the following holds.

**Theorem 3.13.** The pair of the strategies \( (T_1, T_2) \) define a unique Nash equilibrium as follows.

(1) If \( X_0 \in (U_2)^c \), both firms exit at time 0.

(2) If \( X_0 = x \in U_1 \setminus U_2 \), firm 2 exits at time 0. Firm 1 exits at time \( T_1^0(x) \) if the next moment of observation \( \tau \) does not happen by time \( T_1^0(x) \). If \( \tau > T_1^0(x) \), firm 1 continues the game of the lonely firm starting at \( x = X_\tau \), with \( \tau \) as the new starting time moment.

(3) If \( X_0 = x \in U_1 \cap U_2 \), then
   - if the next disaster happens at \( \tau \leq T_2(x) \), the game starts anew;
   - if the next disaster happens at \( T_2(x) < \tau \leq T_1(x) \), firm 2 exits at time \( T_2(x) \), and firm 1 starts as the lonely firm at the new initial moment \( \tau \), at the level \( X_\tau \) of the shock;
   - if the next disaster happens at \( T_1(x) < \tau \), firm 2 exits at time \( T_2(x) \), and firm 1 follows at time \( T_1^0(x) \).
References


