Affirmative Action as a Large Contest

Aaron L. Bodoh-Creed and Brent R. Hickman

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Abstract

We develop a model of affirmative action as a large contest wherein students with heterogeneous underlying abilities compete for seats at vertically differentiated colleges that use color-sighted affirmative action policies to evaluate applicants. Students make costly human capital investments before applying, and these investments are both intrinsically productive and serve as signals of ability to colleges. We use a continuum model to approximate the outcomes of a game with large, but finite, sets of colleges and students. First, we show that (legal) admissions preference schemes and (illegal) quotas are, in fact, outcome equivalent. Second, we design affirmative action systems that maximize welfare, close the black-white test gap, and achieve fair outcomes.

Keywords: Affirmative Action; all-pay auctions; admission preferences; quotas; racial achievement gap; approximate equilibrium.

JEL subject classification: D44, C72, I20, I28, L53.

1 INTRODUCTION

Beginning with the Kennedy administration, the US government has mandated affirmative action (AA) practices in various areas of the economy where it has influence, including education, employment, and procurement. AA has been widely implemented outside the United States as well, in places such as Malaysia, Northern Ireland, India, and South Africa. Throughout this paper we focus on affirmative action with the American higher education system, but our techniques apply more broadly. The objectives of AA, as articulated by policy makers, include the intrinsic benefits of a diverse population within

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1For an in-depth discussion of AA implementations around the world, see Sowell [49].
a college or firm as well as the broader goal of counteracting competitive disadvantages for racial minorities due to past institutionalized discrimination. Despite its intentions however, much debate has arisen over the implications of AA for incentives, especially in college admissions. Supporters claim that AA motivates under-represented minority students to achieve at higher levels by placing within reach seats at top universities—an outcome previously seen by many as unattainable. Critics of AA make the opposite claim: by lowering admission standards AA creates incentives for minorities to exert less effort in preparation for college.

Modeling the effect of affirmative action on agent incentives and the resulting endogenous outcomes has proven extremely difficult. Our first contribution is a purely methodological one. We propose a contest-based model of matching and investment incentives under AA where there are two incentives to invest in human capital (HC). First, HC is an intrinsically productive asset, which results in a productive channel of investment incentives. Second, observable HC levels are a signal to potential match partners that the student has greater innate ability than his peers, which causes a competitive channel of investment incentives. Students’ costs are privately known but their race and HC output are commonly observable. Our model allows us to consider heterogeneity amongst the quality of colleges’ and students’ innate ability, include an endogenous human capital accumulation choice, and allow for imperfect information on the part of colleges admitting applicants. In short, our model allows us to address many of the most important welfare questions posed in the prior literature in a structure that we believe better represents the universe of incentives at play within an affirmative action regime.

The allocation of college seats is determined by a rank-order mechanism that maps each student’s HC choice into a college seat, possibly taking race into consideration as well. Although contests are typically very difficult to analyze, we prove that when the number of students and college seats is large the contest equilibria can be approximated using a relatively simple limit game where a continuum of students compete for a continuum of college seats. We use the equilibria of the limit game to approximate how affirmative action alters the incentives facing the agents and the effects of these distortions.

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2In the Supreme Court cases on affirmative action (e.g., U. of California Regents v. Bakke [46]) the colleges made the stronger argument that diverse classes is directly welfare enhancing for both the nonminority and minority students.
3Fryer and Loury [25] have used this “level playing field” argument as a possible rebuttal to their “Myth 3: Affirmative Action Undercuts Investment Incentives.”
4Typically, actual college markets involve enough competitors to make this a reasonable assumption. For example, the US National Center for Education Statistics reported that in 2005 over 1.8 million recent high-school graduates enrolled in college.
5The structure of our contest provides access to a well-developed set of market design tools for characterizing equilibrium behavior. For example, a standard existence theorem proven by Athey [1] implies that a monotone pure-strategy equilibrium exists.
on college assignments and human capital choices.

To understand why considering a large number of agents might simplify the model, consider a student that is applying to elite colleges. If the student wants to ascertain whether she is likely to be admitted to these schools, she does not need to consider the other students who might be applying or the application strategies they are employing. Instead, she simply consults a college guide that describes the qualifications of previously admitted students (e.g., GPA or SAT scores) and compares these statistics with her own application. If she meets these thresholds, then she can have confidence that she has a reasonable chance at admission. Although this example and our main analysis are focused on college admissions, the basic insight regarding the tractability and applicability of limit games as approximations to real-world contests generalizes far outside the context of college admissions and affirmative action.

Our second contribution addresses public policy and legal issues surrounding affirmative schemes in American colleges. We study three alternative canonical mechanisms: a color-blind mechanism, where no preferential treatment is given; a racial quota, which splits the competition into two separate contests by earmarking seats at each college for each race group; and an admission preference system, where minority HC output is marked up for comparisons with non-minorities. We show that the racial quota and admissions preference schemes are outcome equivalent. In other words, given any equilibrium of any racial quota (admissions preference system), one can design an admissions preference system (racial quota) with an equilibrium that results in the same HC choices and school assignments. From an analytical viewpoint, this result is useful because it lets us limit our analysis to racial quotas, which are easier to analyze since the equilibrium assignment is assortative within each group in an obvious way.

On a more practical level, the equivalence of racial quota and admission preference systems raises difficult questions regarding the legality of these schemes. Since the 1978 Supreme Court Case Regents of the University of California v. Bakke, constitutional jurisprudence has held that racial quotas violate the equal protection clause of the 14th amendment to the U.S. constitution since some students are prevented from competing for some of the seats. In contrast, admissions preference schemes may be acceptable under constitutional law since any student can compete for any seat, albeit the competition is not on a level playing field. However, our result calls into question the bright-line delineation of quotas and admissions preference systems developed by the 1978 opinion

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6Both of the authors are in favor of affirmative action schemes. Our argument is purely regarding the consistency (or lack thereof) of prior legal precedent, not an attempt to argue for (or against) the efficiency or ethics of affirmative action.

7Later cases, which we discuss in depth in section 5 have refined the standards that acceptable admissions preference schemes must meet.
of Justice Powell. The later 2003 judgements of Grutter v. Bollinger et al. and Gratz et al. v. Bollinger et al. turned on whether the University of Michigan had implemented an admissions preference scheme that resulted in a de facto racial quota, which highlights the legal importance of the perceived differences between quotas and admission preferences. To the extent the constitutionality of an affirmative action scheme hinges on the outcomes of these systems, then one might have to conclude that both racial quotas and admissions preferences ought to have the same legal status.

Our final major contribution is to use our limit games to analyze the design of affirmative action schemes. We calibrate our model using data from Hickman [31], which allows us to describe welfare in terms of the dollar value of the college premium each student receives. Using our result on mechanism equivalence, we describe the optimal affirmative action scheme as a racial quota designed to achieve a particular objective. In other words, designing affirmative action schemes can be reduced to solving an optimal control problem with the student incentives included as constraints. One of benchmarks we use for our designed contests is the first best outcome wherein an omniscient social planner assigns all of the students assortatively and the students choose the optimal level of HC to acquire ex post. In addition we use the outcome of a color-blind admissions policy and a representative quota that reserves a fraction of seats for minorities at each college that equals the fraction of minority students in the total pool of applicants.

Our first result is an analysis of the social surplus maximizing affirmative action scheme. One might have thought that the color-blind scheme would minimize distortions and maximize welfare since the color-blind outcome is necessarily assortative, but this viewpoint ignores the fact that the competitive channel incentivizes wasteful acquisition of human capital just to compete for a seat. To limit this waste, the social surplus maximizing affirmative action scheme allocates the worst college seats to the highest cost demographic group - in other words, the minority students are allocated to the worst colleges. The first best scheme generates a college premium of $14,622, the optimal affirmative action scheme a college premium of $13,724, and a color-blind scheme yields a college premium of $13,440.

While the optimal affirmative action scheme may be second best, it results in extremely unfair outcomes for the two demographic groups. This motivates us to consider a fairness criterion. Interestingly, it is possible to equalize the welfare between the two groups, but the average college premium is reduced to $13,351.

Finally, we ask to what extent the SAT test gap between minority and nonminority students can be closed solely through the use of the incentives provided by affirmative action. The test gap is the result of many factors including differing qualities of early childhood care, primary schools and differing availability of complementary inputs to
schooling (e.g., health care, nutrition). We do not believe that an affirmative action scheme is necessarily the most effective method for closing the test gap, but we believe it is useful to place an upper bound on the potential effectiveness of this policy lever. That all being said, our results indicate that the test gap could be reduced from 160 points as it stands today to around 75 points and yield an average social surplus of $13,240.

The remainder of this paper has the following structure: in section 2, we briefly discuss the relation between this work and the previous literature on AA. In section 3, we give an overview of the full model of competitive human capital investment and describe the college assignment contest we study. In section 4, we introduce the limit model with a continuum of students and prove that equilibria of the limit model are approximate equilibria of the finite model of section 3. In section 5, we prove that quota and admissions preference schemes admit the same set of equilibrium outcomes and discuss the practical ramifications of this result. In section 6, we turn to the design of affirmative action schemes and use a calibrated version of the model to describe the welfare maximizing, test gap minimizing, and fairness maximizing affirmative action regimes.

2 PREVIOUS LITERATURE

Since our paper is largely theoretical in nature, this literature review focuses on the theory side of the affirmative action literature. There is also rich empirical literature on the effects of affirmative action programs, and we encourage the interested reader to see the summary provided in Hickman [31].

Previous economic theory has studied AA and effort incentives, but existing models either exhibit important limitations or do not facilitate empirical analysis and the development of data-driven counterfactuals. Fain [18] and Fu [26] study models in the spirit of all-pay contests with complete information (i.e., academic ability types are ex ante observable) where two students compete for a college seat. Extrapolating the two-player insights to real-world settings is difficult because the framework implicitly assumes that all minority students, even the most gifted ones, are at a disadvantage to even the least talented non-minorities. More realistically, both talented and challenged students exist among all groups, making considerations of performance incentives more complex. Franke [19] extends the contest idea to include more than 2 agents, but at the cost of focusing on a specific form of affirmative action program.

The bilateral matching literature has also touched on incentives under AA, an early example being Coate and Loury [15], which studies both human capital investment and

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8Schotter and Weigelt [48] study a similar setting in the laboratory. Their results suggest that no equity-achievement tradeoff exists.
achievement gaps. This paper considers two strategic groups of agents, firms and job applicants. Job applicants make a binary choice to either become qualified at some cost or remain unqualified given a privately known cost of becoming qualified. Firms then observe a noisy signal of the potential employee’s choice and decides to assign the applicant to one of two positions. Coate and Loury [14] provides mild conditions under which discriminatory equilibria exist, analyzes the complex effects of affirmative action on equilibrium outcomes, and demonstrates that the equilibrium beliefs about the typical qualifications of minority applications can be worsened under an affirmative action program.

There are a number of fundamental distinctions between bilateral matching models and a contest framework. Most obviously, the contest framework takes the “prizes” (e.g., jobs, school placements) and how these prizes are assigned as exogenously given. In effect, the colleges in our model are nonstrategic actors. The benefit of the contest approach is that it is easier to incorporate heterogeneity on the part of the applicants (e.g., job seekers, students), allow the applicants to employ a richer action space, and allows us to incorporate heterogeneity in the prizes awarded to the agents (e.g., allow some firms/schools to be more desirable than others). We believe that these features bring us closer to empirical data and real-world counterfactuals, but we acknowledge that firm and college decision-making is an interesting and important component of these markets.

Chan and Eyster [11] focuses on the effect of affirmative action bans on a single school when admission can be conditioned on student traits correlated with race. Epple et al. [17] analyzes a similar question, but considers a set of vertically differentiated colleges. Both papers describe how colleges bias their admissions policy to encourage diversity. These papers assume student quality is fixed and exogenous, and so necessarily can’t say much about the general equilibrium effect of the admission policy changes on student incentives. Fryer et al. [23] partially addresses student incentives by including a binary effort choice in the spirit of Coate and Loury [14], but Fryer et al. [23] simplifies the setting by assuming colleges are homogenous.

Chade, Lewis, and Smith [10] studies a matching model of college admissions with heterogeneous colleges. However, academic achievement is exogenous, and the analysis focuses on the role of information frictions within the market (e.g., such as when SAT scores are a noisy signal of student ability), as well as the strategic behavior of colleges in setting admissions standards. Our framework is a frictionless matching market, but academic achievement is endogenous. In that sense, our work and Chade, Lewis, and Smith [10] may be considered complementary for understanding the role of broad market forces in college admissions.

Although many of these papers make similar points, our conclusions regarding stu-
dent welfare, fairness, and closing the black-white test gap would be impossible without incorporating (1) heterogenous colleges, (2) heterogenous and innate student quality, and (3) endogenous human capital accumulation choices by the students. Most of the papers above include flexible specifications for 1 or 2 of these components, but no prior paper includes rich models of all 3.

Fryer [21] touches on our analysis of the equivalence of quoate and admissions preference schemes. Fryer [21] studies a model of workplace affirmative action wherein firms that wish to maximize profit are subject to an affirmative action mandate imposed by the government and enforced by an auditor. Fryer [21] finds that firms facing a pool of applicants with few minorities will act as if they are subject to a quota on the number of minorities they must hire, which Fryer [21] argues implies an equilibrium equivalence between quota and hiring preference systems. In our setting the students are the strategic actors, and our equivalence result is in some ways stronger - not only is the racial balance at colleges in the two systems the same, but the endogenous quality of the students is identical. However, the fundamental public policy points are very similar.

Finally, our methodology analyzes approximate equilibria played by a large number of agents, which has been a prominent theme in the industrial organization and microeconomic theory literature. Due to the broad scope of this literature, we provide only a brief survey and a sample of the important papers related to the topic. Early papers focused on conditions under which underlying game-theoretic models could be used as strategic microfoundations for general equilibrium models (e.g., Hildenbrand [32] and [33], Roberts and Postlewaite [45], Otani and Sicilian [44], and Jackson and Manelli [36]). Other early papers focused on conditions under which generic games played by a finite number of agent approach some limit game played by a continuum of agents (e.g., Green [28] and [29], Sabourian [47]). A more recent branch of this literature applies these ideas to simplify the analysis of large markets and mechanisms with an eye to real-world applications (e.g., Swinkels [50], Cripps and Swinkels [16], McLean and Postlewaite [41], Budish [6], Kojima and Pathak [40], Fudenberg, Levine and Pesendorfer [24], Weintraub, Benkard and Van Roy [51], Krishnamurthy, Johari, and Sundarajan [38], and Azevedo and Leshno [5]). Many of the approximation results we use are based on Bodoh-Creed [21].

Of these papers on approximate equilibria and large games, we would like to single out the contemporaneously developed Olszewski and Siegel [43] as particularly relevant. Both this paper and Olszewski and Siegel [43] use the equilibria of contests played by a
continuum of agents as an approximation of a contest played by a large, but finite, set of contenders. Although there are a large number of differences in the models used (e.g., the agent utility functions), the largest differences between our works concern the informational assumptions and the applications. This paper focuses on an incomplete information setting and focuses our analysis on the difference (or lack thereof) between various affirmative action schemes and how to design optimal affirmative action programs. Olszewski and Siegel [43] analyze a complete information setting and use their framework to design effort-maximizing prize structures.

3 The Finite Model

This section describes a general contest model wherein agents make a costly investment that plays the dual role of acting as a productive input and determining one’s prospects in the contest. The model described below can apply to various market settings characterized by matching with non-transferrable utility where workers and firms assortatively match and worker quality is determined by endogenous HC. In the context of the college admissions application we have in mind, the agents are students that are competing for college seats by exerting effort to accrue human capital prior to the college admissions process.

We model the market as a Bayesian game where high school students are characterized by an observable demographic classification—minority or non-minority—and a privately-known cost type that governs HC production. Students compete for the right to occupy seats at colleges of differing quality and the seats are allocated to students according to a pre-specified rank-order mechanism which is a function of measured HC and demographic classification. Agents can observe the set of potential match partners (colleges) and the form of the sorting mechanism before making decisions, but they must incur a non-recoverable cost to invest in HC before entering the market. A student’s ex-post payoff is the utility derived from placing at a given college with her acquired human capital, minus her investment cost.

In this section we lay out the model when the number of students and colleges are finite. Although we spend most of the paper working with a limit approximation of this model, the finite model and the limit approximation share the same underlying primitives described in this section. A secondary goal is to highlight the difficulty of working with the finite model directly, which highlights the power of our limit approximation.
3.1 Agents, Actions, and Payoffs

The set of all agents (high-school students) is denoted \( K = \{1, 2, \ldots, K\} \), but there are two demographic subgroups, \( M = \{1, 2, \ldots, K_M\} \) (minorities) and \( N = \{1, 2, \ldots, K_N\} \) (non-minorities), where \( K_M + K_N = K \) and demographic class is observable. Each agent has a privately-known cost type \( \theta \in [\underline{\theta}, \overline{\theta}] \subset \mathbb{R}^+ \), and each student views his or her opponents’ unobservable types as independent random variables \( \Theta \) whose realizations follow group-specific distributions, or \( \Theta \sim F_i(\theta), i = M, N \). For convenience, I denote the unconditional type distribution for the overall population by \( F_K(\theta) \). The reader should assume throughout that high values of \( \theta \) are associated with students that have a high cost of accruing human capital.

There is a set of fixed match partners (colleges) \( P_K = \{p_1, p_2, \ldots, p_K\} \), where \( p_k \in [\underline{p}, \overline{p}] \subseteq \mathbb{R}^+ \) denotes the quality level of the \( k \)-th college, and \( p_k \neq p_l, \forall k \neq l \). It shall be useful in what follows to denote the \( k \)-th order statistic by \( p(k : K) \) so that \( \min_k \{P_K\} = p(1 : K) \) and \( \max_k \{P_K\} = p(K : K) \). The colleges are passive in our model, and each college accepts the students assigned at that college through the admissions contest.

Each agent’s strategy space, \( S = [s, \infty) \subset \mathbb{R}^+ \) is the set of feasible HC levels that can be attained. These are assumed to be observable, say through a standardized examination such as the ACT or SAT. Human capital \( s \) is the minimum level required to participate in the market; in the current context, this would be a minimum literacy threshold required to attend college. In skilled labor markets \( s \) might represent a college degree, and investment above \( s \) could be interpreted as points above the minimum passing college GPA.

Agents value both college quality and HC. The gross match utility derived from being placed at college \( k \) for a student with type \( \theta \) and HC \( s \) is given by \( U(p_k, s, \theta) \). However, in order to acquire HC \( s \in S \), an agent must incur a cost given by \( C(s; \theta) \), which depends on both her unobservable type and her HC investment level. The total utility for a student of type \( \theta \) that choose human capital level \( s \) and is assigned to college \( k \) is

\[
U(p_k, s, \theta) - C(s; \theta)
\]

\[\text{The assumption that there are two groups is not restrictive, as results for the two-group case extend straightforwardly (but tediously) to the case with } L > 2 \text{ groups.}\]

\[\text{Costs can arise in various ways: they could come from a labor-leisure tradeoff, where } \theta \text{ indexes one’s preference for leisure; they could represent psychic costs, indexed by } \theta \text{, involved in exerting effort to learn new concepts, where the more able students learn with the least effort; or costs could be interpreted as monetary investment required for study aides (e.g., computers, tutors, and private education), where } \theta \text{ represents the severity of the budget constraint that determines one’s consumption–investment tradeoff.}\]
3.2 Allocation Mechanisms

We now describe the contest that allocates students to colleges. Letting $s_i$ denote student $i$’s human capital level and $s_{-i}$ the vector of all other players’ actions, we let $P_j(s_i, s_{-i}), j = \mathcal{M}, \mathcal{N}$, be an allocation function that describes the college to which student $i$ is assigned. Note that we have deliberately allowed the allocation function to depend on student $i$’s demographic classification to reflect potential discrimination between members of the two groups. With this function in mind, the ex post payoff for student $i$ is then

$$\Pi(s_i, s_{-i}; \theta) = U[P_j(s_i, s_{-i}), s_i, \theta_i] - C(s_i; \theta_i),$$

The baseline color-blind admission rule (involving no diversity preference) is represented by the following allocation mechanism which yields positive assortative matching:

$$P^\text{cb}_\mathcal{M}(s_i, s_{-i}) = P^\text{cb}_\mathcal{N}(s_i, s_{-i}) = P^\text{cb}(s_i, s_{-i}) = \sum_{k=1}^{K} p(k : K) \mathbb{1}[s_i = s(k : K)]. \quad (1)$$

In the above expression, $\mathbb{1}$ is an indicator function equaling 1 if its argument is true and 0 otherwise.

We concentrate on two canonical forms of AA that have received attention due to wide implementation: quotas and admission preferences. A race quota is the practice of earmarking seats for each demographic group. Within the current modeling environment, this is equivalent to a set of $K_M$ seats being reserved for minorities and then allowing an assortative match within each group. A representative quota is one which the minority and nonminority students are allocated similar pools of seats. In general, race quotas can earmark very different college seats to the different groups of agents.

Let $P_\mathcal{M} = \{p_{M1}, p_{M2}, \ldots, p_{MK_M}\}$ and $P_\mathcal{N} = \{p_{N1}, p_{N2}, \ldots, p_{NK_N}\}$ denote the sets of seats earmarked for minorities and non-minorities, respectively, and let $s_\mathcal{M} = \{s_{M1}, s_{M2}, \ldots, s_{MK_M}\}$ and $s_\mathcal{N} = \{s_{N1}, s_{N2}, \ldots, s_{NK_N}\}$ denote the group-specific human capital profiles. Then a

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12 One could also envision more exotic forms of quota systems in which seats are earmarked to serve part of the minority mass. For simplicity, the current discussion of quotas shall refer only to a full quota rule in which, for example, a 20% minority population mass implies 20% of seats are reserved at each college. This follows the spirit of Indian reservations laws.

13 In the limit model with a continuum of college seats, a representative quota allocates an identical distribution of seats to each group.
quota allocation mechanism is represented by the functions

\[
P^q_M(s_i, s_{-i}) = \sum_{m=1}^{K_M} p_M(m : K_M) \mathbb{1}[s_i = s_M(m : K_M)], \text{ and}
\]

\[
P^q_N(s_i, s_{-i}) = \sum_{n=1}^{K_N} p_N(m : K_M) \mathbb{1}[s_i = s_N(n : K_N)].
\]

(2)

The defining characteristic of a quota is a split of the competition into two separate games where students compete only within their own race group.

An admission preference allows the members of each group to compete against each other, but the human capital choices of the members of each group are treated differently. In practice, many American admissions committees are thought to treat observed minority applicants’ SAT scores (a commonly used measure of HC) as if they were actually higher when evaluating them against their non-minority competitors. More formally, an admission preference is a markup function \(\tilde{S} : S \rightarrow \mathbb{R}_+\) through which minority output is passed to produce a set of transformed HC levels, \(\tilde{s} = \{s_{N1}, \ldots, s_{NK_N}, \tilde{s}(s_{M1}), \ldots, \tilde{s}(s_{MK_M})\}\), and allocations are given by the following group-specific functions:

\[
P^q_M(s_i, s_{-i}) = \sum_{k=1}^{K} p(k : K) \mathbb{1}[\tilde{S}(s_i) = \tilde{s}(k : K)], \text{ and}
\]

\[
P^q_N(s_i, s_{-i}) = \sum_{k=1}^{K} p(k : K) \mathbb{1}[s_i = \tilde{s}(k : K)].
\]

(3)

Regardless of whether admissions are color-blind or follow some form of AA, ties between competitors are assumed to be broken randomly.

Before investing time and resources into HC production, agents observe the set of seats \(R_K\), the admission rule \(P_j\), \(r \in \{cb, q, ap\}\) and \(j \in \{M, N\}\), and the number of competitors from each group \(K_M\) and \(K_N\). Under the payoff mapping \(\Pi(\theta, s_i, s_{-i})\) induced by a particular admission rule, students optimally choose their achievement level based on their type and the types of potential match partners, taking into account opponents’ optimal behavior. The model defined above fits the mold of a contest, which can be thought of as an asymmetric, multi-object, all-pay auction with single-unit bidder demands and bid preferences.

\[14\]There has been a fair amount of empirical research estimating a substantial average admission preference for minorities at elite American colleges; e.g., Chung, Espenshade and Walling [13], and Chung and Espenshade [12]. Hickman [31] employs a similar empirical measure for the aggregate US college market using the theory developed here, and finds evidence of a substantial admission preference even at lower-ranked colleges.
3.3 Model Assumptions

We now outline a series of assumptions on our model primitives. This series of assumptions serves three goals:

1. Establish the existence of a monotone equilibrium.
2. Justify the use of a centralized, assortative matching structure.
3. Insure the model is sufficiently well-behaved that our limit approximation is valid.

Although we highlight how the assumptions tie into goals (1) and (2), we defer discussion of (3) until the next section.

**Assumption 1.** For each \((\theta, s) \in [\theta, \overline{\theta}] \times \mathbb{R}_+\), we have \(C_s(s, \theta) > 0\), \(C_\theta(s, \theta) > 0\), and \(C_{ss}(s, \theta) > 0\)

**Assumption 2.** \(U_p(p, s, \theta) > 0\), \(U_s(p, s, \theta) \geq 0\), \(U_\theta(p, s, \theta) \leq 0\), and \(U_{ss}(p, s, \theta) \leq 0\)

Assumptions 1 and 2 are regularity conditions on utility so that individuals’ decision problems will have easily defined global maximizers. In words, 1 states that costs are strictly increasing in human capital level \(s\) and cost type \(\theta\). Moreover, the cost function is (weakly) convex in \(s\) for any given \(\theta\). Assumption 2 states that utility is differentiable, strictly increasing in college quality \(p\), weakly increasing and concave in human capital level \(s\) and weakly decreasing in cost type \(\theta\).

**Assumption 3.** \(U(p, s, \theta) - C(s; \theta) = 0\) and \(\arg \max_s U(p, s, \theta) - C(s; \theta) \leq s\)

Assumption 3 is a boundary condition for our model. First, the assumption requires that the lowest type of student is indifferent between participating in the college admissions contest and not attending college, which normalizes the utility of not attending college to 0. This has the benefit of allowing us to interpret \(U(p, s, \theta) - C(s; \theta)\) as the equilibrium college premium of the students. Second, the assumption demands that the minimally qualified student who chooses to attend college does not have an interest in acquiring more HC. While this is not usually assumed, we find that the assumption is supported in the data we use to calibrate our model in section 6.

**Assumption 4.** There exists \(s\) such that \(U(p, s, \theta) - C(s; \theta) < U(p, s, \theta) - C(p; \theta)\)

Assumption 4 requires that there exist a human capital level so large that even the lowest type, the type that gets the largest value from human capital and has the lowest cost for acquiring human capital, would rather not invest in human capital at all and be assigned to the worst school. This implies that we can limit our analysis to human capital levels within \(s \in [s, \overline{s}]\). Unless otherwise stated, we simply let \(S = [s, \overline{s}]\).
Assumption 5. $U_{ps}(p, s, \theta) \geq 0$ and $U_{p\theta}(p, s, \theta) \leq 0$

Assumption 5 requires (weak) positive complementarity between student HC and college quality and that student and college qualities are substitutes. This assumption is not strictly necessary for any technical reason and will not play a direct role in the analysis to follow. Rather, the assumption implies that the decentralized matching equilibrium is characterized by positive assortative matching, which justifies the use of the centralized rank-order mechanisms we have defined.

Assumption 6. $U_{s\theta}(p, s, \theta) \leq 0$ and $C_{s\theta}(s, \theta) > 0$

Assumption 7. $F_M(\theta)$ and $F_N(\theta)$ have continuous and strictly positive densities $f_M(\theta)$ and $f_N(\theta)$, on a common support $[\theta, \overline{\theta}]$.

Assumption 8. $F_P(p)$ has a continuous and strictly positive density $f_P(p)$ on support $[\overline{p}, \overline{p}]$.

Assumptions 6 is key for existence of a monotone pure-strategy equilibrium. Assumption 7 states that marginal benefits of HC are decreasing and marginal costs of HC are increasing in a student’s type. Assumptions 7 and 8 are standard regularity conditions on the type distributionst that are supported by the calibration data.

Assumption 9. $\tilde{S}(s)$ is strictly increasing and differentiable almost everywhere.

Most importantly, assumption 9 assumes that the markup function $\tilde{S}$ is a strictly increasing function, meaning that it does not rearrange rank orderings within demographic groups. In other words, holding race fixed, admissions officers always prefer to enroll a student with more HC.

We focus on (group-wise) symmetric equilibria of both the finite game and the limit game. An equilibrium of the game $\Gamma(K_M, K_N, F_M, F_N, P_K, r, U, C, S)$ is a set of achievement functions $\sigma_j^r : [\theta, \overline{\theta}] \rightarrow R_+$, $j = M, N$ which generate optimal choices of human capital $s = \sigma_j^r(\theta)$ given that all other agents follow the equilibrium strategy. It will be convenient at times to denote the inverse equilibrium achievement functions by $\psi_j^r(s) \equiv \left(\sigma_j^r\right)^{-1}(s) = \theta$.

4 Limit Games and Approximate Equilibria

In this section, we demonstrate existence, uniqueness, and monotonicity of the Bayes-Nash equilibrium of the game defined above. We then discuss some technical difficulties involved in characterizing the equilibrium and prove the main methodological result of the paper, that when $K$ is large the equilibrium can be well approximated by considering an agent’s limiting decision problem as the number of players and college seats approaches infinity.
4.1 Equilibrium Approximation

For large $K$, the equilibrium is analytically and computationally difficult because an agent’s decision problem is a complicated function of the order statistics of opponents’ cost types. To illustrate this point, consider the color-blind case from the perspective of player 1 in the minority group. In equilibrium, her expected utility as a function of her cost type and human capital choice is:

$$\Pi_j(s; \theta) = \sum_{k=1}^{K} U(p(k : K), s, \theta) \times \sum_{k_M \leq \min \{k_M, k_N\}, k_N = k - 1 - k_M} \left( \binom{K_M - 1}{k_M - 1} F_M \left( \sigma^{-1}_M[s] \right)^{K_M - k_M} \left[ 1 - F_M \left( \sigma^{-1}_M[s] \right) \right]^{k_M - 1} \right) \times \left( \binom{K_N}{k_N} F_N \left( \sigma^{-1}_N[s] \right)^{K_N - k_N} \left[ 1 - F_N \left( \sigma^{-1}_N[s] \right) \right]^{k_N - 1} \right) - C(s; \theta).$$

Agents know that their ex-post payoff depends on their rank within the HC distribution, and by monotonicity this is the same as their rank within the realized cohort of opponents. Thus, the above expression for student 1’s expected payoff is a weighted average of all match utilities given some $s$ and her fixed type $\theta$, where the weight on the $k$th least desirable seat is her probability of being the $k$th lowest order statistic among $K$ competitors. The expressions involving $F_M$ and $F_N$ give the respective probabilities of this happening, and the second summation operator is to account for the fact that there are potentially many $(k_M, k_N)$ pairs satisfying the conditions which would make player 1 the $k$th order statistic overall.

Intuitions based on the law of large numbers would suggest that as the market become large (i.e., as $K \to \infty$), the distribution of realized types and human capital choices ought to approach some limit measure. If this intuition is true, then the mapping between human capital choices and school assignment in this limit game, which we call the limit assignment mapping, ought to provide a good description of the outcomes in games with a sufficiently large, but finite, set of players. This would suggest that an individual agent could come very close to maximizing his or her utility in a large finite game by optimizing against the limit assignment mapping.

To see how this simplifies the student’s problem (and as a result, our analysis), consider the plight of a college applicant in the United States. Do would-be college students go to elaborate lengths to determine who else is applying, where those students are applying, and what the other students’ qualifications are? Of course not - to determine whether an application is likely to be accepted, the would-be college student can simply
look at data on the grades and SAT scores of previous entering classes. In our model, the SAT score of previously accepted classes is akin to knowing the equilibrium mapping between human capital levels and school assignments. Conveniently, metrics such as SAT scores of incoming freshman are almost always stable from year to year.

There are two technical issues that must be resolved before this insight can be formally established. First, we must describe the sense in which the limit game is tied to a sequence of games with increasingly many players. Denote the sequence of finite games as \( \{ \Gamma(K_M, F_M, K_N, F_N, \mathcal{P}_K, r, U, C, S) \}_{K_M + K_N = 2}^{\infty} \). Assume that \( \frac{K_M}{K} \to \mu \in (0, 1) \) as \( K \to \infty \), which implies that \( \mu \) is the asymptotic mass of the minority group. The utility function of the students is the same in the finite and the limit game. Finally, college seats are modeled as being generated by nature as independent draws from a compact set \([p, \overline{p}] \subset \mathbb{R}^+\) according to a distribution \( F_P(p) \).

In the limit game, there is a measure \( \mu \) continuum of minority students with types distributed exactly as \( F_M \). There is also a measure \( 1 - \mu \) continuum of nonminority students with types distributed exactly as \( F_N \). Finally, there is a measure 1 continuum of college seats distributed exactly as \( F_P \). The students are assigned to schools as in the finite game. For example, a student in the color-blind admissions game is assigned to a school seat with the same quantile as the student’s chosen human capital level. We discuss the limit assignment mappings in detail in section 4.2.

The second technical issue to resolve is whether or not the limit model is continuous. Our intuition that consulting the limit assignment mapping is a useful tool for optimizing in the finite model assumes that a human capital choice in the limit model yields almost the same outcome as it would in a finite game with many player. It is not obvious that this will be the case since contest models are intrinsically rife with discontinuities. In addition, discontinuities could be generated endogenously through the equilibrium strategy. The hardest part of the task before us is proving that neither of these kinds of discontinuities can occur in equilibrium.

Since the distributions of schools and student types are commonly known in the limit game (i.e., the limit game is one of complete information), we can describe the limit assignment mapping as \( P : S \to \mathcal{P} \), but we remind the reader that the realized assignment function is endogenous with respect to the equilibrium strategy, \( \sigma \). On that note, the Nash equilibria of the complete-information limit game are defined as follows:

\[ \text{15 Although we assume throughout that the convergence is deterministic, one can think of the agents in the finite games as being created by nature by assigning each agent to group } M \text{ with probability } \mu \in (0, 1), \text{ after which she draws a cost type from the corresponding distribution.} \]
**Definition 1.** \( \sigma \) is a Nash equilibrium of the limit game if

\[
\sigma(\theta) \in \arg\max_s U[P(s), s, \theta] - C(s; \theta)
\]

Now we define the notions of an approximate equilibrium that we will be using throughout this paper.

**Definition 2.** Given \( \varepsilon > 0 \), an \( \varepsilon \)-approximate equilibrium is a \( K \)-tuple \( s^\varepsilon = (s_1^\varepsilon, \ldots, s_K^\varepsilon) \) such that for all agents \( i \) and human capital choices \( s' \) we have

\[
U[P(s_i), s_i, \theta_i] - C(s_i; \theta_i) + \varepsilon \geq U[P(s'_i), s'_i, \theta_i] - C(s'_i; \theta_i)
\]

**Definition 3.** Given \( \delta > 0 \), a \( \delta \)-approximate equilibrium is a \( K \)-tuple \( s^\delta = (s_1^\delta, \ldots, s_K^\delta) \), such that there exists an equilibrium \( s^* = (s_1^*, \ldots, s_K^*) \), where \( \|s^\delta - s^*\|_{sup} < \delta \).

From this point on, discussion will focus on the approximate equilibrium, so I shall drop the \( \infty \) superscript for notational ease. Moreover, to avoid tedious verbosity, I shall now refer to the approximate equilibrium functions simply as “the equilibrium,” unless the context requires more specificity. In such cases I shall refer to the equilibrium of a finite game as a “finite equilibrium” and I shall denote related functions with \( K \) listed as a parameter. Thus, \( \Pi^r_j(s; \theta, K) \) will denote the equilibrium payoff function for a \( K \) student game, \( \sigma^r_j(\theta; K) \) will denote its equilibrium, and \( \psi^r_j(s; K) \) will denote the inverse equilibrium. In contrast, \( \Pi^r_j(s; \theta) \) denotes the unique limiting payoff function, \( \sigma^r_j(\theta) \) denotes its maximizer, and \( \psi^r_j(s) \) denotes the inverse maximizer.

In order to show our approximation results are not vacuous, before proceeding we provide a result showing that equilibria of the finite game exist. In section 4.2 we prove that the limit game admits a unique symmetric pure-strategy equilibrium using the differential equations used to describe the limit game equilibrium.

**Theorem 1.** In the college admissions game \( \Gamma(K_M, F_M, K_N, F_N, P_K, r, U, C, S) \) with \( r \in \{cb, q, ap\} \), under assumptions 7-9 there exists a symmetric pure-strategy equilibrium \( (\sigma^r_M(\theta), \sigma^r_N(\theta)) \).

### 4.2 Approximate Equilibria in College Admissions

In this section we illustrate how the equilibria can be described using differential equations derived from the first order conditions faced by the agents. In addition to being easy to compute, the differential equation approach also implies the existence and uniqueness of the symmetric equilibria of the limit game.

---

\(^\text{16}\)Admittedly, indexing finite games only by the total number of players (rather than by \((K_M, K_N)\) pairs) is an abuse of notation, but for large \( K \) the abuse is lessened, as the set of likely possibilities eventually collapses to a singleton.
First, consider a color-blind admissions program. Recall that a color-blind allocation rule means simple positive assortative matching of seats and human capital levels. We denote the endogenous distribution of human capital levels chosen by the students as $G_{cb}^j(s), j = M, N$ and $r \in \{cb, q, ap\}$. Using this notation, we can describe the match of students to colleges as:

$$P_{cb}^M(s) = P_{cb}^N(s) = P_{cb}(s) = F_P^{-1} \left[ G_{cb}^M(s) \right]$$

$$= F_P^{-1} \left[ \mu G_{cb}^M(s) + (1 - \mu) G_{cb}^N(s) \right]$$

$$= F_P^{-1} \left[ 1 - \left( \mu F_M \left[ \psi_{cb}(s) \right] - (1 - \mu) F_N \left[ \psi_{cb}(s) \right] \right) \right].$$

The intuition is simple: quantiles of the population HC distribution $G_{cb}^j(s)$ are mapped into the corresponding quantiles of the distribution $F_P$. Since limiting payoffs do not depend on race, it follows that $\sigma_{cb}^M(\theta) = \sigma_{cb}^N(\theta) = \sigma_{cb}(\theta)$; hence, the lack of subscripts on the inverse strategies in the third line.

The limiting payoff for agent type $\theta$ with HC $s$ is $\Pi_{cb}^L(s; \theta) = U \left[ P_{cb}(s), s, \theta \right] - C(s; \theta)$. Differentiating, we get the following first-order condition (henceforth, FOC):

$$\frac{\partial U}{\partial s} + \frac{\partial U}{\partial P_{cb}(s)} \cdot \frac{dP_{cb}(s)}{ds} = \frac{dC}{ds}$$

The above expression concisely organizes the different aspects of the investment trade-off being made by students. It states that the total cost of human capital investment (the right-hand side) must be exactly offset by the total benefits (the left-hand side), which can in turn be decomposed into two parts. First, there is the direct value accrued to the student of having human capital level $s$, represented by the term $\partial U/\partial s$; this is the productive channel of investment incentives. There is also the indirect benefit of choosing $s$, which comes from the resulting match with college seat $p$. This indirect benefit is itself decomposed into two parts. An increase in one’s human capital level will improve the quality of one’s match partner by $dP_{cb}(s)/ds$, which increments utility by the margin $\partial U/\partial P_{cb}(s)$; the product of these two terms represents the competitive channel of investment incentives.

Theorists with experience in asymmetric auctions may find this statement puzzling, but one must keep in mind that it merely applies to limiting payoffs. In a two-player finite game, differing investment behavior arises from the fact that a minority competitor faces a profile of opponents of each type numbering $(K_N, K_M - 1)$, whereas a non-minority competitor faces profile $(K_N - 1, K_M)$, and since costs are asymmetrically distributed across groups, a minority and a non-minority with the same private cost will have differing expectations of their standing in the distribution of realized competition. However, the difference between their expected ranks quickly vanishes as the number of players gets large.
For the moment, suppose types were observable to a benevolent planner that could match students to colleges assortatively and allow them to invest *ex post*, in which case the FOCs would be  

\[ U_2 \left( P_{i}^{\ast} (\theta), s, \theta \right) = C'(s; \theta). \]

The additional investment resulting from the competitive channel of incentives is effectively the price that society pays in order to resolve principal-agent asymmetry of information, and produce socially beneficial assortative matching. We refer to the outcome generated by the social planner’s omniscient assignment and the ex post investments of the students as the *First Best Outcome*. For concreteness, the following theorem compares the color-blind policy and the first-best outcome, but similar results could be generated for any of our admission schemes.

**Theorem 2.** *HC investment for all types in the color-blind admissions scheme exceeds the full-information outcome where the social planner observes costs, assortatively matches seats to \( \theta \)'s, and investment happens *ex-post*.*

The basic intuition for the result is clear: by shutting down the positive channel, the first-best outcome reduces the incentive for students to acquire human capital. The net effect is to reduce the human capital obtained for every type of student. Our first goal is simply to point out the fact that when human capital is used as the basis of assignment to schools as well an intrinsically valuable investment, students will choose to invest more in human capital. The welfare effects of this are ambiguous. If the higher human capital levels do not produce any externalities, then the competitive channel is merely wasteful signalling.

Explicitly differentiating results in the following differential equation for investment:

\[
\frac{d\sigma^{cb}(\theta)}{d\theta} = -\frac{U_1 \left[ P^{cb}[\sigma^{cb}(\theta)], \sigma^{cb}(\theta), \theta \right] \cdot f_k(\theta)}{f_p \left( F_p^{-1} \left[ 1 - F_k(\theta) \right] \right) \cdot (C'[\sigma^{cb}(\theta); \theta] - U_2 \left[ P^{cb}[\sigma^{cb}(\theta)], \sigma^{cb}(\theta), \theta \right])},
\]

\[ \sigma^{cb}(\theta) = s \quad \text{(boundary condition)}. \]

The boundary condition comes from the fact that a player of type \( \theta \) will always be matched with the lowest seat in a monotone equilibrium, so she cannot do better than to simply choose HC level \( s \) to complement \( p \). Note that, given the assumptions on the model primitives, the initial value problem defined by (5) results in a strictly decreasing function.

I now depart from the baseline color-blind model, and derive the approximate equilibrium in the presence of race-conscious admission policies beginning with quotas. Recall that a quota system in the finite game can be thought of as selecting \( K_M \) seats and setting them aside for allocation to group-\( M \) agents. In the limit game we denote the cumulative distribution function describing the distribution of seats reserved for the minority
(nonminority) students as \( Q_M, Q_N \) where these measures are subject to the following feasibility constraint

\[
\text{For all } p \text{ we have } \mu Q_M(p) + (1 - \mu) Q_N(p) = F_P(p)
\]

In effect, the minority and nonminority students compete in distinct contests for different pools of seats. These distinct contests yield group-specific allocation functions of the form

\[
P^q_j(s) = Q^{-1}_j \left[ G^q_j(s) \right] = Q^{-1}_j \left( 1 - F_j \left[ \psi^q_j(s) \right] \right), \quad j \in \{M, N\}.
\]

As in the color-blind case, the quantiles of the group-specific human capital distributions are mapped into the corresponding quantiles of \( Q_j \). By taking the total derivative of the student’s equilibrium utility functions, we find the following differential equation describing the equilibrium when \( Q_j \) have full support.

\[
\frac{d\sigma^q_j(\theta)}{d\theta} = -\frac{U_1 \left( P^q_j[\sigma^q_j(\theta)], \sigma^q_j(\theta), \theta \right) \cdot f_j(\theta)}{f_P \left( F^{-1}_P \left[ 1 - F_j(\theta) \right] \right) \cdot \left( C'[\sigma^q_j(\theta); \theta] - U_2 \left( P^q_j[\sigma^q_j(\theta)], \sigma^q_j(\theta), \theta \right) \right)},
\]

\[
\sigma^q_j(\bar{\theta}) = \bar{s} \quad \text{(boundary condition)}.
\]

When \( Q_j \) does not have full support, then there will be jumps in the equilibrium strategies when \( P^q_j \left[ \sigma^q_j(\theta) \right] \) encounters the left edge of a gap in the support. The interested reader is referred to appendix C for a description of how to identify the location and size of the jumps. Between these jumps, equation 7 describes the equilibrium strategy.

In 1978, the Supreme Court ruled that explicit quotas are unconstitutional in the case of Regents of the University of California v. Bakke. Subsequently, college admissions boards have been forced to seek other means, dubbed admission preferences or the practice of giving minority test scores more credit than non-minority scores, by which to implement AA. As in the finite-agent model, an admission preference rule is modeled as a markup function \( \tilde{S} : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) and seats are matched assortatively with transformed HC, \( \tilde{s} = \{s_{n1}, \ldots, s_{nK_N}, \tilde{S}(s_{m1}), \ldots, \tilde{S}(s_{mK_M})\} \). In other words, minorities are repositioned ahead of non-minority counterparts with investment of \( \tilde{S}(s) \) or less. The limiting allocation function for group \( M \) is

\[
P^{\tilde{s}_M}(s) = F^{-1}_P \left[ (1 - \mu) G_N \left( \tilde{S}(s) \right) + \mu G_M(s) \right] = F^{-1}_P \left[ 1 - \left( 1 - \mu \right) F_N \left[ \psi^{ap}_N(\tilde{S}(s)) \right] + \mu F_M \left[ \psi^{ap}_M(s) \right] \right]
\]
and limiting allocations for group \( N \) are given by

\[
P_{N}^{\text{ap}}(s) = F_{P}^{-1} \left[ (1 - \mu)G_{N}(s) + \mu G_{M} \left( \tilde{S}^{-1}(s) \right) \right] \\
= F_{P}^{-1} \left[ 1 - \left( (1 - \mu)F_{N} \left[ \psi_{N}^{\text{ap}}(s) \right] + \mu F_{M} \left[ \psi_{M}^{\text{ap}}(\tilde{S}^{-1}(s)) \right] \right) \right]. \tag{9}
\]

The intuition for the above expressions is similar as before: limiting mechanisms map the quantiles of a distribution into the corresponding college seat quantiles. For non-minorities, it is a mixture of the distributions of non-minority HC and subsidized minority HC. For minorities, it is a mixture of the distributions of minority HC and \emph{de-subsidized} non-minority HC.

Unlike the color-blind or quota case, it is not possible to describe which kind of student obtains each seat in an admissions preference scheme without first solving for the equilibrium. This feature of the admissions preference mechanism makes it significantly harder to work with than the other two schemes. However, we prove in section 5 that the admissions preference and quota schemes are outcome equivalent - for any admissions preference markup function one can describe a quota scheme that generates the same school assignment and equilibrium human capital choices. Because of this equivalence, for the majority of the paper we work with color-blind and quota systems. However, for completeness, we now provide the differential equations describing the admissions preference equilibrium when \( \tilde{S} \) is differentiable and \( \tilde{S}(\bar{s}) = \bar{s} \).

\[
\left( \psi_{M}^{\text{ap}} \right)'(s) = - \frac{C_{s} \left[ s; \psi_{M}^{\text{ap}}(s) \right] - U_{2} \left[ P_{M}^{\text{ap}}(s), s, \psi_{M}^{\text{ap}}(s) \right]}{U_{1} \left[ P_{M}^{\text{ap}}(s), s, \psi_{M}^{\text{ap}}(s) \right]} \cdot \frac{f_{P} \left[ P_{M}^{\text{ap}}(s) \right]}{f_{M} \left[ \psi_{M}^{\text{ap}}(s) \right]} \\
- \frac{1 - \mu}{f_{M} \left[ \psi_{M}^{\text{ap}}(s) \right]} \left( \psi_{N}^{\text{ap}} \right)' \left( \tilde{S}(s) \right) \tilde{S}'(s)
\]

and

\[
\left( \psi_{N}^{\text{ap}} \right)'(s) = - \frac{C_{s} \left[ s; \psi_{N}^{\text{ap}}(s) \right] - U_{2} \left[ P_{N}^{\text{ap}}(s), s, \psi_{N}^{\text{ap}}(s) \right]}{U_{1} \left[ P_{N}^{\text{ap}}(s), s, \psi_{N}^{\text{ap}}(s) \right]} \cdot \frac{f_{P} \left[ P_{N}^{\text{ap}}(s) \right]}{U_{1} \left[ P_{N}^{\text{ap}}(s), s, \psi_{N}^{\text{ap}}(s) \right]} \\
- \frac{\mu f_{M} \left[ \psi_{M}^{\text{ap}}(\tilde{S}^{-1}(s)) \right]}{(1 - \mu) f_{N} \left[ \psi_{N}^{\text{ap}}(s) \right]} \left( \psi_{N}^{\text{ap}} \right)' \left( \tilde{S}^{-1}(s) \right) \frac{d\tilde{S}^{-1}(s)}{ds}.
\]

These equations are described in terms of the inverse strategy functions for notational ease. If \( \tilde{S} \) is not differentiable or \( \tilde{S}(\bar{s}) \neq \bar{s} \), then the equilibrium may involve jumps. The interested reader is (again) referred to appendix C for a description of how to identify the size and location of these jumps.

Now we use the differential equations describing the equilibria under each affirmative action scheme to prove the existence of a unique, pure-strategy Nash equilibrium. It is
easy to allow for regions where \( Q_i \) lacks support in the quota case because we can prove that the resulting jump over the region without support is uniquely defined. In other words, we can prove the equilibrium is unique for any quota scheme. Unfortunately, we do not yet have a direct result for the admissions preference case where \( \tilde{S} \) is nondifferentiable or \( \tilde{S}(\tilde{\xi}) = \tilde{\xi} \). Fortunately, since we work exclusively with quota systems in our later analysis, this result suffices.

**Theorem 3.** A unique Nash equilibrium exists in the limit model in the following cases

1. The limit game of the admissions preference model with a differential markup function \( \tilde{S} \) that satisfied \( \tilde{S}(\tilde{\xi}) = \tilde{\xi} \) and \( F_P \) has full support.

2. The limit game of the quota system with any feasible choice of \( Q_M \) and \( Q_M' \).

### 4.3 Validating the Approximate Equilibria

Our goal in this section is to prove that the equilibria of the limit game are \( \delta \)-approximate equilibria of admissions games with sufficiently many students. Proving this result amounts to proving that the limit game is continuous in the appropriate sense. With the result in hand, it is relatively easy to prove our approximation result.

We require the following assumption on the markup functions to prove our approximation result for the admissions preference system. The assumption bounds the marginal markup applied to minority student human capital choices, which helps insure that minority student utility functions are continuous in the limit game.

**Assumption 10.** There exists \( \lambda_1, \lambda_2 \in (0, \infty) \) such that for all \( s, s' \in S, s > s' \), we have

\[
\lambda_2 (s - s') > \tilde{S}(s) - \tilde{S}(s') > \lambda_1 (s - s')
\]

Note that although our theorem is stated for the case of the admissions preference game, it obviously applies to the color blind game since that is a special case of the admissions preference system. Moreover, the result (almost trivially) extends to the quota system since the arguments would apply to each of the contests separately.

**Theorem 4.** Consider the admissions preference game and assume one of the cases of theorem 3 holds. Under assumptions 1-8, assumption 10 and given \( \epsilon, \delta > 0 \), there exists \( K^* \in \mathbb{N} \) such that for any \( K \geq K^* \) we have the following:

(i) \( \sigma^j_i, i \in M, N \) and \( j \in \{cb, q, ap\} \), is an \( \epsilon \)-approximate equilibrium of the \( K \)-agent game.

(ii) \( \sigma^j_i, i \in M, N \) and \( j \in \{cb, q, ap\} \), is a \( \delta \)-approximate equilibrium for the \( K \)-player game.
Proving point (i) is in essence a claim about the continuity of the equilibrium payoffs of the game. When many students follow the equilibrium strategy of the limit game, then the realized distribution of human capital and college seats will (with high probability) be approximately the same as the distributions realized in the limit game. If the student utility functions are continuous, then these small differences have a negligible effect on the agent utility for each possible action (and so the maximum utility changes only slightly).

Proving point (ii) is more subtle. First, we show that the continuity of the student utility functions implies that the equilibrium correspondence is upper hemicontinuous. In other words, an exact equilibrium of the admissions game with many players must be close to some equilibrium of the limit game. However, it could be the case that there are equilibria of the limit game that are unlike any equilibrium of the finite game (i.e., lower hemicontinuity of the equilibrium correspondence might fail). To rule this out, we use the fact that the limit game has a unique equilibrium to prove that the equilibrium correspondence is in fact continuous, which is equivalent to the claim made by (ii).

5 Mechanism Equivalence

In this section we make our argument that the quota and admissions preferences systems are, in fact, equivalent to each other. In any equilibrium of any affirmative action scheme, the agents in the limit game repond optimally to the assignment mapping, \( P(s) \), that describes how HC choices lead to college assignments. We prove our equivalence result by showing that if an assignment mapping \( P(s) \) is generated by some equilibrium of an admission preference (quota) system, then there exists an equilibrium of some quota (admissions preference) system that yields the same \( P(s) \). Since the \( P(s) \) are the same under each system, the optimal agent responses must also be the same. Note that the notion of equivalence we use implies that not only are the same measures of minority and nonminority students assigned to each school, but the students at each school choose the same level of human capital under both systems.

Theorem 5. Consider some \( P_i(s) : \mathcal{S} \to \mathcal{P} \), \( i \in \mathcal{M}, \mathcal{N} \). \( P_i(s) : \mathcal{S} \to \mathcal{P} \), \( i \in \mathcal{M}, \mathcal{N} \) is the result of an equilibrium of some quota system if and only if there is an equilibrium of an admissions preference system that also yields these assignment functions and admits the same equilibrium.

Note that we have not proven that any choice of \( P_i(s) \) can be implemented by either a quota or an admissions preference scheme. For example, if \( P_i(s) \) is strictly decreasing, then it cannot be implemented by any incentive compatible mechanism.

We have only conjectured that admissions preference schemes with discontinuous equilibria admit a unique equilibrium. If this conjecture is false for some admission preference markup function \( \tilde{S} \), theorem 5 will be unaffected. However, it will be the case that each of the equilibria induced by \( \tilde{S} \) will be equivalent to different quota systems.
Recall that under a quota system the equilibrium assignment of student types to colleges within each group is assortative, which means the only unknown endogenous quantity is the HC accumulation strategy. Given an admission preferences system, both the equilibrium human capital choices and the school assignment need to be computed, which makes the admissions preference schemes more difficult to study. Theorem 5 is useful from a methodological perspective because it shows that there is no loss in generality from focusing solely on outcomes that can be realized using quota schemes.

This equivalence relies on the distribution of student types being fixed and known. If there are aggregate shocks to the distribution of student types, the equivalence will no longer hold unless the quota and admissions preference schemes are allowed to be functions of the realized distribution of applicant types. For example, under a quota scheme that is not responsive to the distribution of college students, a fixed number of minority student are enrolled even in the event that the minority student population is significantly better than expected. In contrast, an admissions preference scheme may allow, the additional high quality minority applicants to be enrolled. Large aggregate shocks in the distribution of undergraduate applicants seems unlikely, but it is easy to imagine aggregate shocks such as economic cycles that would affect the pool of applicants to other kinds of educational programs such as MBA programs.

From a legal perspective, theorem 5 throws light on why it has proven so difficult for the court to draw a line between constitutionally permissible and impermissible affirmative action systems. As we discuss in more depth below, the Supreme Court jurisprudence has clearly ruled that quotas violate the 14th amendment’s guarantee of equal protection because nonminority students cannot compete for the seats reserved for minority applicants. However, the Court has given guidelines under which admissions preference schemes are acceptable, although the members of the court have disagreed as to the extent to which admissions preference schemes are functionally different from quotas.

The cornerstone of supreme court jurisprudence regarding affirmative action is the 1978 case University of California Regents v. Bakke. Justice Powell’s opinion established that the government has a compelling interest in encouraging diversity in university admissions founded on principles of academic freedom and a university’s right to take what actions it feels necessary to provide a high quality education to its students. Given this compelling interest, universities are free to implement affirmative action programs, although these programs must be narrowly tailored and are subject to a rigorous “strict scrutiny” standard of review.

Legal challenges to affirmative action schemes (e.g., Gratz et al. v Bollinger, Grutter v. Bollinger) turn on whether the affirmative action schemes are narrowly
tailored, a term that refers to whether the affirmative action scheme places the smallest possible burden on disadvantaged groups. In an amicus curiae brief to the case of University of California Regents v. Bakke, Harvard College describes their admissions in the following terms, which Justice Powell used as a canonical example of a narrowly tailored affirmative action system:

"In Harvard College admissions, the Committee has not set target quotas for the number of blacks, or of musicians, football players, physicists or Californians to be admitted in a given year. But that awareness [of the necessity of including more than a token number of black students] does not mean that the Committee sets a minimum number of blacks or of people from west of the Mississippi who are to be admitted. It means only that, in choosing among thousands of applicants who are not only 'admissible' academically but have other strong qualities, the Committee, with a number of criteria in mind, pays some attention to distribution among many types and categories of students." (U. of California Regents v. Bakke [46], 438 U.S. 317)

Justice Powell held up the Harvard admissions program as a canonical example of a narrowly tailored, and hence constitutional, affirmative action system. The medical school at the University of California - Davis, the respondent in University of California Regents v. Bakke, used an admissions quota scheme, which Justice Powell ruled was unconstitutional.

Justice Powell’s decision in University of California Regents v. Bakke said little directly about acceptable or unacceptable outcomes of an affirmative scheme, but the opinion provides suggestion about constitutionally acceptable procedurea for implementing an affirmative action system. The keys to a constitutionally approved affirmative scheme are:

1. All applicants are in competition for all seats.

2. To the extent that “odious” distinctions between applications, such as race, are a factor in admissions, the consideration of the applicants must be individualized.

---

20 As Justice Powell wrote:

"... race or ethnic background may be deemed a "plus" in a particular applicant's file, yet it does not insulate the individual from comparison with all other candidates for the available seats." (Page 438 U.S. 318)

21 Return to Justice Powell’s opinion:

“This kind of program [referring to the Harvard College admissions system] treats each applicant as an individual in the admissions process. The applicant who loses out on the last
The first consideration rules out explicit quotas and most universities in the United States adapted their admissions policies accordingly in the post-Bakke era. However, Justice Powell’s opinion acknowledges that even admissions preference schemes that treat applicants as individuals can serve as “... a cover for a functionally equivalent quota system” (U. of California Regents v. Bakke [46], 438 U.S. 219).

The 2003 cases Gratz et al. v. Bollinger et al. and Grutter v. Bollinger et al. were the first affirmative action cases addressed by the Supreme Court following the ruling in University of California Regents v. Bakke. These cases turned on whether the University of Michigan admissions preference schemes are narrowly tailored. Interestingly, although Powell’s 1978 opinion appeared to say little about acceptable or unacceptable outcomes of affirmative action, the justices in both of these cases looked to the outcomes to judge the extent to which the systems function as defacto quotas.

Gratz et al. v. Bollinger et al. addressed whether the admissions preference scheme used by the University of Michigan College of Literature, Science, and the Arts (LSA) met the narrow-tailoring criteria. The admissions preference scheme used by the LSA attributed points to applicants based on (for example) academic performance, athletic ability, Michigan residency, and race. Applicants that received more than 100 of the maximum possible 150 points received immediate admissions, while those with fewer points were accepted later in the admissions process or not at all. Minorities received an additional 20 points, while applicants with significant leadership or public service achievements received at most 5 points.

The court ruled the LSA admissions preference scheme unconstitutional for two reasons. First, the across-the-board attribution of 20 points based solely on minority status was not individualized enough to qualify as narrowly tailored. Second, “...virtually available seat to another candidate receiving a "plus" on the basis of ethnic background will not have been foreclosed from all consideration for that seat simply because he was not the right color or had the wrong surname. It would mean only that his combined qualifications, which may have included similar nonobjective factors, did not outweigh those of the other applicant. His qualifications would have been weighed fairly and competitively, and he would have no basis to complain of unequal treatment under the Fourteenth Amendment.” (Page 438 U.S. 319 of Bakke)

22Justice Powell also ruled out multitrack quotas, which would be required (in general) for a quota system to replicate an admissions preference scheme:

“Nor would the state interest in genuine diversity be served by expanding petitioner’s twotrack system into a multi-track program with a prescribed number of seats set aside for each identifiable category of applicants.” (Page 438 U.S. 316 of Bakke)

23In her dissent, Justice Ginsburg argued that LSA was simply articulating its policy clearly and unambiguously.

“If honesty is the best policy, surely Michigan’s accurately described, fully disclosed College affirmative action program is preferable to achieving similar numbers through winks, nods, and disguises.” (p. 305)
all [minority freshman applicants] who are minimally qualified are admitted...” (Gratz v. Bollinger [27], 539 U.S. 278), which means that these students are defacto not competing with nonminority applicants for admission. To summarize, the concurring justices argue in their opinion that although LSA did not formally use a quota, the results were functionally the same.

Grutter v. Bollinger et al. revolved around the admissions process of the University of Michigan Law School (Law School), which the Supreme Court ruled constitutional. The key difference between the LSA’s policy and the Law School’s is that each applicant to the law school is given individualized review without points attributed to particular traits of the applicant. However, if one reads the dissenting opinions, Justices Scalia and Rehnquist made separate arguments that the Law School admissions process was functionally equivalent to a quota:

“I join the opinion of The Chief Justice. As he demonstrates, the University of Michigan Law School’s mystical “critical mass” justification for its discrimination by race challenges even the most gullible mind. The admissions statistics show it to be a sham to cover a scheme of racially proportionate admissions.” (Scalia, p 346 - 347 of Grutter)

“... the ostensibly flexible nature of the Law School’s admissions program that the Court finds appealing... appears to be, in practice, a carefully managed program designed to ensure proportionate representation of applicants from selected minority groups.” (Justice Rehnquist in Grutter v. Bollinger [30], 539 U.S. 385)

In the end theorem 5 implies that attempts to differentiate between unconstitutional quotas and constitutional admissions preferences on the grounds of the outcomes produced will likely prove futile. For supporters of affirmative action, the equivalence of sophisticated quotas and affirmative action schemes might suggest that all of these systems ought to be constitutional, and universities should be free to use which ever system helps them better achieve their diversity goals. For opponents of affirmative action, this may provide an argument that no admissions scheme that takes race into account ought to be able to pass constitutional muster. Those that have been approved, such as the Law School’s, were supported because the opacity and ambiguity of the admissions preference scheme obscured the equivalence to a quota.
6 Optimal Affirmative Action Schemes

The previous sections of this paper had focused on describing properties of exogenous affirmative action schemes. In this section we study how to design an optimal affirmative action scheme by solving an optimal control problem. We focus on using the affirmative action system to increase average student welfare, close the test gap between minority and nonminority students, and achieve maximally fair outcomes.

The control, $u(p) \in [0, 1]$, represents the fraction of seats at school $p$ that are allocated to minority students. Because the index variable is the college, $p$, all of the variables in our problem must be written as functions of $p$. The state variables of our control problem are the equilibrium strategies of the students, $\sigma_M(p)$ and $\sigma_N(p)$, and the log of the type of the student from each group that is assigned a seat at college $p$, $X_M(p) = \ln \theta_M(p)$ and $X_N(p) = \ln \theta_N(p)$. We let $L$ denote the objective function of the problem. Our optimal control problem can be written

$$L(\sigma_M, X_M, \sigma_N, X_N, u)$$

such that

$$\dot{X}_M(p) = -\frac{u * f_P(p)}{\mu f_N^X(X_M(p))}$$

$$\dot{X}_N(p) = -\frac{(1 - u) * f_P(p)}{(1 - \mu)f_N^X(X_N(p))}$$

$$\dot{\sigma}_M(p) = \frac{U_p(p, \sigma_M, X_M)}{C_s(\sigma_M, X_M) - U_s(p, \sigma_M, X_M)}$$

$$\dot{\sigma}_N(p) = \frac{U_p(p, \sigma_N, \theta_N)}{C_s(\sigma_N, \theta_N) - U_s(p, \sigma_N, \theta_N)}$$

$$\int_0^p u(p) f_P(p) dp = \mu$$

$$\sigma_M(p) = \sigma_N(p) = 520, \ X_M(p) = X_N(p) = \ln 7.324$$

Equations 11 - 14 denote the laws of motion for the state variables, and equation 15 insures that enough seats are allocated to minorities that the entire measure $\mu$ of minorities obtains a college seat. Equation 16 provides boundary conditions for our state variables that are derived from Hickman [31]. We impose the boundary condition $\sigma_j(p) = \xi = 520$ regardless of whether or not both groups are assigned seats at the worst school, $p$. We are, in effect, assuming that both groups are assigned at least a vanishingly small fraction of a seat at every college.\(^{24}\)

\(^{24}\)We do this primarily so that the optimal control problems are tractable. An alternative to our approach would be to allow the boundary conditions for each group to be defined by indifference conditions.
We used the estimated utility functions and type distributions from Hickman [31] to calibrate our model. Hickman [31] uses the US News and World Report college quality index as his metric for college quality, $p$. A student’s human capital level, $s$, is represented by the student’s SAT score. For the utility functions we used

\[
U(p, s, X) = \rho(p, s) * u(p)
\]

\[
\rho(p, s) = -0.176 + 0.000774s - 0.00000049s^2 + 0.00076ps
\]

\[
u(p) = 40134p^{0.536}
\]

where $\rho$ represents the probability of graduating from college and $u$ is the college premium conditional on graduating. The cost function for the model is

\[
C(s, X) = \theta e^{0.013X(s-\bar{s})}, \bar{s} = 520
\]

The college qualities and student types are all roughly normally distributed, so we calibrated our model by assuming these quantities are exactly normally distributed with the mean and standard deviations computed from the nonparametric estimates of Hickman [31]. The resulting distributions were:

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Standard Deviation</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>College quality</td>
<td>0.564</td>
<td>0.190</td>
<td>[0.087, 0.9]</td>
</tr>
<tr>
<td>Minority Student Type</td>
<td>3.42</td>
<td>2.000</td>
<td>[-2.83, 7.32]</td>
</tr>
<tr>
<td>Nonminority Student Type</td>
<td>1.67</td>
<td>2.034</td>
<td>[-2.83, 7.32]</td>
</tr>
</tbody>
</table>

For numerical stability we truncated the ranges of these variables by removing the worst 0.5% of colleges, 2.6% of minority students, and 0.2% of nonminority students as well as the best 3.8% of colleges and 1.35% of nonminority students.

Before discussing the particular questions we try to answer, we would like to discuss a few weaknesses of our approach. Since we pose our questions in the form of a calibrated model, the answers implicitly assume the model is accurate and that the differences between the underlying data and the calibration are not significant. Because of

25 As pointed out by Fryer and Loury [22], many of the theoretical results on the consequences of affirmative actions systems are contingent on assumptions about the underlying economic primitives. Like the previous literature, our results are specific to the functional forms and values we have assumed. Our hope is that the calibration exercise has put the results in close contact with the real world.

26 We solved all of our problems using the GPOPS-2 optimal control package, which solves the problems using spectral methods. Because the laws of motion describing our problem are stiff differential equations, numerical stability raised significant issues. Although it would have been possible in theory to use the nonparametric estimates of $F_P$, $F_M$, and $F_N$ directly, as a practical matter significant difficulties with the stability of the algorithm became obvious.
the continuity of the model, we do not think the use of a calibrated model will change
the qualitative nature of our results, but one ought to keep the calibration in mind when
interpreting the high precision results we provide.

Another deficiency of our model is that we need to assume that school quality is fixed
and exogenous. Although it is outside of the scope of this paper to provide a model that
endogenizes school quality, it is well known that there are spillovers between students
in many contexts that make school quality a function of which students enroll. The
spillovers may be a function of student characteristics (e.g., Hoxby [35]) or student effort
choices (e.g., Fruewirth [20]). In addition, if we take at face value the briefs filed by the
universities in Regent of the University of California v. Bakke [46], Gratz v. Bollinger
[27], and Grutter v. Bollinger [30], universities believe that student welfare is directly
enhanced by diversity amongst the student population at individual universities. All of
these effects are, in the context of our model, included in the quality of the university.

Since these reallocations of the students may influence the underlying distribution of
school qualities, we must treat our results as something like a partial equilibrium. In
other words, we believe our results point in the right direction, but are not the entire
story. We discuss the issue of endogenous school quality in more depth in section 6.4.

6.1 Welfare Maximization

Our first goal is to study the affirmative action program that maximizes the social surplus
of the students. Recall that the (unachievable) first best outcome for the college admis-
sions problem would be to assortatively assign each student to the appropriate college
and then have each student chooses the utility maximizing human capital level given
his or her school assignment. One might have thought that an undistorted, color-blind
scheme would be an obvious choice for second-best since the matching remains assor-
tative under such a scheme. We find, however, that because a color-blind admissions
scheme has wasteful signaling through the competition channel, distorting the allocation
of seats can improve welfare by dampening the competition channel enough to yield a
net benefit.

Before we proceed with our analysis, we should be perfectly clear that our notion
of welfare excludes many factors that one might consider to be important aspects of the
benefits (or costs) of affirmative action. The notion of welfare we use is completely based
on the expected college premium of entering college freshman. Our estimates do not
include less tangible benefits that might be accrue to the students as a result of socializing
or studying with members of the other group. Moreover, we do not include the social
gains that might be had from decreasing the inequality of the salary or education level of
the two groups. Finally, to the extent one might view redressing past discrimination as
a component of the welfare gains or losses of affirmative action, these concerns are also excluded from our analysis.

Given these caveats, our goal is to make two broad points. First, the allocation of seats that maximizes our metric for student welfare results in complete segregation. The magnitude of this distortion reveals the importance of the welfare losses from the wasteful competition for seats in a color-blind system. Second, arguments regarding the merits of affirmative action (including in the Supreme Court cases discussed above) often compare whether the proposed benefits of diverse classrooms is worth the misallocation caused by affirmative action. This argument seems to implicitly understand deviations from color-blind admissions as harmful to welfare. Since our analysis shows that we can do significantly better than a color-blind policy, the real hurdle is whether welfare gains from diversity is more important than the losses from wasteful signaling that our welfare optimal scheme prevents.

For the social surplus problems, we use the following objective function

\[
\int_{p} \{ u \ast [U(p, \sigma_M, X_M) - C(\sigma_M, \theta_M)] + (1 - u) \ast [U(p, \sigma_N, \theta_N) - C(\sigma_N, \theta_N)] \} f_P(p) dp
\]

Since the objective function and the equations of motion are linear in \(u\), we know immediately that the solution will have a bang-bang structure. In other words, the social surplus maximizing affirmative action scheme will generically involve complete segregation - all of the seats in each school will be allocated to one of the two groups. However, it could be that the two groups are allocated essentially identical schools. For example, it could be that each time a school \(p\) is allocated to nonminority students, minority students are allocated a school with a quality very close to \(p\).

However, the social surplus maximizing scheme assigns all of the low quality college seats to minority students and reserves the best colleges for nonminority students, a result depicted in figure \[\text{1}\]. Figure \[\text{1}\] includes the allocation under the color-blind scheme for comparison. A disproportionate fraction of seats at low quality colleges are assigned to minority students under a color-blind scheme because the population of minority students contains a large fraction of high costs students relative to the population of nonminority students.

We compare the welfare from the first best, optimal, color-blind, and representative

\[27\text{At this point we would like to reemphasize that both authors are strong supporters of affirmative action. Our experience as teachers and as students has caused us to believe that the benefits of diverse classrooms to students are very likely to be welfare improving relative to either color-blind admissions or the optimal distortion we describe below.}\]
quota schemes in the following table.

<table>
<thead>
<tr>
<th></th>
<th>Social Surplus (Thousands of Dollars)</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Best</td>
<td>14,622</td>
</tr>
<tr>
<td>Optimal</td>
<td>13,724</td>
</tr>
<tr>
<td>Color Blind</td>
<td>13,440</td>
</tr>
<tr>
<td>Representative Quota</td>
<td>13,370</td>
</tr>
</tbody>
</table>

First, note that the losses from the competitive channel in a color-blind system amount to almost $1200 per student per year. An optimal scheme is able to recover only about $300 of these losses. Finally, and perhaps most surprisingly, the welfare loss of moving from a color-blind system to a representative quota is rather small at just $70 per student per year.

In order to explore the relationship between social surplus and fairness, we revisit the social surplus maximization control problem, but we restrict the degree to which schools can be segregated by insisting that $u(p) \geq u$. If $u = 0$, the affirmative action program will maximize social surplus. If $u = \mu$, then a representative quota results. In figure 1 we have plotted the average utility for the minority and nonminority students as functions of $u$. Note that the average utility between the two groups differs even in the case $u = \mu$ because the distributions of types is different between the groups.
As we can see from figure 2, the optimal affirmative action scheme is, in effect, a large transfer from the minority to the nonminority students. When we consider moving from a representative quota to a fully optimal affirmative action scheme, the average welfare of the minority students drops from $12,489 to $5,646, while the welfare of the nonminority students rises from $13,564 to $15,506. The larger magnitude of the effect on the minority students is due to the relative size of the groups. This large disparity in the welfare of the two groups in the optimal affirmative action program suggests that fairness considerations ought to play a large role in our design choice.

6.2 Minimizing the Test Gap

As reported in Hickman [31], as of 1996 the median SAT score of nonminority student is 1030, while the median minority student score was just 870. We will now consider to what extent this 160 point gap can be closed by redesigning the affirmative action scheme. Of course, the test gap is almost certainly the result of many different causes including differing education levels of the student’s parents, disparities in school quality between the two groups, differences in nutrition, different beliefs about the salary impact of attending college, etc. Altering the college assignment process will probably have a long-run endogenous impact on many of these factors. We are optimistic and conjecture that these long-run effects will in general be complementary to our design - for example, encouraging minority students to invest in HC will result in the next generation of parents being better educated. Therefore, we view our calculations as a lower bound on the general equilibrium impact of a redesigned affirmative action scheme.
Again we compare four different affirmative action schemes. In this context, test gap optimal indicates a scheme that minimizes the test gap irrespective of the effects on social surplus. This is equivalent to an objective function of the form

\[
\left( \int_{p}^{P} \frac{u}{\mu} \sigma_{M}(p)f_{P}(p)dp - \int_{p}^{P} \frac{1-u}{1-\mu} \sigma_{N}(p)f_{P}(p)dp \right)^{2}
\]

which is simply the squared difference in the average human capital level. Since the objective is no longer linear in \( u \), the solution will not be a bang-bang result. The allocation of college seats that minimizes this objective is shown in figure 3.

As one can see, the test gap minimizing allocation gives more or the best and worst college seats to the minority students with a smattering of seats at midquality colleges. When we compare the test gap for the different affirmative action schemes, we find:

<table>
<thead>
<tr>
<th>Test Gap (SAT Points)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test Gap Optimal</td>
</tr>
<tr>
<td>First Best</td>
</tr>
<tr>
<td>Welfare Optimal</td>
</tr>
<tr>
<td>Color Blind</td>
</tr>
<tr>
<td>Representative Quota</td>
</tr>
</tbody>
</table>

Therefore, the best we can hope for in terms of closing the gap in the short run with our affirmative action instrument is to reduce it to 75 SAT points, which is about half of the
gap that existed as of 1996.

Closing the test gap has significant effects on the inequality between the groups. We numerically computed the boundary of the set of feasible social surplus and test gap combinations.28 As one can see in figure 4, closing the test gap requires significant transfers from the nonminority students to the minority students.

![Figure 4](image)

As we move from a welfare to a test gap optimal regime, the minority students’ welfare increases from $5,646 to $17,345 and the majority students’ welfare drops from $15,506 to $12,357. Again, fairness concerns are clearly a significant issue. That being said, the average social surplus drops much less, from $13,724 to $13,240.

### 6.3 Welfare and Fairness

Now we move onto the third of our numerical studies, the relationship between welfare and fairness. As our fairness objective, we simply use the squared difference in the average welfare of the minority and nonminority students:

\[
\left( \int_{p}^{\bar{p}} \frac{\partial u}{\partial \mu} \left[ U(p, \sigma_M, X_M) - C(\sigma_M, \theta_M) \right] f_P(p) dp - \int_{p}^{\bar{p}} \frac{1}{1 - \mu} \left[ U(p, \sigma_N, X_N) - C(\sigma_N, \theta_N) \right] f_P(p) dp \right)^2
\]

By optimizing with respect to a convex combination of this objective and social surplus we can trace out the set of feasible fairness and welfare maximizing outcomes. The result is displayed in figure 5.

---

28Formally this was done by solving optimal control problems with an objective function that was a convex
As is obvious from the plot, the gap between the average payoffs of the two groups can be completely closed. If the gap is closed, all students receive a college premium of $13,351. Closing the gap requires overrepresenting the minority students in the very best and worse schools, although the division is less extreme than in the test gap minimizing solution.

6.4 Endogenous School Quality

Throughout this analysis we have assumed that school quality is fixed and exogenous and, in particular, independent of the students that are assigned to each college. Proposing a model of the endogenous determination of school quality is beyond the scope of this paper, but this endogeneity is clearly relevant for the numerical experiments we run. For example, if the social surplus maximizing affirmative action scheme greatly distorts the human capital choices of students at a given college, then it is easy to imagine that the quality of the college would change, and one might justifiably wonder what this neglected endogenous effect this has on the social surplus.

As noted above, it is not our goal to propose a model of the determinants of school quality. However, it is straightforward to use our results to assess the average human capital level of the students assigned to each college under each of our affirmative actions plans. In the following figure we plot the average human capital level at each college.

---

29 All of the students with each group choose the same human capital as the other members of their demographic group at a given college. However, since the human capital level at a college can differ across groups, we must account for this difference through averaging.
Three facts leap out from a cursory inspection of figure 6. First, the human capital choices at each school are almost the same in the color-blind and representative quota schemes. Second, the first-best outcome yields strictly lower human capital levels than the representative quota or the color-blind scheme. The gap between the color-blind and the first-best human capital choices is because the first-best outcome is generated by shutting down the (wasteful) competition incentive to accrue human capital.

Finally, the welfare optimal outcome entails radically different human capital levels than any other scheme. For low quality schools to which minorities are assigned, we see relatively low cost minority students competing intensely for available seats, which drives the human capital levels at low quality colleges up. If college quality is related to the human capital choices of the accepted students, this suggests that the “bad colleges” to which minorities are relegated under the social surplus maximizing scheme might improve significantly in the long-run. Symmetrically, the middling quality colleges to which high cost nonminority students are assigned show a significant drop in average human capital, which suggests these schools may drop in quality in the long-run. However, the effects of the welfare-optimal assignment at the lowest- and highest-quality colleges is essentially the same as under the color-blind or representative quota system, which is a result of the fact that the same students are competing for these seats under all three regimes.

Given the nonmonotonicities in the human capital levels of the welfare optimal assignment scheme, a more informative perspective is yielded by looking at the CDF of the
human capital distribution, which is portrayed in figure 7. As expected, the distribution of human capital levels under the first-best scheme is first order stochastically dominated by the other regimes, again reflecting the shutdown of the wasteful competition incentive for human capital accumulation. Similarly, the CDF for the welfare maximizing scheme lies to the left of the CDFs for the color-blind and representative quota schemes, which implies that the welfare maximizing scheme is eliminating some wasteful human capital accumulation. Relative to figure 6, however, the differences in the aggregate human capital distribution generated by the welfare maximizing assignment appears similar to the color-blind and representative quota schemes.

7 CONCLUSION

The purpose of this paper has been to introduce a new model of affirmative action and use it to enrich the debate around the differences between quota and admissions preference systems as well as informing the design of affirmative action schemes. Modeling affirmative action programs is challenging since one must consider the human capital investment decisions of students, heterogeneity in underlying quality on both sides of the market, and the strategic decisions of universities given the information they are presented. While many prior papers in the literature gain tractability by simplifying various components of the problem (e.g., assuming all college are homogenous or student quality...
is innate), we take the opposite tack and consider a market with a continuum of agents. The continuum approximation greatly simplifies the analysis and allows us to produce a large number of novel results.

Our first application of the model is to study the difference between quota-based and admissions preference-based affirmative action systems, and we find that in fact there is no difference in the equilibrium outcomes produced. It is not particularly surprising that one can achieve the same diversity levels under each scheme - for example, one might imagine duplicating the effect of a quota by giving minority applicants a small boost when evaluating their applications. However, we believe it is surprising that the equivalence holds over both the diversity at individual colleges and the human capital accumulation decisions of those admitted.

Moreover, our equivalence formalizes comments made in the jurisprudence regarding the difference between quotas and affirmative action schemes. The legality of affirmative action turns on what was perceived by Justice Powell as a sharp difference between quotas and well designed admissions preference systems. Later opinions about the legality of different admissions preference schemes hinged on how closely the respective justice thought the admissions preference scheme mimicked a quota. Our analysis suggests that drawing a sharp line between quotas and admissions preference schemes based on outcomes will be futile. Not being legal scholars, we have little to say about the eventual legal ramifications for universities that employ color-sighted affirmative action programs.

Our last contribution is to design optimal affirmative action programs using a calibration of our model drawn from Hickman [31]. We show that social surplus maximization entails segregating the minority students into the worst schools, which helps alleviate wasteful signaling of quality through excessive human capital accumulation choices. Unfortunately, this design causes huge transfers from the minority students for a small average gain for the nonminority students.

We also consider the limits of using affirmative action at the college level as a tool for closing the black-white test gap. Our analysis suggests that the gap can be reduced by more than 50%, but at the cost of wildly unfair outcomes. In this case, minority students receive large transfers from nonminority students since reducing the test gap requires giving the minority students a large fraction of seats at the best (and worst) schools.

Lastly, we consider equalizing the welfare of the two groups, which we consider a metric of fairness. We are able to completely equalize the payoffs of minority and nonminority students at a relatively modest cost in terms of average welfare.

Three other interesting directions for further research exist. First, if we could incorporate a model of how student assignment endogenously influences college quality, we could discuss with more confidence the general equilibrium implications of massive
changes in the assignment of students to schools. Unfortunately, we are unaware of any structural models we could use as the basis for this research agenda.

Second, the current model focuses on student behavior, conditional on participation in the college market, but there is another interesting group of individuals to consider as well: those whose college/work-force decisions may be affected by a given policy. This question could be addressed by formalizing the “supply-side,” being comprised of potential colleges and firms who may enter the market and supply post-secondary education services or unskilled jobs. Such a model might illustrate how the marginal agent (i.e., the individual indifferent between attending college and entering the workforce) is affected by a given college admission policy. This would help to characterize the effect of AA on the total mass of minorities enrolling in college.

Finally, the eventual goal for this line of research should be to answer the question of how AA helps or hinders the objective of erasing the residual effects of institutionalized racism. This will require a dynamic model in which the policy-maker is not only concerned with short-term outcomes for students whose types are fixed, but also with the long-run evolution of the type distributions. Empirical evidence suggests that academic competitiveness is determined by factors such as affluence, as well as parents’ education. If AA affects performance and outcomes for current minority students in a certain way, then the next question is what effect it might that have on their children’s competitiveness when the next generation enters high school? If a given policy produces the effect of better minority enrollment and higher achievement in the short-run, then one might conjecture that a positive long-run effect will be produced. However, given the mixed picture on the various policies considered in this paper, it seems evident that a long-run model is needed in order to give meaningful direction to forward-looking policy-makers. It is our hope that the theory developed here will serve as a basis for answering these important questions in the future.

**References**


A Proofs

A.1 Proving Theorem

Since the equilibrium strategies are strictly decreasing (proposition), we know immediately that the equilibrium strategy must be almost everywhere differentiable. We now prove that there is a lower bound on the derivative of the equilibrium strategy, which implies that the distribution of human capital in any equilibrium must be nonatomic. Moreover, it implies if we look at sequences of equilibrium strategies, the resulting limit strategy generates a nonatomic distribution of human capital.

**Lemma 1.** There exists $\omega < 0$ such that for any equilibrium strategy, $\sigma(\theta)$, of either a finite game or the limit game, we have $\frac{d}{d\theta} \sigma(\theta) < \omega$ at points where the strategy is differentiable.

**Proof.** We prove our lemma for the color-blind game, but the proof extends directly to the admissions games with quotas by treating each group separately. Finally, the proof technique easily extends to the score function game, although the notation becomes cumbersome since one must accommodate both groups of students and account for the variation in $\tilde{S}$.

Suppose there is no such upper bound on the derivative, which means that for any $\omega < 0$ there exists $\theta$ such that $\sigma'(\theta) > \omega$. From the a.e. differentiability of $\sigma$, there exists an interval $[\theta_L, \theta_U]$ such that $\sigma'(\theta) > \omega$ for all $\theta \in [\theta_L, \theta_U]$ where $\sigma'(\theta)$ exists. Without loss of generality, we assume $\sigma'(\theta)$ exists at $\theta_L$. Let $s_L = \sigma(\theta_L)$ and $s_U = \sigma(\theta_U)$, and note that $0 < s_L - s_U < \omega(\theta_L - \theta_U)$. Since $\sigma$ must be decreasing, we have

$$\Pr\{\sigma(\theta) \in [s_U, s_L]\} = F(\theta_U) - F(\theta_L)$$
Rearranging this we find
\[
\frac{\Pr\{\sigma(\theta) \in [s_U, s_L]\}}{s_L - s_U} = \frac{F(\theta_U) - F(\theta_L)}{\sigma'(\theta_U) - \sigma'(\theta_L)} > \frac{-1}{\omega} \frac{F(\theta_U) - F(\theta_L)}{\theta_U - \theta_L}
\]

Let \( \eta_\theta = \inf_\theta f(\theta) > 0 \). Taking limits we find
\[
\lim_{s_U \to s_L} \frac{\Pr\{\sigma(\theta) \in [s_U, s_L]\}}{s_L - s_U} > \frac{-1}{\omega} \lim_{\theta_U \to \theta_L} \frac{F(\theta_U) - F(\theta_L)}{\theta_U - \theta_L} = -\frac{1}{\omega} f(\theta_L) > -\frac{1}{\omega} \eta_\theta > 0
\]

This means that in intervals where \( \sigma'(\theta) \) is close to 0, the “density” of individuals making the associated human capital choices is arbitrarily large. We call such a point of high density a pseudo-atom.

Let \( \delta_p = \sup \delta p < \infty \). Increasing the human capital choice from \( s_L \) to \( s_U \) yields a minimal benefit of increasing the rank of one’s school in the limit game by
\[
-\frac{1}{\omega} \frac{\eta_\theta}{\delta_p} (s_L - s_U)
\]

If we let \( \eta_U = \min U_1(p,s,\theta) > 0 \), then the utility benefit must be at least
\[
-\frac{1}{\omega} \frac{\eta_\theta}{\delta_p} \eta_U (s_L - s_U)
\]

Let the maximum marginal cost of human capital that we can observe in any equilibrium be denoted
\[
\delta_C = \max_{s \in S, \theta \in \Theta} C_1(s, \theta) < \infty
\]

This means the cost of deviating from \( s_U \) to \( s_L \) is bounded from above by
\[
(s_L - s_U) \delta_C
\]

For such a deviation to be suboptimal, we must have
\[
(s_L - s_U) \delta_C \geq -\frac{\eta_U \eta_\theta}{\omega} \frac{\eta_U (s_L - s_U)}{\delta_p \delta_C}
\]

which requires
\[
\sigma(\theta) \leq \omega \leq -\frac{\eta_\theta \eta_U}{\delta_p \delta_C}
\]

In the game with \( K \) students, the formation of pseudoatoms is probabilistic. Consider an agent with type \( \theta_U \) who in equilibrium chooses \( s_U = \sigma^K(\theta) \). Suppose such a student
considers increasing her human capital choice to \( s_L \). For each student she leap frogs, her school placement improves by at least \( (\delta_p)^{-1} K \), which generates a utility benefit of at least

\[
\frac{\eta_U}{\delta_p} \frac{1}{K}
\]

For each student, there is a probability of at least \(-\frac{\eta_U}{\omega} (s_L - s_U)\) of observing a human capital choice in \([s_U, s_L]\), which yields a lower bound on the expected benefit of the deviation equal to

\[
\frac{\eta_U}{\delta_p} \frac{1}{K} E[i]
\]

where \( i \) is distributed binomially with \( K \) draws using a parameter equal to \(-\frac{\eta_U}{\omega} (s_L - s_U)\).

Given the distribution of \( i \), we can write

\[
\frac{\eta_U}{\delta_p} \frac{1}{K} E[i] = \frac{\eta_U}{\delta_p} \frac{1}{K} \left[ -\frac{\eta_U}{\omega} (s_L - s_U) K \right] = -\frac{\eta_U}{\delta_p} \frac{\eta_U}{\omega} (s_L - s_U)
\]

The remainder of the argument proceeds as above.

**Theorem 4.** Consider the admissions preference game and assume one of the cases of theorem 3 holds. Under assumptions 1 - 8, assumption 10, and given \( \epsilon, \delta > 0 \), there exists \( K^* \in \mathbb{N} \) such that for any \( K \geq K^* \) we have the following:

(i) \( \sigma^j, j \in \{cb, q, ap\} \), is an \( \epsilon \)-approximate equilibrium of the \( K \)-agent game.

(ii) \( \sigma^j, j \in \{cb, q, ap\} \) is a \( \delta \)-approximate equilibrium for the \( K \)-player game

**Proof.** We prove our theorem through a series of lemmas steps. These lemmas are primarily necessary for the application of the theorems in Bodoh-Creed [4]. Since our proofs are entirely in terms of payoffs for each \((\theta, s)\) pair as a function of distributions of effective human capital choices (defined below), our proof applies to all of our games.

Let \( \Delta_K(X) \) denote the set of empirical measures generated \( K \) draws from the set \( X \). \( \pi_{\mathcal{N}}^{K_N} \) denote the empirical measure of human capital choices for the nonminority students when there are \( K_N \) such students in the economy. \( \pi_{\mathcal{M}}^{K_M} \) denotes the empirical measure of subsidized human capital choices (i.e., the distribution of \( \tilde{S}(s) \)) for the nonminority students when there are \( K_M \) such students in the economy. We use the notation \( \Pi(s, \theta; \pi_{\mathcal{N}}^{K_N}, \pi_{\mathcal{M}}^{K_M}, K) \) to refer to the expected utility in the \( K \)-agent game of an agent of type \( \theta \) that chooses human capital level \( s \) given \( \pi_{\mathcal{N}}^{K_N} \) and \( \pi_{\mathcal{M}}^{K_M} \). Let \( \Pi(s, \theta; \pi_{\mathcal{N}}^{K_N}, \pi_{\mathcal{M}}^{K_M}) \) refer to the utility received by an agent of type \( \theta \) that chooses human capital level \( s \) given \( \pi_{\mathcal{N}} \) and \( \pi_{\mathcal{M}} \) in the limit game.

First we note that since all of the parameters in our model (e.g., \( S, \Theta, P, \Delta_{K_N}(S) \), etc.) are drawn from compact spaces, any claims of continuity of \( U, P \), etc. immediately
implies uniform continuity. Uniform continuity will appear several times in the proofs that follow.

**Lemma 2.** For all $\varepsilon > 0$ there exists $K^*$ such that for all $K > K^*$ and all $(s, \theta; \pi^K_N, \pi^K_M) \in \mathcal{S} \times \Theta \times \Delta_{K_N}(S) \times \Delta_{K_M}(S)$ we have

$$\|\Pi(s, \theta; \pi^K_N, \pi^K_M, K) - \Pi(s, \theta; \pi^K_N, \pi^K_M)\| < \varepsilon$$

**Proof.** We have assumed that the empirical distribution of seats in the $K$-student economy, $F^K_K$, converges uniformly to the distribution of seats in the limit game, $F_P$. Therefore, for any $\gamma > 0$ there exists $K'$ such that $\|F^K_K(p) - F_P(p)\| < \gamma$ for all $p$ and all $K > K'$. There are two slightly different cases to consider. Suppose exactly one student has selected human capital $s$, which means

$$\frac{K_M}{K} \pi^K_M(s) + \frac{K_N}{K} \pi^K_N(s) = \frac{1}{K}$$

Let the quantile of the student be denoted $q_s = \frac{K_M}{K} \pi^K_M([q, s]) + \frac{K_N}{K} \pi^K_N([q, s])$. This student will be assigned to the school $(F^K_P)^{-1}(q_s)$. Since $f_P$ is strictly positive, there exists $\lambda > 0$ such that $f_P(\circ) > \lambda$, which means

$$\left\|\left((F^K_P)^{-1}(q_s) - (F_P)^{-1}(q_s)\right)\right\| < \frac{\gamma}{\lambda}$$

The remainder follows from the uniform continuity of the agent’s utility function.

Now consider the case where multiple students declare the same human capital level. In this case we know that all such students will be randomly assigned to schools between $(F^K_P)^{-1}(\bar{q}_{s})$ and $(F^K_P)^{-1}(\bar{q}_{s})$ where $[\bar{q}_{s}, \bar{q}_{s}]$ is the set of quantiles spanned by these students. By a very similar argument, the expected utilities (with the expectation taken only over the random assignment to seats) must be close in the $K$ agent and the limit economy for $\gamma$ sufficiently small.

We now prove that the limit game is continuous over the space of measures of human capital chosen by the minority and nonminority students. If the distributions of human capital admitted atoms, then continuity would fail since small changes in the $s$ chosen would result in large changes in the school assignment. Fortunately, lemma [1] proves that the equilibrium strategies have an upper bound on their derivative, which implies that pseudo-atoms cannot exist. Let $\Sigma^R$ be the set of strategies that adhere to the bound prescribed by lemma [1]. Let $\Delta^R(S)$ denote the space of pushforward measures generated by a strategy $\sigma \in \Sigma^R$ and the distribution of either the minority or nonminority students,
\( F_{\mathcal{M}}(\theta) \) or \( F_{\mathcal{N}}(\theta) \). The set \( \Delta^R(\mathcal{S}) \) is a tight family of measures and is compact as a result (Theorem 15.22 of Aliprantis and Border [3]).

**Lemma 3.** \( \{\Pi(s, \theta; \pi_{\mathcal{N}}, \pi_{\mathcal{M}})\}_{(s, \theta)} \) is uniformly equicontinuous in \((\pi_{\mathcal{N}}^{K\mathcal{N}}, \pi_{\mathcal{M}}^{K\mathcal{M}}) \in \Delta^R(\mathcal{S}) \times \Delta^R(\mathcal{S})\) when the spaces of measures are endowed with the weak-* topology.

**Proof.** Recall that all of the measures in \( \Delta^R(\Theta) \) are of bounded density, which implies that the measures are nonatomic. When restricted to a set of nonatomic measures, the Kolmogorov and weak-* topologies are identical, and so we will work with properties of the Kolmogorov metric.

Let \((\pi_{\mathcal{M}}, \pi_{\mathcal{N}}) \in \Delta^R(\mathcal{S}) \times \Delta^R(\mathcal{S})\). The use of the Kolmogorov metric is crucial for our result since the quintiles associated with each human capital choice \( s \), namely \( q_s = \mu \pi_{\mathcal{M}}([s, s]) + (1 - \mu) \pi_{\mathcal{N}}([s, s]) \), are uniformly continuous with respect to \( \pi_{\mathcal{M}} \) and \( \pi_{\mathcal{N}} \) under the Kolmogorov metric. Therefore the school assigned to a student for any given level of human capital, \((F_{\mathcal{P}})^{-1} (q_s)\), is continuous in \( \pi_{\mathcal{M}} \) and \( \pi_{\mathcal{N}} \). Since we formally require equicontinuity of the utility across all type-action pairs, we need \( F_{\mathcal{P}} \) to be bounded from below and \( U_{1}(p, s, \theta) \) to be bounded from above. The former implies that small changes in \( q_s \) do not cause arbitrarily large changes in the assigned school, while the later insures that small changes in the assigned school do not cause arbitrarily large changes in utility. The uniform equicontinuity comes from the compactness of \( \mathcal{S} \times \Theta \times \Delta^R(\mathcal{S}) \times \Delta^R(\mathcal{S}) \).

Given lemmas 2 and 3, our first claim follows immediately from theorem 8 of Bodoh-Creed [4]. We now prove our second claim.

Consider a sequence of exact equilibrium strategies of the \( K \)-agent game, \( \{\sigma^K : \Theta \to \Delta(\mathcal{A})\}_{K=1}^{\infty} \), and suppose \( \sigma^K \to \sigma^\infty \) in the sup-norm. Let \( d_{LP}^S \) denote the Levy-Prokhorov metricization of the weak-* topology over the space \( \Delta(\mathcal{S}) \). Let \( \pi^K (\pi^\infty_\Theta (\omega)) \) denote the pushforward measure given the measure over the type space \( \pi^\Theta \) and the strategy \( \sigma^K (\sigma^\infty) \). \( \pi^K (\pi^\infty_\Theta (\omega)) \) denotes the distribution of human capital levels generated by the strategy \( \sigma^K (\sigma^\infty) \) when that strategy is played by all of the agents in the limit game. Since \( \sigma^K \to \sigma^\infty \) in the sup-norm, for any \( \delta > 0 \) we can choose \( K^* \) large enough that \( d_{LP}^A(\pi^\infty_\Theta, \pi^K_\Theta) < \delta \) for all \( K > K^* \). Where appropriate, we will delineate between the human capital levels of the minority and nonminority students with the associated subscript. Note that from lemma 1, \( \pi^K_{\infty, \mathcal{N}}, \pi^K_{\infty, \mathcal{M}}, \pi^K_{\infty, \mathcal{N}} \) and \( \pi^K_{\infty, \mathcal{M}} \) are elements of \( \Delta^R(\mathcal{S}) \).

Suppose our theorem is false and \( \sigma^\infty \) is not an exact equilibrium of the limit game. Then there exists a set \( \mathcal{D} \subset \Theta \) of measure \( \rho > 0 \) (under \( \pi^\Theta \)) such that for all \( \theta \in \mathcal{D} \), \( s^\infty \in \text{supp}[\sigma^\infty(\theta)] \) and some \( \varepsilon > 0 \)

\[
\Pi(s^\infty, \theta; \pi^\infty_{\mathcal{N}}, \pi^\infty_{\mathcal{M}}) + \varepsilon < \sup_{s \in \mathcal{S}} \Pi(s, \theta; \pi^\infty_{\mathcal{N}}, \pi^\infty_{\mathcal{M}})
\]  

(17)
Since \( \{\Pi(s, \theta; \pi_N, \pi_M)\}_{\theta \in \Theta} \) is uniformly equicontinuous in the weak-* topology over \( \Delta^R(S) \), we have that for any \( \varepsilon > 0 \) we can choose \( K \) sufficiently large that

\[
\sup_{s \in S} \Pi(s, \theta; \pi^K_N, \pi^K_M) + \varepsilon \geq \sup_{s \in S} \Pi(s, \theta; \pi^\infty_N, \pi^\infty_M)
\]  

(18)

**Lemma 4.** For any \( \varepsilon > 0 \) there exists \( K^* \) such that for all \( K > K^* \), any exact equilibrium of the \( K \)-agent game is an \( \varepsilon \)-approximate equilibrium of the limit game.

**Proof.** Combining the convergence of utility functions (lemma 2) and the continuity of the utility function in the limit game (lemma 3), we can appeal to theorem 9 of Bodoh-Creed [4] for our result.

From lemma 4 for any \( \varepsilon > 0 \) there is a sufficiently large \( K \) such that for any \( s^*_K \in \text{supp}[\sigma^K(\theta)] \)

\[
\Pi(s^*_K, \theta; \pi^K_N, \pi^K_M) + \varepsilon \geq \sup_{s \in S} \Pi(s, \theta; \pi^K_N, \pi^K_M)
\]  

(19)

But from the uniform equicontinuity of \( \{\Pi(s, \theta; \pi^\infty_N, \pi^\infty_M)\}_{\theta \in \Theta} \) and \( \sigma^K \to \sigma^\infty \) we have for large enough \( K \) that for any \( s^*_\infty \in \text{supp}[\sigma^\infty(\theta)] \)

\[
\varepsilon \geq \left\| \Pi(s^*_K, \theta; \pi^K_N, \pi^K_M) - \Pi(s^*_\infty, \theta; \pi^\infty_N, \pi^\infty_M) \right\|
\]  

(20)

But this entails

\[
\Pi(s^*_\infty, \theta; \pi^\infty_N, \pi^\infty_M) + 3\varepsilon \geq \Pi(s^*_K, \theta; \pi^K_N, \pi^K_M) + 2\varepsilon \geq \sup_{s \in S} \Pi(s, \theta; \pi^K_N, \pi^K_M) + \varepsilon
\]  

(21)

Since we can choose \( \varepsilon \) arbitrarily small (in particular \( 3\varepsilon < \varepsilon \)) by selecting \( N \) sufficiently large, this contradicts the assertion that \( \sigma^\infty \) is not an exact equilibrium of the limit game. Since this holds for all such sequences of strategies, we have from Theorem 17.16 of Aliprantis and Border [3] that the equilibrium correspondence is upper hemicontinuous.

Our claim could still fail if there were equilibria of the limit game that were not close to any equilibrium of arbitrarily large finite game. In other words, the argument might fail if the equilibrium correspondence were not lower hemicontinuous in \( K \). However, the equilibrium of the limit game is uniquely defined by the differential equations described in section 4.2. Since there is a unique equilibrium of the limit game, it must be the case that the sequence \( \{\sigma^K : \Theta \to \Delta(A)\}_{K=1}^\infty \) converges in the sup-norm to the unique equilibrium of the limit game.
A.2 Remaining Proofs

The remaining results in the paper are produced roughly in the order they appear in the main body.

**Theorem 1.** In the college admissions game \( \Gamma(K_M, F_M, K_N, F_N, P_K, r, U, C, S) \) with \( r \in \{cb, q, ap\} \), under assumptions [1, Theorem 3] there exists a symmetric pure-strategy equilibrium \((\sigma_M^r(\theta), \sigma_N^r(\theta))\).

**Proof.** Existence and monotonicity is a straightforward application of Athey [1, Theorem 3] who establishes these conditions in a general class of auction-related games to which our model belongs. The relation between the grade distributions and the achievement functions follows immediately from the fact that achievement is a strictly decreasing function of private cost types.

**Theorem 2.** HC investment for all types in the color-blind admissions scheme exceeds the full-information outcome where the social planner observes costs, assortatively matches seats to \( \theta \)'s, and investment happens ex-post.

**Proof.** Let \( \sigma^{cb} \) be the equilibrium strategy under color-blind admissions and \( \sigma^{FI} \) be the ex-post investment strategy in the full-information benchmark. The boundary condition for both problems is the same:

\[
\sigma^{cb}(\bar{\theta}) = \sigma^{FI}(\bar{\theta}) = \bar{s}
\]

Since the first order condition must hold at \( \theta = \bar{\theta} \) in the color-blind scheme and the full-information benchmark, we have

\[
\frac{\partial U(P(s), \bar{s}, \bar{\theta})}{\partial p} \cdot \frac{dP(s)}{ds} + \frac{\partial U(P(s), \bar{s}, \bar{\theta})}{\partial s} = \frac{\partial U(P(s), \bar{s}, \bar{\theta})}{\partial s} = \frac{dC(s; \bar{\theta})}{ds}
\]

This can only hold if \( \frac{dP(s)}{ds} = 0 \), which requires \( \frac{\partial \sigma^{cb}(\theta)}{\partial \theta} \to -\infty \) as \( \theta \to \bar{\theta} \). Since the derivative of \( \sigma^{FI} \) is bounded from below, we know for an interval of \( \theta \) in the neighborhood of \( \bar{\theta} \) that \( \sigma^{cb}(\theta) \geq \sigma^{FI}(\theta) \) where the inequality is strict for \( \theta \neq \bar{\theta} \) in that neighborhood.

Now assume that \( \sigma^{cb}(\theta^*) = \sigma^{FI}(\theta^*) \) for some \( \theta < \bar{\theta} \). In this case, we have, as per the argument above, that \( \frac{dP(s)}{ds} = 0 \) and so the derivative of \( \sigma^{cb} \) becomes unbounded as \( \theta \to \theta^* \), which implies \( \sigma^{cb}(\theta) > \sigma^{FI}(\theta) \) for \( \theta \) sufficiently close to, but less than, \( \theta^* \). Therefore, it cannot be the case that \( \sigma^{cb} \) and \( \sigma^{FI} \) ever cross - \( \sigma^{cb} \) is always weakly greater than \( \sigma^{FI} \).

**Theorem 3.** A unique Nash equilibrium exists in the limit model in the following cases

1. The limit game of the admissions preference model with a differential markup function \( \tilde{S} \) that satisfied \( \tilde{S}(\bar{s}) = \bar{s} \) and \( F_P \) has full support.
2. The limit game of the quota system with any feasible choice of $Q_M$ and $Q_M$.

Proof. First consider case and let $\sigma_i^{ap}$ be defined by equation and is therefore continuous. Standard results on differential equations imply that $\sigma_i^{ap}$ is uniquely defined. First note that equation can be treated the game as a direct mechanism wherein the agent’s problem is to choose a declared type $\tilde{\theta}$ where $i = M, N$

$$\max_{\tilde{\theta}} U(P_i^{ap}(\tilde{\theta}), \sigma_i^{ap}(\tilde{\theta}), \theta) - C(\sigma_i^{ap}(\tilde{\theta}), \theta)$$

and computing the first order conditions with respect to $\tilde{\theta}$ and recognizing that in equilibrium $P_i^{ap}(\tilde{\theta}) = P_i^{ap}(\sigma_i^{ap}(\tilde{\theta}))$. This derivation reveals that the local incentive compatibility conditions are already “built in” to the differential equations. It remains to prove that local incentive compatibility implies global incentive compatibility. Suppose $\tilde{\theta} < \theta$ and for compactness let $P_i^{ap}(\tilde{\theta}) = \tilde{p} > p = P_i^{ap}(\theta)$ and $\sigma_i^{ap}(\tilde{\theta}) = \tilde{s} > s = \sigma_i^{ap}(\theta)$. We then have from local incentive compatibility that

$$U_p(p, s, \theta) \frac{dP_i^{ap}(\tilde{\theta})}{\tilde{\theta}} + U_s(p, s, \theta) \frac{d\sigma_i^{ap}(\tilde{\theta})}{\tilde{\theta}} = C_s(s, \theta) \frac{d\sigma_i^{ap}(\tilde{\theta})}{\tilde{\theta}}$$

Since $\tilde{\theta} < \theta$, we have from assumption that $U_p(p, s, \tilde{\theta}) > U_p(p, s, \theta)$, and from assumption that $U_s(p, s, \tilde{\theta}) > U_s(p, s, \theta)$ and $C_s(s, \tilde{\theta}) < C_s(s, \theta)$. The value of the first order condition that results if type $\tilde{\theta}$ deviated from truthfulness upwards by declaring $\tilde{\theta} = \theta$ is

$$U_p(p, s, \tilde{\theta}) \frac{dP_i^{ap}(\tilde{\theta})}{\tilde{\theta}} + U_s(p, s, \tilde{\theta}) \frac{d\sigma_i^{ap}(\tilde{\theta})}{\tilde{\theta}} > C_s(s, \tilde{\theta}) \frac{d\sigma_i^{ap}(\tilde{\theta})}{\tilde{\theta}}$$

which, since there is an inequality, implies declaring $\tilde{\theta} = \theta$ cannot be optimal if the true type is $\tilde{\theta}$. Similar arguments imply that deviating downward from truthfulness also cannot be optimal.

The above argument assumes a direct mechanism structure, and as a final step we must rule out cases where it might be optimal for the agent to choose a human capital level outside of the range of $\sigma_i^{ap}$. For example, consider $\tilde{s} > s = \sigma_i^{ap}(\tilde{\theta})$. Let $p = P_i^{ap}(\tilde{\theta})$. We know from our first order conditions that

$$U_p(p, s, \theta) \frac{dP_i^{ap}(\tilde{\theta})}{\tilde{\theta}} + U_s(p, s, \theta) \frac{d\sigma_i^{ap}(\tilde{\theta})}{\tilde{\theta}} = C_s(s, \theta) \frac{d\sigma_i^{ap}(\tilde{\theta})}{\tilde{\theta}}$$

\[30\] We have written the first order condition with the derivatives defined using limits from the right (i.e., using sequences contained in $\Theta$).
Increasing from $s$ to $\bar{s}$ does not change the school $p$, and so it can only be optimal if

$$U_s(p, \bar{s}, \theta) = C_s(\bar{s}, \theta)$$  \hfill (23)

Since $U_{ss} < 0$, $C_{ss} > 0$, and $U_s(p, s, \theta) \leq C_s(s, \theta)$, equation (23) cannot hold, which means deviating to $\bar{s}$ cannot be optimal.

Now consider case 2 and assume that $\sigma^q_i$ be defined by equation 7 over the intervals where $Q_i$ has support. Again, by standard results in differential equations, if $[p, p']$ is an interval where $Q_i$ has support and type $\theta$ is assigned to college $p$ with human capital level $\sigma^q_i(\theta) = s$, then the strategy is uniquely defined by equation 7 for all types that are (in equilibrium) assigned a seat at a college in $[p, p']$. The only way there could exist two equilibria $\sigma^q_i$ and $\tilde{\sigma}^q_i(\theta)$ is if there is some discontinuity at $\theta$ shared by both strategies such that

$$\lim_{\varepsilon \to 0^+} \sigma^q_i(\theta + \varepsilon) \neq \lim_{\varepsilon \to 0^+} \tilde{\sigma}^q_i(\theta + \varepsilon)$$

In other words, the jump over an interval in which $Q_i$ lacks support is not uniquely defined. We prove that the first such jump in the strategy must be uniquely defined. An (omitted) induction step using essentially the same structure can be used to prove that all of the jumps must be uniquely defined.

Suppose that $Q_i$ lacks support over the interval $[p, p']$ and $Q_i$ has full support over both $[p, p]$ and $[p', p' + \delta]$ for some $\delta > 0$. $\theta$ satisfy $P^q_i(\sigma^q_i(\theta)) = p$ - in other words, $\theta$ is where the first jump must occur. In equilibrium it must be the case that $\theta$ is indifferent about whether to make this jump, so

$$U(p, s, \theta) - C(\sigma^q_i(\theta), \theta) = U(p', s', \theta) - C(s', \theta)$$

where $s = \sigma^q_i(\theta)$ and $s' = \lim_{\varepsilon \to 0^+} \sigma^q_i(\theta + \varepsilon)$. Suppose there was a second equilibrium $\tilde{\sigma}^q_i$ starting at type $\theta$ such that $s'' = \lim_{\varepsilon \to 0^+} \tilde{\sigma}^q_i(\theta + \varepsilon) > s'$. Then it must be the case that

$$U(p, s, \theta) - C(\sigma^q_i(\theta), \theta) = U(p', s', \theta) - C(s', \theta) = U(p', s'', \theta) - C(s'', \theta)$$

But since $U_{ss}$ is strictly concave and $C_{ss}$ is strictly convex, this cannot be true. Therefore $s'$ is uniquely defined.

**Theorem 5.** Consider some $P_i(s) : S \to P$, $i \in \mathcal{M,N}$. $P_i(s) : S \to P$, $i \in \mathcal{M,N}$ is the result of an equilibrium of some quota system if and only if there is an equilibrium of an admissions preference system that also yields these assignment functions and admits the same equilibrium strategies.
Proof. Suppose \( \overline{P}_i : \mathcal{S} \to \mathcal{P} \), \( i \in \mathcal{M}, \mathcal{N} \), is the result of an equilibrium of some quota system and denote the equilibrium strategies \( \sigma_i^\mathcal{N} : \Theta \to \mathcal{S} \). Since \( \overline{P}_i \) and \( \sigma_i^\mathcal{N} \) are strictly monotone, the functions are invertible. Let \( P^{ap}(s) \), the assignment function under admissions preferences for the nonminority students, be \( P^{ap}(s) = \overline{P}_i \). Since the assignment functions are the same for the nonminority students, \( P^{ap} \) and \( \overline{P}_i \) generate identical decision problems for the nonminorities. Therefore, if \( \sigma_i^\mathcal{N} \) was an equilibrium for nonminority students under \( P^{ap} \), then \( \sigma_i^{ap}(\theta) = \sigma_i^\mathcal{N}(\theta) \) will be an equilibrium for the nonminority students under \( P^{ap}(s) \).

To construct the outcome equivalent score function, let

\[
\tilde{S}(s) = (\overline{P}_i)^{-1}(P_M(s))
\]

A minority student who chooses human capital level \( s \) will then be assigned to college

\[
P^{ap}(\tilde{S}(s)) = P^{\mathcal{N}}_i(\tilde{S}(s)) = P^{\mathcal{N}}_i((\overline{P}_i)^{-1}(P_M(s))) = P^\mathcal{M}_i(s)
\]

Since the assignment functions are the same for the minority students, \( P^{ap} \) and \( P^\mathcal{M}_i \) generate identical decision problems for the minorities. Therefore, if \( \sigma^\mathcal{M}_i \) was an equilibrium for minority students under \( P^\mathcal{M}_i \), then \( \sigma^{ap}_M(\theta) = \sigma^\mathcal{M}_i(\theta) \) will be an equilibrium for the nonminority students under \( P^{ap}(s) \).

Now suppose \( P^{ap} : \mathcal{S} \to \mathcal{P} \) with score function \( \tilde{S} \) is the result of an equilibrium of some admissions preference system and denote the equilibrium strategies \( \sigma_i^{ap} : \Theta \to \mathcal{S}, \) \( i \in \mathcal{M}, \mathcal{N} \). To define the equivalent quota system, we need to define allocations of seats to each group. Let these distributions be denoted \( Q_i, i \in \mathcal{M}, \mathcal{N} \), and define them as

For all \( p \) let \( Q_M(p) = 1 - F_i \left[ \psi_i^{ap} \left( \tilde{S}^{-1} \left( (P^{ap})^{-1} (p) \right) \right) \right] \)

For all \( p \) let \( Q_N(p) = 1 - F_i \left[ \psi_i^{ap} \left( (P^{ap})^{-1} (p) \right) \right] \)

Note that the total measure of nonminority students choosing \( s \) and minority students choosing \( \tilde{S}^{-1}(s) \) under \( P^{ap} \) (in equilibrium is)

\[
1 - \mu F_M \left[ \psi_M^{ap} \left( \tilde{S}^{-1} \left( (P^{ap})^{-1} (p) \right) \right) \right] - (1 - \mu) F_N \left[ \psi_i^{ap} \left( (P^{ap})^{-1} (p) \right) \right] = \\
\mu Q_M(p) + (1 - \mu) Q_N(p) = F_P(p)
\]

which implies that under either system nonminority students choosing \( s \) and minority students choosing \( \tilde{S}^{-1}(s) \) is assigned college \( p \). Written more formally, \( P^\mathcal{N}_i(s) = P^{ap}(s) \) and \( P^\mathcal{M}_i(s) = P^{ap}(\tilde{S}(s)) \). As argued above, since the decision problems for the agents are the same, the equilibrium strategies in the original admissions preference scheme and the
B The Outcomes of Affirmative Action Under Pure Competition

Our goal in this section is to make to identify conditions under which encouragement and discouragement effects occur in our model. Discouragement effects describe situations wherein competition within a contest intensifies, which causes low ability competitors to exert less effort while at the same time high ability competitors work harder. Since affirmative action schemes (most notably admissions preference schemes) in effect make minority students better competitors, one might expect discouragement effects within the nonminority students. In addition, there will be a symmetric encouragement effect within the minority student population.

Our notion of changing competition is captures by examining changes in the distribution of the types of competitors that can be ordered using a single crossing condition applied to the density functions of the relevant distributions.

**Definition 4.** Two functions $f(\theta)$ and $g(\theta)$ satisfy the strict single crossing condition (SCC) when for any $\theta > \tilde{\theta}$ we have $f(\tilde{\theta}) \geq g(\tilde{\theta})$ implies $f(\theta) > g(\theta)$. We denote this relation $g \succ_{\text{SCC}} f$.

Keeping in mind that students with low values of $\theta$ choose higher levels of human capital, if $f_1$ and $f_2$ represent distributions of student types, then $f_1 \succ_{\text{SCC}} f_2$ implies that $f_1$ represents a stronger student population.

Now we turn to our analysis of the discouragement effect. We first consider a color-blind admission policy and analyze how students react to changes in the distributions of types of the other students. Our first result implies a form of the discouragement effect. When there are fewer high ability students (and hence more low ability competitors), the low ability students make greater investments in human capital.

**Theorem 6.** Consider two population cost distributions, $F_1(\theta)$ and $F_2(\theta)$, that have have full support over $[\theta, \bar{\theta}]$. Assume $f_1(\theta) \prec_{\text{SCC}} f_2(\theta)$. Under a color-blind admissions policy, there exists a point $\theta^*$ such that $\sigma_1^b(\theta) < \sigma_2^b(\theta)$ for each $\theta \in (\theta^*, \bar{\theta})$.

**Proof.** In the color-blind admissions scheme, we need not differentiate between the minority and nonminority students. Furthermore, I will drop the “cb” superscript for notational ease. Denote the equilibrium investment functions under $F_1$ and $F_2$ by $\sigma_1$ and $\sigma_2$, respectively. Note that since the distributions of types are different, the school allocated to each type in equilibrium will also depend on the distributions $F_1$ and $F_2$, and we denote the associated assignment functions as $P_1$ and $P_2$. 

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Using the notation of a revelation mechanism, the FOC is now
\[ U_p(P(\theta), \sigma_i(\theta), \theta)P'_i(\theta) + U_s(P(\theta), \sigma_i(\theta), \theta)\sigma'_i(\theta) = C_s(\sigma_i(\theta), \theta)\sigma'_i(\theta) \]

We can rewrite the first order condition as
\[ \sigma'_i(\theta) = -\frac{U_p(P_i(\theta), \sigma_i(\theta), \theta)f_i(\theta)}{f_P(1 - F_i^{-1}(\theta)) [C_s(\sigma_i(\theta), \theta) - U_s(P(\theta), \sigma_i(\theta), \theta)]} \]

For the lowest ability student, \( \theta = \bar{\theta} \), we can write
\[ \sigma'_i(\theta) = -\frac{U_p(P_i, \bar{s}, \bar{\theta})f_i(\bar{\theta})}{f_P(\bar{p}) [C_s(\bar{s}, \bar{\theta}) - U_s(P, \bar{s}, \bar{\theta})]} \tag{24} \]

where we use the fact that \( P(\bar{\theta}) = \bar{p} \), and we have the boundary condition \( \sigma_i(\bar{\theta}) = \bar{s} \) for \( i = 1, 2 \). Since for SCC to hold we must have \( f_1(\bar{\theta}) < f_2(\bar{\theta}) \), these facts and equation (24) imply \( \sigma'_2(\bar{\theta}) < \sigma'_1(\bar{\theta}) < 0 \). This in turn means that \( \sigma_1(\theta) < \sigma_2(\theta) \) within a neighborhood of \( \bar{\theta} \) by differentiability (recall also that \( \sigma_i \) is strictly decreasing and rises in the leftward direction from the boundary point).

Our second result provides condition that there is at most one crossing-point between \( \sigma_1^b \) and \( \sigma_2^b \). If there is no crossing point, then all of the students increase their human capital choices under \( F_1 \). If there is a crossing point, then we see both an encouragement and a discouragement effect. In other words, bad students choose higher levels of human capital, good students acquire less human capital, and the gap between the best and worst students contracts. Our proof proceeds by ordering the slope of the equilibrium strategy as function of the underlying primitives.

**Theorem 7.** Assume all of the conditions of theorem 6 hold. In addition we will assume that the distribution of seats is potentially different, and we delineate between the distributions using the notation \( F_{p,1} \) and \( F_{p,2} \). In addition, assume:

1. \( U_{ps} = 0 \)
2. \( \frac{U_p(F_{p,1,2}^{-1}(1 - F_i(\theta)), \bar{s}, \theta)f_i(\theta)}{f_{p,1,2}(1 - F_i(\theta))} \) and \( \frac{U_p(F_{p,1,2}^{-1}(1 - F_2(\theta)), \bar{s}, \theta)f_2(\theta)}{f_{p,1,2}(1 - F_2(\theta))} \) satisfy the strict single crossing condition

**Theorem 8.** Under a color-blind admissions policy, there exists at most a single point \( \theta^* \in (\bar{\theta}, \bar{\theta}) \) such that \( \sigma_1^b(\theta) < \sigma_2^b(\theta) \) for each \( \theta \in (\theta^*, \bar{\theta}) \) and \( \sigma_1^b(\theta) < \sigma_2^b(\theta) \) for each \( \theta \in [\bar{\theta}, \theta^*) \).

**Proof.** The first part of our result follows the proof of theorem 6 and is omitted. What remains is to show that the point \( \theta^* \), if it exists, must be unique.
Suppose there are more than two points where \( \sigma_1(\theta) = \sigma_2(\theta) \) and let \( \theta^* \) denote the largest such point and \( \theta^{**} \) the second largest. Let \( \sigma_1(\theta^*) = \sigma_2(\theta^*) = s^* \) and \( p_i^* = P_i(\theta^*) = F_{P_i}(1 - F_i(\theta^*)) \). Since \( \sigma_1 \) must cross \( \sigma_2 \) from above (recall, \( \sigma_1(\theta) < \sigma_2(\theta) \) in \((\theta^*, \bar{\theta})\)), the former must have a more negative. Combining these facts yields

\[
\sigma'_1(\theta^*) = -\frac{U_p(p^*_1, s^*, \theta^*) f_1(\theta^*)}{f_{P,1}(p^*_1) [C(s^*, \theta^*) - U_s(p^*_1, s^*, \theta^*)]} < -\frac{U_p(p^*_2, s^*, \theta^*) f_2(\theta^*)}{f_{P,2}(p^*_2) [C(s^*, \theta^*) - U_s(p^*_2, s^*, \theta^*)]} = \sigma'_2(\theta^*)
\]

Note that since \( U_{ps} = 0 \) we have \( C_s(s^*, \theta^*) - U_s(p^*_1, s^*, \theta^*) = C_s(s^*, \theta^*) - U_s(p^*_2, s^*, \theta^*) \). Therefore, it must be the case that

\[
\frac{U_p(p^*_1, s^*, \theta^*) f_1(\theta^*)}{f_{P,1}(p^*_1)} > \frac{U_p(p^*_2, s^*, \theta^*) f_2(\theta^*)}{f_{P,2}(p^*_2)} \tag{25}
\]

Now we will argue from contradiction that \( \sigma_1 \) and \( \sigma_2 \) can cross at most once. Suppose there were a second point \( \theta^{**} < \theta^* \) such that \( \sigma_1(\theta^{**}) = \sigma_2(\theta^{**}) \). In this case, we would have \( \sigma_1 \) crossing \( \sigma_2 \) from below, which would imply \( \sigma_1 \) is less steep than \( \sigma_2 \), which would require

\[
\sigma'_1(\theta^{**}) = -\frac{U_p(p^{**}, s^{**}, \theta^{**}) f_1(\theta^{**})}{f_{P,1}(p^{**}) [C_s(s^{**}, \theta^{**}) - U_s(p^{**}, s^{**}, \theta^{**})]} > -\frac{U_p(p^{**}, s^{**}, \theta^{**}) f_2(\theta^{**})}{f_{P,2}(p^{**}) [C_s(s^{**}, \theta^{**}) - U_s(p^{**}, s^{**}, \theta^{**})]} = \sigma'_2(\theta^{**})
\]

where \( p^{**} \) and \( s^{**} \) are defined analogously. Again, assumption 1 implies \( C_s(s^{**}, \theta^{**}) - U_s(p^{**}, s^{**}, \theta^{**}) = C_s(s^{**}, \theta^{**}) - U_s(p^{**}, s^{**}, \theta^{**}) \). For \( \sigma'_1(\theta^{**}) > \sigma'_2(\theta^{**}) \) to be true, it must then be the case that

\[
\frac{U_p(p^{**}, s^{**}, \theta^{**}) f_1(\theta^{**})}{f_{P,1}(p^{**})} < \frac{U_p(p^{**}, s^{**}, \theta^{**}) f_2(\theta^{**})}{f_{P,2}(p^{**})}
\]

But this is contradicted by assumption 2. Therefore, \( \sigma_1 \) and \( \sigma_2 \) can cross at most once. \( \square \)

Theorem 7 implies that when faced with competing against a more capable student population, low ability students (i.e., students with high values of \( \theta \)) will reduce their human capital investment. This is an example of the discouragement effect which is a common feature of the contests literature: when a given cost type \( \theta \) falls far enough behind in the sense that the mass of lower-cost competitors \( F_2(\theta) \) is large, his incentives for investing in costly effort become weak. If there is a crossing point \( \theta^* \), then we also find an encouragement effect that causes low-cost individuals (\( \theta \) sufficiently low) to invest more aggressively to keep the larger mass of their like opponents at bay. This means that an increase in the degree of competition will increase the disparity in achievement between top students and the ones at the bottom.
Theorem 7 may also be particularly relevant in light of recent developments in US higher education. It is well known that the number of foreign students seeking post-secondary education in the United States has been steadily on the rise for several decades. According to a press release by the Institute of International Education, between 2000 and 2010 the number of foreign students enrolled rose by 32% to 723,277.\footnote{http://www.iie.org/en/Who-We-Are/News-and-Events/Press-Center/Press-Releases/2011/2011-11-14-Open-Doors-International-Students} If the set of foreign students entering the American higher education market are more competitive than the average domestic student—say, due to selection induced by travel costs or higher tuition fees than in their countries of origin, for example—so that the type distribution takes a dominance shift, then the model predicts that the increased competition will spur on the best and brightest domestic students to greater levels of achievement, while having the opposite effect on higher-cost domestic students.

Another interesting aspect of the theorem is that the crossing point for $\sigma_1$ and $\sigma_2$ must occur to the left of the crossing point for $f_1$ and $f_2$. If something is known about the distributions, then this fact can be used to bound the masses of students who increase or decrease investment, without computing the equilibrium, as in the following example.

**Example 1.** Assume $U(p,s,\theta) = p$, let $F_2$ be an arbitrary distribution $F(\theta)$ satisfying Assumption\footnote{Although $f_1(\theta) = 0$ now, it is still strictly positive at the upper end of the support, so the logic of the proof for Theorem 6 still holds.} above, with density $f(\theta)$, and let $F_1 = F(\theta)^\tau$, $\tau > 1$. It is easy to verify that $F_1$ LRD $F_2$, they both share a common support, and $f_1(\theta) = \tau F(\theta)^{\tau-1} f(\theta)$\footnote{The interior crossing point of the densities is now defined by $\tau F(\tilde{\theta})^{\tau-1} f(\tilde{\theta}) = f(\tilde{\theta})$, which implies $F(\tilde{\theta}) = \left(\frac{1}{\tau}\right)^{\frac{1}{1-\tau}}$. Therefore, Theorem 7 shows that the function $b(\tau) = (1/\tau)^{1/(\tau-1)}$ provides a lower bound (upper bound) on the mass of students who decrease (increase) investment under $F_2$, relative to $F_1$. For example, if $\tau = 2$ then at least 50% of students will decrease investment. Moreover, as $\tau \to \infty$, the degree of dominance of $F_1$ over $F_2$ becomes very severe, and we have $\lim_{\tau \to \infty} b(\tau) = 1$, so a mass of students approaching one decrease investment.} The interior crossing point of the densities is now defined by $\tau F(\tilde{\theta})^{\tau-1} f(\tilde{\theta}) = f(\tilde{\theta})$, which implies $F(\tilde{\theta}) = \left(\frac{1}{\tau}\right)^{\frac{1}{1-\tau}}$.

Therefore, Theorem 7 can also tell us something about the incentive effects induced by a change in allocation mechanism. Given the assumption that $f_N <_{LRD} f_M$, instituting a quota in place of color-blind admissions is like putting each race group into a new competition where the cost distributions take an LRD shift. To see why, note that $f_N <_{LRD} f_M$ implies $f_N <_{LRD} f_K <_{LRD} f_M$ where $F_K(\theta) = \mu F_M(\theta) + (1 - \mu) F_N(\theta)$ denotes the population cost distribution. Of course, theorem\footnote{Theorem 7 requires some assumptions on the combination of $U_p$, $f_P$, and $f_i$. The following theorem provides one example of how these requirements could be jointly satisfied to yield results based on our comparative statics.} requires some assumptions on the combination of $U_p$, $f_P$, and $f_i$. The following theorem provides one example of how these requirements could be jointly satisfied to yield results based on our comparative statics.
We assume in the following theorem that the minority and nonminority students are subject to a representative quota, which means that the minority and nonminority students are assigned the same distributions of seats. In effect, this quota demands that the population of students at each school be representative of the population at large.

**Theorem 9.** Assume a proportional quota is used, \( U(p, s, \theta) = p, f_p = 1 \), and \( f_N \) and \( f_M \) satisfy the strict single crossing condition with \( f_M(\overline{\theta}) > f_N(\overline{\theta}) \). Then there exist \( \theta^*_M, \theta^*_N, \theta^*_{MN} \in (\underline{\theta}, \overline{\theta}) \) such that

- \( \sigma^q_M(\theta) < (>) \sigma^c(\theta) \) for each \( \theta < (>) \theta^*_M; \)
- \( \sigma^q_N(\theta) > (>) \sigma^c(\theta) \) for each \( \theta < (>) \theta^*_N; \) and
- \( \sigma^q_N(\theta) > (>) \sigma^q_M(\theta) \) for each \( \theta < (>) \theta^*_{MN}. \)

**Proof.** We need only prove that the conditions of theorem \( \square \) hold. The first condition holds by assumption. The LRD condition requires simply \( f_N <_{LRD} f_M \). Since this is assumed to hold, theorem \( \square \) yields the desired result. \( \square \)

In words, the policy shift will induce the best and brightest minority students to decrease achievement, while the best and brightest non-minorities will invest more aggressively, relative to color-blind incentives. One can think of this effect as coming from a shift in the severity of competition as explained above: for the former group the distribution of competition becomes less intense, taking a LRD shift under a quota, but for the latter the cost distribution is now dominated by the one they face in the color-blind world. At some point, however, there is a cutoff above which discouragement effects become amplified for high-cost non-minorities and mitigated among high-cost minorities, because a given type \((\theta, N)\) now faces a larger mass of lower-cost competitors under a quota, whereas the opposite is true for \((\theta, M)\).

Alternatively these effects can also be thought of as arising from a change in the supply of available seats of different varieties. Specifically, the set of top seats available to non-minority students is restricted under a quota, so with an increase in scarcity, the effective equilibrium market price of \( s = \sigma_N(\theta) \) naturally rises, with the opposite being the case for minorities. Similarly, there is an increase in the supply of low-quality seats allocated to non-minorities, and a corresponding decrease for minorities, producing the opposite effects for high-cost individuals.

Finally, Theorem \( \square \) can also tell us something about how different admissions policies can affect inequality in the stock of HC.\(^{33}\) Part (iii) demonstrates how achieving more

\(^{33}\)As it pertains to racial inequality, this phenomenon, dubbed the Black-White Test Score Gap, has received considerable attention from social scientists. See Jencks and Phillips \(^{12}\) for an in-depth discussion.
racially diverse college campuses may come at the cost of exacerbating racial inequality in the stock of high HC: whereas the lowest cost types from both groups, i.e., \( \theta \leq \theta^*_{MN} \), invest the same under color-blind admissions, there now appears an achievement gap among a minority and a non-minority of the same cost type under a quota. On the other hand, the reverse happens for high cost types: a quota diminishes average achievement inequality partially through higher investment by minorities, and partially through diminished investment by non-minorities, conditional on a given type. Interestingly, part (i) implies that a quota decreases the level of inequality within the minority group, and part (ii) shows that, for non-minorities, a quota will have the effect of increasing inequality.

C Identifying Discontinuities in the Equilibrium

In this appendix we briefly describe how to identify discontinuities in the equilibrium. This section will be of practical interest primarily to practitioners who wish to use equations 7 and 10 to compute equilibria numerically.

First let us consider quota schemes, where jumps are caused by gaps in the support of \( Q_j \). The size of the jump must make the types on the edge of the gap indifferent about making the jump. Formally written, suppose \([p_L, p_U]\) is an interval such that \( Q_j([p_L, p_U]) = 0 \) and for all \( \epsilon > 0 \) we have \( Q_j([p_L - \epsilon, p_U]) > 0 \) and \( Q_j([p_L, p_U + \epsilon]) > 0 \). Let \( \theta \) be such that \( P(\sigma^q_j(\theta)) = p_L \). Then it must be that for \( s = \limsup_{\theta' \to \theta} \sigma^q_j(\theta) \) (i.e., the human capital choice on the other side of the jump) we have

\[
U(p_L, \sigma^q_j(\theta), \theta) - C(\sigma^q_j(\theta), \theta) = U(p_U, s, \theta) - C(s, \theta)
\]

Now we discuss how to identify gaps in an admissions preference scheme. Gaps in an admissions preference system are caused by kinks or discontinuities in \( \tilde{S} \). When these issues arise, the marginal incentives for both groups will change. We handle each possible issue in turn.

First consider a kink in \( \tilde{S} \) at \( s \) such that \( \frac{d}{ds} \tilde{S} \) jumps at \( s \). Without loss of generality, assume that the strategies are lower semicontinuous \(^{34}\) and let \( \theta_N = \psi_N^q(\tilde{S}(s)) \) and \( \theta_M = \psi_M^q(\tilde{S}(s)) \), and consider the first order conditions that would have to hold at \( s \) if the strategies

\(^{34}\)Since the edges of these jumps are described by an indifference condition, we could just as easily construct an upper semicontinuous equilibrium.
are continuous

For $i = N$, $U_p(P_{ap}(\overline{S}(s'), \overline{S}(s'), \theta_N)) \left. \frac{dP_{ap}(s)}{ds} \right|_{s=\overline{S}(s')} + U_s(P_{ap}(\overline{S}(s'), \overline{S}(s'), \theta) = C_s(\overline{S}(s'), \theta)$

For $i = M$, $U_p(P_{ap}(\overline{S}(s'), s', \theta) \left. \frac{dP_{ap}(s)}{ds} \right|_{s=\overline{S}(s')} + U_s(P_{ap}(\overline{S}(s'), s, \theta) = C_s(\overline{S}(s'), \theta)$

For both group’s strategies to be continuous, we would need the first order conditions across the discontinuity in $\frac{d}{ds} \overline{S}$ to be continuous. In other words, we would require

$$\frac{dP_{ap}(s)}{ds} = \frac{dP_{ap}(s)}{ds} \frac{d\overline{S}(s)}{ds}$$

which is clearly impossible at the discontinuity in $\frac{d\overline{S}(s)}{ds}$.

To resolve this problem, one of the groups must jump. We will construct an equilibrium where the minority students jump, but the construction and the test for the validity of the construction is symmetric in the case where the nonminority students jump. If the minority student strategy exhibits a jump, then it must be the case that the first order condition for the nonminority students is smooth across the discontinuity, so $\frac{dP_{ap}(s)}{ds}$ must be continuous. Although convoluted, we write the equation for $\frac{dP_{ap}(s)}{ds}$ below for clarity:

$$\frac{dP_{ap}(s)}{ds} = -\frac{\Phi}{f_P(1-\Phi)} \text{ where } \Phi = \frac{(1-\mu)f_N(\psi_{ap}(s))}{(\sigma_{ap}^N)'(s)} + \mu f_M(\sigma_{ap}^M(\overline{S}^{-1}(s))) \frac{(\sigma_{ap}^M)'(s)\overline{S}'(s)}{(\sigma_{ap}^N)'(s)\overline{S}'(s)}$$

and we let $(\sigma_{ap}^i)'$ is infinity if group $i$ has jumped across that level of human capital. In other words, if the minority student’s jump, it must be that $(\sigma_{ap}^N)'(s)$ drops discontinuously to keep $\frac{dP_{ap}(s)}{ds}$ constant at $s$. For the duration of the minority student jump, we can use the differential equations [10] to describe the nonminority student strategy. This construction is successful if in the gap we have

$$U_p(P_{ap}(\overline{S}(s)), s, \theta) \frac{dP_{ap}(s)}{ds} \frac{d\overline{S}(s)}{ds} + U_s(P_{ap}(\overline{S}(s)), s, \theta) \geq C_s(s, \theta)$$

If the inequality is reversed, then it must be the case that nonminority student jump and minority students do not. Finally, we need to define the size of the jump in the minority student strategy. Suppose the minority strategy is lower semicontinuous and we have $\sigma_{ap}^M(\theta) = s$ (i.e., $\theta$ is the type of minority student that jumps). To define the jump we
need to compute $s'$ such that

$$U(P^{ap}(\tilde{S}(s)), s, \theta) - C(s, \theta) = U(P^{ap}(\tilde{S}(s')), s', \theta) - C(s', \theta)$$

and let the minority strategy jump to $s'$ at $\theta$. Again, it will often be the case that the first order conditions for the two groups will not align and

$$\frac{dP^{ap}(s)}{ds} \neq \frac{dP^{ap}(s)}{ds} \frac{d\tilde{S}(s)}{ds}$$

If this occurs, it is treated as noted above. It is easy to construct examples where the groups repeatedly jump and never compete at the same college. For example, if $\tilde{S}(s) = s + \Delta, \Delta > 0$, the equilibrium has this structure.

Second, assume that there is a discontinuity in $\tilde{S}$ at human capital level $s$. Since we have assumed $\tilde{S}$ is increasing, $\tilde{S}$ must jump upwards. Let $\sigma^q_j(\theta) = s$ and assume $\sigma^q_j$ is lower semicontinuous at $\theta$.\(^{35}\) In this case, there must be a jump in the equilibrium strategy of the minority students that is defined by

$$U(P^{ap}(\tilde{S}(s)), s, \theta) - C(s, \theta) = U(P^{ap}(\tilde{S}(s')), s', \theta) - C(s', \theta)$$

and we let the value of the minority student strategy jump to $s'$ at $\theta$. Again, if the first order conditions cannot both line up, then we will have to allow one of the groups’ strategies to jump again, which requires the construction techniques outlined above.

\(^{35}\)The construction would be essentially the same if we chose to let $\sigma^q_j$ be upper semicontinuous at $\theta$.\(^{60}\)