Distributional Perfect Equilibrium in Bayesian Games with Applications to Auctions

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Abstract

In second-price auctions with interdependent values, bidders do not necessarily have dominant strategies. Moreover, such auctions may have many equilibria. To use the concept of trembling hand perfect equilibrium as a tool to rule out the less intuitive equilibria, we define the notion of distributional perfect equilibrium for Bayesian games with infinite type and action spaces. We prove that every Bayesian game has a distributional perfect equilibrium if the information structure of the game is absolutely continuous and the payoffs are equicontinuous. We apply distributional perfection to a class of symmetric second-price auctions with interdependent values and observe that a specific type of equilibrium is perfect, while many of less intuitive equilibria are not.

JEL Codes. C72.

Keywords. Trembling hand perfect equilibrium, Bayesian games with infinite type and action spaces, Second-price auctions with interdependent values.

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1 Introduction

In private value second-price auctions each bidder has a dominant strategy in which he bids his own valuation of the object. However, in second-price auctions with interdependent values bidders might not have any dominant strategy while there may exist many equilibria in undominated strategies. Multiplicity of equilibria in auction models is studied in Milgrom [21], Birulin [10], and Bikhchandani et al. [7], [8]. The approaches taken in these papers to distinguish between multiple equilibria are mostly “no ex-post regret” and “ex-post efficiency”. Birulin [10] shows that “no ex-post regret” criterion fails to rule out many non-intuitive equilibria. Moreover, although efficiency is helpful to differentiate between multiple equilibria, it does not suggest a strategic behavior from bidders’ point of view. Therefore, a selection tool is desired in such Bayesian games to rule out the less intuitive equilibria. In this paper we develop the concept of trembling hand perfect equilibrium for Bayesian games with infinite type and action spaces as a refinement of Bayesian Nash equilibrium.

Simon and Stinchcombe in [25] extend trembling hand perfect equilibrium to games with infinite action spaces for normal form games in two main approaches. The first approach is a direct generalization of Selten’s original definition, and it is based on the notion of completely mixed strategy, i.e. a strategy which assigns a positive probability to each nonempty open set. This approach gives rise to the definitions of weak perfect and strong perfect equilibrium. The second approach, referred to as the finitistic approach, uses the notion of an $\varepsilon$-perfect equilibrium in finite approximations of the original game. Bajoori et al. [4] examined these two approaches in further detail and provided an improved definition of the finitistic approach. More importantly, however, their results implies a severe critique on the finitistic approach altogether. Therefore, in this paper our definition of perfection for Bayesian games is in line with the first approach in Simon and Stinchcombe [25].

We consider a Bayesian game with $n$ players, where each player’s type space is a separable metric space. After receiving the types, players simultaneously choose an action from a compact metric action space, then each player gets a payoff that may depend on all the types and all the actions chosen by the players. Following Milgrom and Weber [22], we assume that the prior probability measure on the product of the type spaces is absolutely continuous with respect to the product measure of its marginal probabilities,
also we assume that the payoffs are equicontinuous. Milgrom and Weber in [22] prove the existence of an equilibrium in distributional strategies for such Bayesian games.¹ As it is not clear how players would play the game when using their distributional strategies, we define a similar equilibrium concept in behavioral strategies: A behavioral strategy profile is called “distributional equilibrium” if its induced distributional strategy profile is an equilibrium.

To extend the concept of trembling hand perfect equilibrium to Bayesian games with infinite action and type spaces, we define “distributional perfect equilibrium” as a behavioral strategy profile such that for its induced distributional strategy profile the conditions of trembling hand perfect equilibrium are satisfied. We prove that in the Bayesian game described above distributional perfection is a refinement of distributional equilibrium. Furthermore, we show the existence of distributional perfect equilibrium and that the set of distributional perfect equilibria is sequentially closed in the Tychonoff topology when taking the strong topology on the set of probability measures on action spaces.

Note that in finite games perfect equilibrium assigns no mass to weakly dominated strategies. However, this property is not compatible for infinite games, as Example 2.1 by Simon and Stinchcombe in [25] presents a game with infinite action spaces in which the unique equilibrium of the game is in weakly dominated strategies. Instead, the weaker property of limit undominatedness² is used for strategies in infinite games which is somewhat stronger than limit admissibility introduced in Simon and Stinchcombe [25]. We show that a distributional perfect equilibrium is in limit undominated strategies. We also discuss perfection in finite Bayesian games, to see that distributional perfection is equivalent to perfect equilibrium in finite Bayesian games defined in Bajoori et al. [5], if every player receives each type from his type space with strictly positive probability.

As we mentioned before, one of the applications of distributional perfection is in auctions with interdependent values, because such auctions may have many equilibria, even in undominated strategies. We study symmetric second-price auctions with interdependent values for two bidders where the valuation function of each bidder is strictly increas-

¹There are many other studies on the existence of equilibrium in Bayesian games, for example: Athey [2], Mcadams [17], Reny [23], Jackson et al. [14],[15], Mertens [20], Meirowitz [18], Mallozzi et al. [19], Kim and Yannelis [26], Zandt and Vives [28], Balder [6].

²This property was introduced in Bajoori et al. [4].
ing in his own type. We use distributional perfection to get a smaller set of equilibria and rule out the strange and the less intuitive ones. To explain this idea consider the following example.

**Example 1.1** Consider a 2-bidder second-price auction with interdependent values $v_1 = t_1 + \frac{1}{2} t_2$ and $v_2 = \frac{1}{2} t_1 + t_2$, where the types $t_1, t_2$ for bidder 1 and 2 respectively, are drawn independently from $[0, 1]$ according to the uniform distribution. Then, each bidder simultaneously submits a bid from the set $[0, \frac{3}{2}]$. The strategy profile $(b_1, b_2)$ in which $b_i = \frac{3}{2} t_i$, for every bidder $i = 1, 2$, is a Bayesian Nash equilibrium. However, there are many other equilibria in this auction. As an example consider the following class of equilibria. For every $s_1, s_2 \in [0, 1]$ where $s_1 < s_2$:

\[
\hat{b}_1(t_1) = \begin{cases} 
    s_2 + \frac{1}{2} t_1 & \text{if } t_1 \in [s_1, s_2] \\
    \frac{3}{2} t_1 & \text{otherwise,}
\end{cases}
\]

\[
\hat{b}_2(t_2) = \begin{cases} 
    s_1 + \frac{1}{2} t_2 & \text{if } t_2 \in [s_1, s_2] \\
    \frac{3}{2} t_2 & \text{otherwise.}
\end{cases}
\]

This class of equilibria is depicted in Figure 1. In this strategy profile, in the case both types are drawn from the set $[s_1, s_2]$, bidder 1 wins the auction and pays the amount $\hat{b}_2(t_2) = s_1 + \frac{1}{2} t_2$, which is his minimum ex post valuation in this case. Note that in this case bidder 1 overbids, while bidder 2 underbids. This class of equilibria leads to an inefficient outcome, hence we would like to be able to rule them out by the help of some kind of refinement.

In Proposition 5.1, we will show that the strategy profile $(b_1, b_2)$ is distributional perfect. Moreover, in Proposition 5.3 we will see that a large subclass of strategy profiles $(\hat{b}_1, \hat{b}_2)$ are not distributional perfect. The reason for this observation is that in the latter equilibria, bidder 1 might bid even higher than his maximum valuation and bidder 2 might bid even lower than his minimum valuation. Now, if each bidder takes making mistakes by his opponent serious, then he knows any bid from the set $[0, \frac{3}{2}]$ is possible. This prevents bidder 1 from overbidding, since by overbidding he may get a negative payoff at the end of the day. Similarly, by taking mistakes into account by bidder 2, he would not underbid anymore, as he can increase his chance of winning and getting a strictly positive payoff.
By taking a different approach, Bajoori et al. [5] define perfect equilibrium for Bayesian games only based on behavioral strategies and without making use of distributional strategies so called behavioral perfection. Behavioral perfection is a more straightforward way of defining perfect equilibrium in Bayesian games than distributional perfection and it is conceptually preferable. Moreover, it has the advantage that perfection can be verified for every type of each player. However, in developing the theory, particularly in proving the existence, distributional strategies are easier to work with.

The remainder of the paper is structured as follows. Section 2 describes the model in detail. In section 3 distributional perfect equilibrium is introduced and the results related to this concept are provided, which the most important one is the existence result in Theorem 3.4 Section 4 discusses the relation between distributional perfection and the approach taken in Bajoori et al. [5]. In section 5 the concept of distributional perfection is applied to a class of second-price auctions with interdependent values.

2 The model

Let $\Gamma$ denote the following Bayesian game: There are $n$ players. The set $T_i$ of player $i$’s types is a complete, separable metric space with a metric $d_{T_i}$ and the set $A_i$ of player $i$’s actions is a compact metric space with a metric $d_{A_i}$. Let $\mathcal{T}_i$ and $\mathcal{A}_i$ be the induced Borel $\sigma$-fields on $T_i$ and $A_i$, respectively. Moreover, let $T = \times_{i=1}^{n} T_i$ and $A = \times_{i=1}^{n} A_i$. Suppose $\mu$ is a probability measure on the product $\sigma$-field $\mathcal{T} = \otimes_{i=1}^{n} \mathcal{T}_i$ with marginal probability $\mu_i$ on $\mathcal{T}_i$ for every player $i$. We assume that each $\mu_i$ is a completely mixed probability measure. Recall that a probability measure is completely mixed if it assigns strictly positive weight to every nonempty open set. Player $i$’s payoff function $\pi_i : T \times A \to \mathbb{R}$ is bounded and jointly measurable.
We impose two more assumptions on Bayesian game $\Gamma$, following Milgrom and Weber [22]. First, the measure $\mu$ is absolutely continuous with respect to $\hat{\mu} = \times_{i=1}^{n} \mu_i$. This property is called Absolutely Continuous Information. Second, the payoffs are equicontinuous, that is for every player $i$ and every $\varepsilon > 0$, there is a set $E \in \mathcal{T}$ such that $\mu(E^c) < \varepsilon$ and the family $\{\pi_i(t, \cdot) \mid t \in E\}$ is equicontinuous. Milgrom and Weber in [22] give detailed explanation of these two assumptions. Absolutely continuous information is a rather weak assumption and it is satisfied when the type spaces are finite or when each player’s type is drawn independently from his type space. As an example this assumption rules out the case of a uniform distribution on the diagonal of the unit square. Although the equicontinuity of payoffs is truly restrictive, it is needed for the existence of distributional equilibrium. However, in the case of finite action spaces the equicontinuity of payoffs holds.

2.1 Behavioral and distributional strategy

A pure strategy for player $i$ in $\Gamma$ is a measurable function $p_i : T_i \to A_i$ and a behavioral strategy for him is a function $\beta_i : T_i \times A_i \to [0, 1]$ such that (1) the section function $\beta_i(t_i, \cdot) : A_i \to [0, 1]$ is a probability measure for every $t_i \in T_i$, (2) the section function $\beta_i(\cdot, B) : T_i \to [0, 1]$ is measurable for every $B \in A_i$. When player $i$ plays according to the behavioral strategy $\beta_i$, for each type $t_i \in T_i$, he chooses his action according to the probability measure $\beta_i(t_i, \cdot)$. A behavioral strategy $\beta_i$ for player $i$ is called pure if there is a pure strategy $p_i$ for player $i$ such that $\beta_i(t_i, \cdot) = D_{p_i(t_i)}(\cdot)$ for every type $t_i \in T_i$ in which $D$ denotes Dirac measure. For simplicity, we denote this pure behavioral strategy by $p_i$.

According to Milgrom and Weber [22], a distributional strategy for player $i$ in a Bayesian game is a probability measure $\gamma_i$ on $T_i \times A_i$ such that the marginal distribution of $\gamma_i$ on $T_i$ equals the probability measure $\mu_i$, i.e., $\gamma_i(U \times A_i) = \mu_i(U)$ for every $U \in \mathcal{T}_i$. Given a behavioral strategy $\beta_i$, the induced distributional strategy $\gamma_i$ is uniquely determined by

$$\gamma_i(U \times B) = \int_{U} \beta_i(t_i, B) \, \mu_i(dt_i)$$

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$^3$Let $\mu$ and $\nu$ be two measures on a $\sigma$-field $\Sigma$. The measure $\mu$ is absolutely continuous with respect to $\nu$ if for every $G \in \Sigma$, $\nu(G) = 0$ implies $\mu(G) = 0$.

$^4$A family $\mathcal{F}$ of real functions on the metric space $X$ is equicontinuous if to every $\varepsilon > 0$ corresponds a $\delta > 0$ such that for every $x, y \in X$ with $d(x, y) < \varepsilon$ we have $|f(x) - f(y)| < \delta$, for every $f \in \mathcal{F}$.
for every rectangle \( U \times B \in \mathcal{T}_i \otimes \mathcal{A}_i \). Conversely, the behavioral strategies corresponding to a distributional strategy \( \gamma_i \) are exactly the regular conditional probabilities. Hence, there is a many-to-one mapping from behavioral strategies to distributional strategies that preserves the players’ ex ante expected payoffs. Notice that in games with finite action and type spaces, if player \( i \) receives each type from his type space with strictly positive probability, then the mapping from behavioral strategies to distributional strategies is one to one.

Let \( \Delta_i \) be the set of all distributional strategies for player \( i \) and \( \Delta = \times_{i=1}^n \Delta_i \). Also, let \( \Pi_i : \Delta \to \mathbb{R} \) be the expected payoff of player \( i \) playing distributional strategy \( \gamma_i \) against a behavioral strategy profile \( \beta \), i.e.

\[
\Pi_i(\gamma_i, \gamma_{-i}) = \int \pi_i(t, a) \gamma_i(da_i | t_i) \gamma_{-i}(da_{-i} | t_{-i}) \mu(dt),
\]

where \( t = (t_1, \ldots, t_n) \in T \) and \( a = (a_1, \ldots, a_n) \in A \), \( \gamma_i(da_i | t_i) \) is a version of conditional probability on \( A_i \) given \( t_i \), and by integrals with respect to \( \gamma_{-i}(da_{-i} | t_{-i}) \) we mean the iterated integrals with respect to \( \gamma_j(da_j | t_j) \) for all \( j \neq i \). As the measure \( \mu \) is absolutely continuous with respect to \( \mu_t \), the Radon-Nikodym theorem implies the existence of a measurable function \( f : T \to \mathbb{R} \) such that for every measurable set \( G \in \mathcal{T} \) we have \( \mu(G) = \int_G f \mu_t \). Therefore, the expected payoff \( \Pi_i \) can be expressed in an easier form as

\[
\Pi_i(\gamma_i, \gamma_{-i}) = \int \pi_i(t, a) f(t) d\gamma_i d\gamma_{-i}.
\]

Let \( \beta_i \) be a behavior strategy corresponding to the distributional strategy \( \gamma_i \), for every player \( i \). Then, by Theorem 10.2.1 in Dudley [11] we conclude that

\[
\Pi_i(\gamma_i, \gamma_{-i}) = \int \pi_i(t, a) f(t) \beta_i(t_i, da_i) \mu_t(dt_i) \beta_{-i}(t_{-i}, da_{-i}) \mu_{-i}(dt_{-i}).
\]

We use \( \Pi_i^b(\beta_i, \beta_{-i} | t_i) \) to denote player \( i \)'s expected payoff given his behavioral strategy \( \beta_i \) and his type \( t_i \) against a behavioral strategy profile \( \beta_{-i} \). Thus,

\[
\Pi_i^b(\beta_i, \beta_{-i} | t_i) = \int \pi_i(t, a) f(t) \beta_i(t_i, da_i) \beta_{-i}(t_{-i}, da_{-i}) \mu_t(dt_t). \]

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5Dudley’s notation and terminology in [11] differs from ours. Part (II) of Theorem 10.2.1. in Dudley [11] with our terminology can be expressed as follows: For every measurable function \( g : T_i \times A_i \to \mathbb{R} \) we have

\[
\int g \, d\gamma_i = \int \int g(t_i, a_i) \beta_i(t_i, da_i) \mu_t(dt_i).
\]
One can also define \( \Pi^b_i(\sigma_i, \beta_{-i} \mid t_i) \) and \( \Pi^b_i(a_i, \beta_{-i} \mid t_i) \) respectively for every probability measure \( \sigma_i \) on \( A_i \) and every action \( a_i \in A_i \). Furthermore, notice that

\[
\Pi_i(\gamma_1, \ldots, \gamma_n) = \int \Pi^b_i(\beta_i, \beta_{-i} \mid t_i) \mu_i(dt_i).
\]

In the special case where player \( i \) chooses a pure behavioral strategy \( p_i \), his expected payoff is

\[
\Pi^b_i(p_i, \beta_{-i} \mid t_i) = \int \pi_i(t, (p_i(t_i), a_{-i})) f(t) \beta_{-i}(t_{-i}, da_{-i}) \mu_{-i}(dt_{-i}).
\]

### 2.2 BNE and distributional equilibrium

A probability measure \( \sigma_i \) on \( A_i \) is called a best response of player \( i \) for type \( t_i \in T_i \) against a behavioral strategy profile \( \tau_{-i} \), if for every probability measure \( \sigma'_i \) on \( A_i \) we have

\[
\Pi^b_i(\sigma_i, \tau_{-i} \mid t_i) \geq \Pi^b_i(\sigma'_i, \tau_{-i} \mid t_i).
\]

The set of such best responses is denoted by \( BR_i(t_i, \tau) \). A behavioral strategy \( \beta_i \) is called a best response of player \( i \) against the behavioral strategy profile \( \beta_{-i} \) for every \( t_i \in T_i \). A behavioral strategy profile \( \beta = (\beta_1, \ldots, \beta_n) \) is a Bayesian Nash Equilibrium (BNE) if \( \beta_i \) is a best response to \( \beta_{-i} \) for every player \( i \).

Distributional strategy \( \gamma_i \) of player \( i \) is a best response against a distributional strategy profile \( \eta_{-i} \), if \( \Pi_i(\gamma_i, \eta_{-i}) \geq \Pi_i(\tilde{\gamma}_i, \eta_{-i}) \), for every distributional strategy \( \tilde{\gamma}_i \) of player \( i \). Define the correspondence \( BR_i : \Delta \rightarrow \Delta_i \) to be the best response correspondence for player \( i \) on the set of distributional strategies. Moreover, define \( BR : \Delta \rightarrow \Delta \), for every \( \gamma \in \Delta \), as \( BR(\gamma) = \times_{i=1}^n BR_i(\gamma) \).

The equilibrium concept in Bayesian games can be defined in a weaker form where each player maximizes his payoff in ex ante probabilities, in other words, by using distributional strategies. This equilibrium concept is used by Milgrom and Weber in [22] and they prove that the Bayesian game \( \Gamma \) described above, contains at least one of them. However, it is not clear how players would play the game if they are given their distributional strategies, while behavioral strategies do not have this problem. Hence, we define a behavioral strategy profile \( \beta = (\beta_1, \ldots, \beta_n) \) to be a distributional equilibrium, if for the induced distributional strategy profile \( \gamma = (\gamma_1, \ldots, \gamma_n) \) we have \( \gamma_i \in BR_i(\gamma) \), for every player \( i \).
Next proposition proves that BNE and distributional equilibrium are equal up to a \( \mu_i \)-measure 1 set of types for every player \( i \).

**Proposition 2.1** For every distributional equilibrium \( \beta \), assume that the set of the best responses of player \( i \) against behavioral strategy profile \( \beta_{-i} \) is not empty.\(^6\) Then, there is a BNE \( \hat{\beta} \) such that for every player \( i \) we have \( \beta_i(t_i, \cdot) = \hat{\beta}_i(t_i, \cdot) \), for \( \mu_i \)-a.e. \( t_i \in T_i \).

**Proof.** Let \( \beta = (\beta_1, \ldots, \beta_n) \) be a distributional equilibrium. Then, for the induced distributional strategy profile \( \gamma = (\gamma_1, \ldots, \gamma_n) \) we have \( \gamma_i \in BR_i(\gamma) \). Now, by Lemma 6.1 we have \( \beta_i(t_i, \cdot) \in BR_i(t_i, \beta) \) for \( \mu_i \)-a.e. \( t_i \in T_i \). In other words, for every \( i \) there exists a set \( S_i \in T_i \) with \( \mu_i(S_i) = 0 \) such that for every \( t_i \in T_i \setminus S_i \) we have \( \beta_i(t_i, \cdot) \in BR_i(t_i, \beta) \). Now, define \( \tilde{\beta} = (\tilde{\beta}_1, \ldots, \tilde{\beta}_n) \), for every \( t_i \in T_i \setminus S_i \), by letting \( \tilde{\beta}_i(t_i, \cdot) = \beta_i(t_i, \cdot) \) and for every \( t_i \in S_i \) selecting any \( \hat{\beta}_i(t_i, \cdot) \in BR_i(t_i, \beta) \). It is clear that \( \tilde{\beta} \) is a BNE. This completes the proof. \( \blacksquare \)

We say a behavioral strategy \( \beta_i \) (weakly) dominates \( \hat{\beta}_i \) if for their induced distributional strategies \( \gamma_i \) and \( \hat{\gamma}_i \) respectively, we have

\[
\Pi_i(\gamma_i, \gamma_{-i}) \geq \Pi_i(\hat{\gamma}_i, \gamma_{-i}),
\]

for all \( \gamma_{-i} \in \Delta_{-i} \), with a strict inequality for some \( \gamma_{-i} \). The behavioral strategy \( \beta_i \) is called (weakly) dominant if it dominates every other behavioral strategy of player \( i \). The behavioral strategy \( \beta_i \) is undominated if there is no other behavioral strategy that dominates \( \beta_i \).

### 3 Distributional perfection

We define perfect equilibrium for Bayesian games in behavioral strategies, because a player given a behavioral strategy knows how to play the game, but this is not clear if he is given a distributional strategy. Nevertheless, we use distributional strategies as a tool to verify whether the behavioral strategy profile satisfies the conditions of trembling hand perfect equilibrium or not. This concept is called distributional perfect equilibrium. Throughout this paper the weak metric (also called Prohorov metric) on a set of probability measures is denoted by \( \rho^w \).

\(^6\)The existence of such behavioral strategy might be derived from measurable selection theorems for example in [1] and [27].
Definition 3.1 A behavioral strategy profile $\beta = (\beta_1, \ldots, \beta_n)$ is called distributional perfect if for the induced distributional strategy profile $\gamma = (\gamma_1, \ldots, \gamma_n)$ there exists a sequence of completely mixed distributional strategy profiles $(\gamma^k)_{k=1}^{\infty} = (\gamma^k_1, \ldots, \gamma^k_n)_{k=1}^{\infty}$ such that for every player $i$: 

(i) $\lim_{k \to \infty} \rho^w(\gamma^k_i, \gamma_i) = 0$, 

(ii) $\lim_{k \to \infty} \rho^w(\gamma^k_i, BR_i(\gamma^k)) = 0$.

Distributional perfection in Bayesian game $\Gamma$ is a refinement of distributional equilibrium. This is proved in the following theorem.

Theorem 3.2 Every distributional perfect strategy profile in Bayesian game $\Gamma$, is a distributional equilibrium.

Proof. Let $\beta = (\beta_1, \ldots, \beta_n)$ be a distributional perfect strategy profile. By Definition 3.1, for the induced distributional strategy profile $\gamma = (\gamma_1, \ldots, \gamma_n)$ there exists a sequence of completely mixed distributional strategy profiles $(\gamma^k)_{k=1}^{\infty} = (\gamma^k_1, \ldots, \gamma^k_n)_{k=1}^{\infty}$ such that for every player $i$ we have $\lim_{k \to \infty} \rho^w(\gamma^k_i, \gamma_i) = 0$ and $\lim_{k \to \infty} \rho^w(\gamma^k_i, BR_i(\gamma^k)) = 0$. Therefore, for every $i$, there exists a sequence $(\tilde{\gamma}^k_i)_{k=1}^{\infty}$ such that $\tilde{\gamma}^k_i \in BR_i(\gamma^k)$ for every $k \in \mathbb{N}$ and $\rho^w(\gamma^k_i, \tilde{\gamma}^k_i) \to 0$ as $k \to \infty$. By the triangle inequality for $\rho^w$ this implies for every $i$ that $\rho^w(\gamma_i, \tilde{\gamma}^k_i) \to 0$ as $k \to \infty$. Now, by Lemma 6.3 we conclude the upper hemicontinuity of the correspondence $BR_i$, for every player $i$. Hence, $\gamma_i \in BR_i(\gamma)$, which implies that behavioral strategy $\beta$ is distributional perfect. 

As we just proved, distributional perfection is a refinement of distributional equilibrium, but it may not be a refinement of BNE. In other words, distributional perfection does not guarantee that player $i$, for every type $t_i \in T_i$, selects a best response against the opponents’ strategies. However, by Proposition 2.1, given the distributional perfect equilibrium, we can construct a BNE that satisfies conditions (i) and (ii) of Definition 3.1 as follows. Let $\beta$ be a distributional perfect strategy profile. Then, if for any type $t_i$ of player $i$, $\beta_i(t_i, \cdot)$ is not a best response against $\beta_{-i}$, he moves to one of his best responses, while this does not change the expected payoffs of the opponents, since it happens only on a measure zero set.

In addition, we prove the existence of distributional perfect BNE in Bayesian game $\Gamma$ and that the set of distributional perfect equilibria is sequentially closed in the topology
\( \omega \), where \( \omega \) is the Tychonoff topology on the set \( \times_{i \in T_i} \Delta(A_i) \) when taking the strong topology on each \( \Delta(A_i) \). The following fixed point theorem is used to prove the existence of distributional perfection. This theorem is derived from Glicksberg [13].

**Theorem 3.3** Let \( S \) be a nonempty compact and convex subset of a locally convex Hausdorff space. Let \( F : S \to S \) be an upper hemicontinuous correspondence with nonempty and convex values. Then, \( F \) has a fixed point.

**Theorem 3.4** In Bayesian game \( \Gamma \), the set of distributional perfect equilibria is nonempty and sequentially closed in the topology \( \omega \).

**Proof.** First we prove that the set of distributional strategy profiles satisfying conditions (i) and (ii) of Definition 3.1 is not empty. For every player \( i \), let \( \nu_i \) be a completely mixed distributional strategy. Note that such a \( \nu_i \) exists, because \( T_i \times A_i \) is separable.

Define for every \( k \in \mathbb{N} \)

\[
\Delta_i(k) = \{ \gamma_i \in \Delta_i \mid \gamma_i(B_i) \geq \frac{1}{k} \cdot \nu_i(B_i), \ \forall B_i \in T_i \otimes A_i \}.
\]

Moreover, let \( \Delta(k) = \times_{i=1}^n \Delta_i(k) \) and \( BR^k_i : \Delta(k) \to \Delta_i(k) \) be the best response correspondence for player \( i \) restricted to \( \Delta_i(k) \)

\[
BR^k_i(\gamma) = \times_{i=1}^n BR^k_i(\gamma),
\]

for every \( \gamma \in \Delta(k) \). We verify the conditions of Theorem 3.3 for the correspondence \( BR^k \).

Since \( T \) is a complete and separable metric space, by Theorem 1.4 in [9] \( \mu \) is a tight measure.\(^8\) This fact together with the compactness of the set \( A \), implies that \( \Delta_i \) is a tight set of probability measures.\(^9\) Now, by Prohorov’s Theorem we conclude that \( \Delta_i \) is a compact metric space with respect to the weak metric. It is easy to see that \( \Delta_i(k) \)

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\(^7\)Distributional strategy \( \gamma_i \) of player \( i \) is a best response restricted to \( \Delta_i(k) \) against a distributional strategy profile \( \eta_{-i} \in \Delta_{-i}(k) \), if \( \Pi_i(\gamma_i, \eta_{-i}) \geq \Pi_i(\tilde{\gamma}_i, \eta_{-i}) \), for every distributional strategy \( \tilde{\gamma}_i \in \Delta_i(k) \) of player \( i \).

\(^8\)Let \( X \) be a metric space with the Borel \( \sigma \)-field \( \Sigma \). A measure \( \mu \) on \( \Sigma \) is tight if for every \( G \in \Sigma \), \( \mu(G) = \sup \{ \mu(K) \mid K \subset G, \text{ and } K \text{ compact} \} \). Moreover, Theorem 1.4 in [9] states that if \( X \) is separable and complete, then each probability measure on \( \Sigma \) is tight.

\(^9\)A family of probability measures on a metric space is tight if for every \( \epsilon > 0 \) there is a compact set \( K \) satisfying \( \mu(K) > 1 - \epsilon \), for every \( \mu \) in the family.
is a closed subset of $\Delta_i$ with respect to the strong metric. Consequently, by Theorem V.3.13 in [12], it is closed with respect to the weak metric. Hence, $\Delta_i(k)$ is compact with respect to the weak metric. Also, one can easily check that $\Delta_i(k)$ is convex. Moreover, upper hemicontinuity of $BR^k$ follows by Lemma 6.3. Now, we apply Theorem 3.3, which leads us to the existence of an equilibrium point $\gamma^k \in \Delta(k)$, for every $k$.

Now, define\(^{10}\)

$$PBR_i(\gamma) = \{(t_i, a_i) \in T_i \times A_i \mid (t_i, a_i) \in \text{supp}(\gamma_i) \text{ and } \gamma_i \in BR_i(\gamma)\}.$$  

It is clear that $\gamma^k_i(PBR_i(\gamma^k)) \geq 1 - \frac{1}{k} \nu_i(T_i \times A_i) = 1 - \frac{1}{k}$. This implies that $\rho^{w}(\gamma^k_i, BR_i(\gamma^k)) \leq \frac{1}{k}$. Furthermore, as $\Delta_i$ is compact with respect to the weak metric, without loss of generality we can assume that there is a distributional strategy profile $\gamma$ such that $\rho^{w}(\gamma^k, \gamma) \to 0$ when $k \to \infty$. Let $\beta$ be a behavioral strategy profile corresponding to $\gamma$. Obviously, $\beta$ is a distributional perfect equilibrium. Hence, the set of distributional perfect equilibria is nonempty.

Now, we prove that the set of distributional perfect equilibria is sequentially closed in $\omega$. Let $(\beta_\ell)_{\ell=1}^\infty$ be a sequence of distributional perfect equilibria in which $\beta_\ell = (\beta_{\ell,1}, \ldots, \beta_{\ell,n})$ and suppose that for every player $i$, the behavioral strategy $\beta_{\ell,i}$ converges to $\beta_i \in \times_{t_i \in T_i} \Delta(A_i)$ in the topology $\omega$ as $\ell \to \infty$. Then, for every $B \in A_i$ the function $\beta_{\ell,i}(\cdot, B)$ on $T_i$ converges pointwise to $\beta_i(\cdot, B)$ as $\ell \to \infty$. This implies that $\beta_i(\cdot, B)$ is measurable for every $B \in A_i$, hence $\beta_i$ is a behavior strategy. Let $\beta = (\beta_1, \ldots, \beta_n)$. We show that $\beta$ is distributional perfect. Let $\gamma_\ell = (\gamma_{\ell,1}, \ldots, \gamma_{\ell,n})$ and $\gamma = (\gamma_1, \ldots, \gamma_n)$ be the induced distributional strategy profiles by $\beta_\ell$ and $\beta$ respectively. It is clear that $\rho^{w}(\gamma_{\ell,i}, \gamma_i) \to 0$ as $\ell \to \infty$, for every $i$. Since for every $\ell$ the strategy profile $\beta_\ell$ is distributional perfect, there exits a sequence of completely mixed distributional strategies $\gamma_{\ell,i}$, for every player $i$, satisfying the conditions of Definition 3.1. Hence, there is a $K^{1}_{\ell,i}$ such that for every $k \geq K^{1}_{\ell,i}$, we have $\rho^{w}(\gamma_{\ell,i}, \gamma_i) < \frac{1}{k}$. Similarly, there is a $K^{2}_{\ell,i}$ such that for every $k \geq K^{2}_{\ell,i}$, we have $\rho^{w}(\gamma_{\ell,i}, BR_i(\gamma_{\ell,i})) < \frac{1}{k}$. Let $K_{\ell,i} = \max\{K^{1}_{\ell,i}, K^{2}_{\ell,i}\}$. Now, the sequence $\gamma^{K_{\ell,i}}_{\ell,i}$ of completely mixed distributional strategies satisfies the conditions of Definition 3.1 for the strategy profile $\gamma$ when $\ell \to \infty$. Namely, for every player $i$ we have

$$\rho^{w}(\gamma^{K_{\ell,i}}_{\ell,i}, \gamma_i) \leq \rho^{w}(\gamma_{\ell,i}, \gamma_i) + \rho^{w}(\gamma_{\ell,i}, \gamma_i) < \frac{1}{\ell} + \rho^{w}(\gamma_{\ell,i}, \gamma_i).$$

\(^{10}\)Recall that $\text{supp}(\mu) = \{x \in X \mid \text{for every open set } G \text{ if } x \in G \text{ then } \mu(G) > 0\}$.
and
\[ \rho^w(\gamma_{\ell,i}^{K_{\ell,i}}, BR_i(\gamma_{\ell}^{K_{\ell,i}})) < \frac{1}{\ell}. \]

Clearly, both \( \rho^w(\gamma_{\ell,i}^{K_{\ell,i}}, \gamma_i) \) and \( \rho^w(\gamma_{\ell,i}^{K_{\ell,i}}, BR_i(\gamma_{\ell}^{K_{\ell,i}})) \) converge to zero when \( \ell \to \infty \). Hence, \( \beta \) is a distributional perfect equilibrium.

In the theorem above, to define \( \omega \) we use the strong topology on each \( \Delta(A_i) \), but not the weak topology. In the weak topology the limit \( \beta_i \) of a sequence of behavioral strategies \( (\beta_{\ell,i})_{\ell=1}^{\infty} \) may not be a behavioral strategy, as we encounter difficulties to show that \( \beta_i(\cdot, B) : T_i \to [0, 1] \) is measurable for every \( B \in A_i \).

As we explained in the introduction, for finite games perfect equilibrium puts weight zero on dominated strategies. However, this property is not compatible for infinite games, since Simon and Stinchcombe [25] in Example 2.1 presents a game with infinite action spaces in which the unique equilibrium of the game is in weakly dominated strategies. Hence, a weaker property of limit undominatedness is used for strategies in infinite games that is defined as follows: Behavioral strategy \( \beta_i \) is limit undominated if for the induced distributional strategy \( \gamma_i \), there is a sequence of undominated\(^{11}\) distributional strategies \( (\gamma_i^k)_{k=1}^{\infty} \) such that \( \rho^w(\gamma_i^k, \gamma_i) \to 0 \) as \( k \to \infty \). The following result shows that a distributional perfect strategy profile is in limit undominated strategies. The proof is simple and similar to the proof of Theorem 5.1 in Bajoori et al. [4].

**Theorem 3.5** Let \( \beta \) be a distributional perfect equilibrium in Bayesian game \( \Gamma \). Then, \( \beta_i \) is limit undominated, for every player \( i \).

In the remark below we see that distributional perfection is equivalent to trembling hand perfect equilibrium in games with finite action and type spaces defined in Bajoori et al. [5].

**Remark.** Consider a Bayesian game in which every player’s action space and type space are finite and every player receives each type from his type space with strictly positive probability. This game satisfies all the assumptions of the game \( \Gamma \) described above, so it is an special case of this class of Bayesian games. A behavioral strategy for player \( i \) in finite games is a function \( \beta_i : T_i \to \Delta(A_i) \), where \( \Delta(A_i) \) is the set of

\(^{11}\) Undominated distributional strategies are defined similarly to undominated strategies for normal form games, since distributional strategies are simply probability measures on the product space of types and actions.
mixed strategies on $A_i$. Behavioral strategy $\beta_i$ is called completely mixed if $\beta_i(t_i)$ is a completely mixed strategy for each type $t_i \in T_i$. A direct generalization of Selten’s perfect equilibrium to finite Bayesian games in Bajoori et al. [5] is defined as a behavioral strategy profile $\beta = (\beta_1, \ldots, \beta_n)$ for which there exists a sequence of completely mixed behavioral strategy profiles $(\beta^k)_{k=1}^{\infty} = (\beta^k_1, \ldots, \beta^k_n)_{k=1}^{\infty}$ with the following properties for every player $i$ and each type $t_i \in T_i$:

(i) $\lim_{k \to \infty} d_E(\beta_i^k(t_i), \beta_i(t_i)) = 0$,

(ii) $\lim_{k \to \infty} d_E(\beta_i^k(t_i), BR_i(t_i, \beta^k)) = 0$,

where $d_E$ is the Euclidean metric.

Now, define $A'_i = T_i \times A_i$ and $\Delta(A'_i)$ to be the set of mixed strategies on $A'_i$. Let $\Delta_d(A'_i) \subseteq \Delta(A'_i)$ to be the set of all player $i$’s distributional strategies. Moreover, the player $i$’s payoff is his ex ante expected payoff, i.e. his expected payoff before he learns his own type. The equivalent definition of distributional perfection in finite Bayesian games is:

A behavioral strategy profile $\beta = (\beta_1, \ldots, \beta_n)$ is called perfect if for the induced distributional strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$, there exists a sequence of completely mixed distributional strategy profiles $(\sigma^k)_{k=1}^{\infty} = (\sigma^k_1, \ldots, \sigma^k_n)_{k=1}^{\infty}$ such that for every player $i$ we have:

(i) $\lim_{k \to \infty} d_E(\sigma_i^k, \sigma_i) = 0$,

(ii) $\lim_{k \to \infty} d_E(\sigma_i^k, BR_i(\sigma^k)) = 0$,

In this finite game, for every distributional strategy there is a unique behavioral strategy and vice versa. Also, for a behavioral strategy profile $\beta = (\beta_1, \ldots, \beta_n)$ if we have $\beta_i(t_i) \in BR_i(t_i, \beta)$ for every $t_i \in T_i$, then obviously for the induced distributional strategy $\gamma = (\gamma_1, \ldots, \gamma_n)$ we have $\gamma_i \in BR_i(\gamma)$, for every player $i$. Conversely, it is easy to see that if $\gamma_i \in BR_i(\gamma)$, then $\beta_i(t_i, \cdot) \in BR_i(t_i, \beta)$, for every $t_i \in T_i$ and every player $i$. Therefore, the latter definition is equivalent to the perfection defined in Bajoori et al. [5] for finite games.
4 A.e. pointwise perfection

In Bajoori et al. [5] a definition of perfect equilibrium was introduced in Bayesian games without using distributional strategies. It is called a.e.-pointwise perfection and it has a close relation with distributional perfection. Recall that a behavioral strategy \( \beta_i \) is completely mixed if the probability measure \( \beta_i(t_i, \cdot) : A_i \to [0, 1] \) is completely mixed, for every \( t_i \in T_i \). A behavioral strategy profile \( \beta = (\beta_1, \ldots, \beta_n) \) is a.e.-pointwise perfect if for every player \( i \) there exists a set \( S_i \in T_i \) of \( \mu_i \)-measure 0 and a sequence of profiles of completely mixed behavioral strategies \( (\beta_k)_{k=1}^\infty = (\beta_1^k, \ldots, \beta_n^k)_{k=1}^\infty \) with the following properties for every player \( i \) and every type \( t_i \in T_i \setminus S_i \):

(i) \( \lim_{k \to \infty} \rho^w(\beta_i^k(t_i, \cdot), \beta_i(t_i, \cdot)) = 0 \),

(ii) \( \lim_{k \to \infty} \rho^w(\beta_i^k(t_i, \cdot), BR_i(t_i, \beta_i)) = 0 \).

In the next proposition we show that a.e.-pointwise perfection is a stronger concept than distributional perfect equilibrium. The opposite direction remains an open question.

**Proposition 4.1** Every a.e.-pointwise perfect strategy profile is distributional perfect.

**Proof.** Let \( \beta = (\beta_1, \ldots, \beta_n) \) be an a.e.-pointwise perfect strategy profile. Then, by the definition, there is a set \( S_i \in T_i \) of \( \mu_i \)-measure 0 and a sequence of completely mixed profiles of behavioral strategies \( (\beta^k)_{k=1}^\infty = (\beta_1^k, \ldots, \beta_n^k)_{k=1}^\infty \) such that for every player \( i \) and every type \( t_i \in T_i \setminus S_i \) conditions (i) and (ii) of this definition are satisfied. For every player \( i \), let \( \gamma_i \) and \( \gamma_i^k \) be the distributional strategies induced by \( \beta_i \) and \( \beta_i^k \) respectively. We prove that conditions (i) and (ii) of Definition 3.1 are satisfied. Take a player \( i \). For condition (i), we show that for any bounded and continuous function \( h : T_i \times A_i \to \mathbb{R} \), we have

\[
\int_{T_i \times A_i} h \, d\gamma_i^k \longrightarrow \int_{T_i \times A_i} h \, d\gamma_i \quad \text{as} \quad k \to \infty.
\]

As \( h \) is bounded and continuous on \( T_i \times A_i \), the function \( h_{t_i}(a_i) = h(t_i, a_i) \) on \( A_i \) is also bounded and continuous for every \( t_i \in T_i \). Let

\[
g^k(t_i) = \int_{A_i} h_{t_i} \beta_i^k(t_i, da_i), \quad g(t_i) = \int_{A_i} h_{t_i} \beta_i(t_i, da_i).
\]

By Theorem 10.2.1 in [11], the functions \( g^k \) and \( g \) are measurable. Moreover, because for every type \( t_i \in T_i \setminus S_i \), \( \lim_{k \to \infty} \rho^w(\beta_i^k(t_i, \cdot), \beta_i(t_i, \cdot)) = 0 \), we have \( g^k(t_i) \longrightarrow g(t_i) \) as
\( k \to \infty, \) for \( \mu_i \)-a.e. \( t_i \in T_i. \) Therefore, by Dominated Convergence Theorem
\[
\int_{T_i} g^k(t_i) \mu_i(dt_i) \longrightarrow \int_{T_i} g(t_i) \mu_i(dt_i) \quad \text{as} \quad k \to \infty.
\]
This implies that
\[
\int_{T_i \times A_i} h \, d\gamma^k_i \longrightarrow \int_{T_i \times A_i} h \, d\gamma_i \quad \text{as} \quad k \to \infty.
\]
Now, we prove that condition (ii) of Definition 3.1 is satisfied. We know that for every \( t_i \in T_i \setminus S_i \) we have \( \rho^w(\hat{\beta}^k_i(t_i, \cdot), \text{BR}_i(t_i, \beta^k)) \to 0 \) as \( k \to \infty. \) Hence, for every \( t_i \in T_i \setminus S_i, \) there exists a sequence \( \hat{\beta}^k_i(t_i, \cdot) \in \text{BR}_i(t_i, \beta^k) \) such that \( \rho^w(\hat{\beta}^k_i(t_i, \cdot), \hat{\beta}^k_i(t_i, \cdot)) \to 0 \) as \( k \to \infty. \) Define \( \hat{\gamma}^k_i \) to be the distributional strategies induced by \( \hat{\beta}^k_i \) for every \( k \in \mathbb{N}. \)
With the same argument as in the previous part, we conclude that \( \rho^w(\gamma^k_i, \hat{\gamma}^k_i) \to 0 \) as \( k \to \infty. \) Since
\[
\Pi_i(\gamma_i, \gamma_{-i}) = \int_{T_i} \Pi^i_\beta(\beta_i, \beta_{-i} \mid t_i) \mu_i(dt_i),
\]
one can easily verify that if \( \hat{\beta}^k_i(t_i, \cdot) \in \text{BR}_i(t_i, \beta^k) \) for every \( t_i \in T_i \setminus S_i, \) then \( \hat{\gamma}^k_i \in \text{BR}_i(\gamma^k). \) Hence, we have \( \lim_{k \to \infty} \rho^w(\gamma^k_i, \text{BR}_i(\gamma^k_i)) = 0. \) So, condition (ii) holds. 

\section{Second-price auctions with interdependent values}

In this section we study symmetric second-price auctions with interdependent values for two bidders and apply distributional perfection in this context. Let \( \Lambda \) be the following sealed-bid second-price auction for a single indivisible object. There are 2 bidders. Prior to bidding, each bidder \( i \) receives a private signal \( t_i \in T_i = [t, T], \) which is called the type of bidder \( i. \) Signals are drawn independently according to a distribution \( \mu_i. \) Bidder \( i \)'s valuation of the object may depend on both types and is denoted by \( v_i(t_1, t_2). \) Each bidder \( i, \) after observing his own type, submits a bid from a set \( A_i \) independently of his opponent, where \( A_i \) is sufficiently large in the sense that it contains the range of \( v_i. \) Given the bids submitted by the bidders, the highest bid wins the auction for the price equal to the second highest bid.\(^{12}\)

Suppose the following assumptions in this auction:

1. \( v_1(t_1, t_2) = v_2(t_2, t_1). \)

\(^{12}\)In the event of a tie for the highest bid, the winner is chosen according to a probability distribution which may depend on the identity of the highest bidder and his type as well. In our setting, a tie will only occur with probability 0, even given the type of one bidder, which makes the specification of this tie-breaking rule irrelevant for the expected profits.
(2) $v_i$ is continuously differentiable.
(3) $v_i$ is strictly increasing in $t_i$ and increasing in $t_j$ for $j \neq i$.

In $\Lambda$ the set of available bids for bidder $i$ is an interval $A_i = [a_i, \overline{a}_i]$, where $v_i(t, \underline{t}) = a_i$ and $v_i(t, \overline{t}) = \overline{a}_i$, for $i = 1, 2$. In such auctions bidders do not necessarily have any dominant strategy, however it may contain many equilibria. As we explained in the introduction, efficiency is the concept which has been used to distinguish between multiple equilibria in this setting. But, from the bidders’ point of view it may not be a strategic behavior. Hence, a refinement is needed in order to rule out the less intuitive equilibria and get a smaller set of solutions. We use distributional perfection for this purpose. In Proposition 5.1 we show that a class of BNEs that was discovered by Milgrom in [21] are distributional perfect and in Proposition 5.3 we prove that distributional perfection rules out another class of equilibria introduced by Birulin in [10].

**Proposition 5.1** Let $\phi : [\underline{t}, \overline{t}] \rightarrow [\underline{t}, \overline{t}]$ be an increasing bijection. Consider the strategy profile $(b_1, b_2)$ in which $b_1(t_1) = v_1(t_1, \phi(t_1))$ and $b_2(t_2) = v_2(\phi^{-1}(t_2), t_2)$. If $v_i(t, \phi(t)) = v_i(\phi(t), t)$, for $i = 1, 2$, then $(b_1, b_2)$ is a distributional perfect BNE. Moreover, $b_i$ is undominated for every bidder $i$.

**Proof.** First we show that $(b_1, b_2)$ is a BNE. Define $w(t) = v_1(t, \phi(t)) = v_2(\phi(t), t)$ for every $t \in [\underline{t}, \overline{t}]$. One can easily see that $w : [\underline{t}, \overline{t}] \rightarrow [a, \overline{a}]$ is an increasing bijection. Suppose that bidder 1 bids $b_1(t_1) = v_1(t_1, \phi(t_1))$. The expected payoff of bidder 2 having type $t_2$ and bidding $p_2$ is:

$$
\Pi_2^0(b_1, p_2 \mid t_2) = \int_{\underline{t}}^{\overline{t}} (v_2(t_1, t_2) - v_1(t_1, \phi(t_1))) \mu_1(dt_1).
$$

As $v_2$ is strictly increasing in the second argument and we know

$$
v_2(t_1, t_2) - v_1(t_1, \phi(t_1)) = v_2(t_1, t_2) - v_2(\phi(t_1), t_1) = v_2(t_1, t_2) - v_2(t_1, \phi(t_1)),
$$

it is clear that if $\phi(t_1) \leq t_2$, then $v_2(t_1, t_2) - v_1(t_1, \phi(t_1)) \geq 0$ and negative otherwise. Hence, the maximum of $\Pi_2^0(b_1, p_2 \mid t_2)$ is obtained by choosing $p_2$ such that $w^{-1}(p_2) = t_1 = \phi^{-1}(t_2)$ which means $p_2 = w(\phi^{-1}(t_2)) = v_2(\phi^{-1}(t_2), \phi^{-1}(t_2)) = b_2(t_2)$. Furthermore, $\Pi_2^0(b_1, p_2 \mid t_2)$ is maximized over $p_2$ if and only if $p_2 = b_2(t_2)$, which implies that bidding $b_2$ is the unique best response against $b_1$ for bidder 2.

Next, we prove that the strategy profile $(b_1, b_2)$ is distributional perfect. Let $\gamma_1$ be the distributional strategy induced by $b_1$ and for every $k \in \mathbb{N}$ define $\gamma_1^k = (1 - \varepsilon_k)\gamma_1 + \varepsilon_k \eta_1,$
where \( \varepsilon_k \in (0, 1) \), \( \varepsilon_k \to 0 \) as \( k \to \infty \), and \( \eta_1 \) is the normalized uniform distributional strategy\(^\text{13}\) on the product space \( T_1 \times A_1 \). Let \( p_2 \) be a pure behavioral strategy for bidder 2 and \( \sigma_2 \) be the induced distributional strategy by \( p_2 \). Then, the expected payoff of bidder 2 choosing \( \sigma_2 \) against \( \gamma^k_1 \) is:

\[
\Pi_2(\gamma^k_1, \sigma_2) = \int \int_{p_2 > a_1} (v_2(t_1, t_2) - a_1) \, d\gamma^k_1 \, d\sigma_2
\]

\[
= \int_{T_2} \int_{p_2 > a_1} (v_2(t_1, t_2) - a_1) \, d\gamma^k_1 \, \mu_2(dt_2)
\]

\[
= \int_{T_2} \Pi^b_2(\beta^k_1, p_2 | t_2) \, \mu_2(dt_2),
\]

where \( \beta^k_1 \) is a behavioral strategy corresponding to \( \gamma^k_1 \) for every \( k \in \mathbb{N} \). To maximize \( \Pi_2(\gamma^k_1, \sigma_2) \) it is enough to maximize \( \Pi^b_2(\beta^k_1, p_2 | t_2) \) over \( p_2 \). We have

\[
\Pi^b_2(\beta^k_1, p_2 | t_2) = (1 - \varepsilon_k) \int_{p_2 > a_1} (v_2(t_1, t_2) - a_1) \, d\gamma_1
\]

\[+ \varepsilon_k \int_{p_2 > a_1} (v_2(t_1, t_2) - a_1) \, d\eta_1
\]

\[= (1 - \varepsilon_k) \int_{T_1} (v_2(t_1, t_2) - v_1(t_1, \phi(t_1))) \, \mu_1(dt_1)
\]

\[+ \varepsilon_k \int_{p_2 > a_1} (v_2(t_1, t_2) - a_1) \, d\eta_1
\]

\[= (1 - \varepsilon_k) \Pi^b_2(b_1, p_2 | t_2) + \varepsilon_k \int_{p_2 > a_1} (v_2(t_1, t_2) - a_1) \, d\eta_1.
\]

Suppose that \( p_2 = b^k_2(t_2) \) maximizes \( \Pi^b_2(\beta^k_1, p_2 | t_2) \) for every \( k \). Also, we know that \( p_2 = b_2 \) is the unique maximizer of \( \Pi^b_2(b_1, p_2 | t_2) \). Then, clearly we have that \( b^k_2 \) converges to \( b_2 \) when \( k \to \infty \). With a similar argument for bidder 2, we can conclude that \((b_1, b_2)\) is distributional perfect.

Finally, the fact that bidding \( b_2 \) is the unique best response for bidder 2 against \( b_1 \), implies that \( b_2 \) is undominated. A similar argument holds for \( b_1 \). \[\square\]

Remark that if for the class of strategy profiles \((b_1, b_2)\) in Proposition 5.1 one can find more than one map \( \phi \) that satisfies the condition \( v_i(t, \phi(t)) = v_i(\phi(t), t) \), for \( i = 1, 2 \), then obviously these strategy profiles are not in dominant strategies. Because, for each

\(^{13}\)That is \( \eta_1(S_1 \times A_1) = \mu_1(S_1) \) for every \( S_1 \in T_1 \) and for a behavioral strategy \( \lambda_1 \) corresponding to \( \eta_1 \) we have \( \lambda_1(t_1, \cdot) \) is the uniform distribution on \( A_1 \) for every \( t_1 \in T_1 \).
Proposition 5.2 Suppose that there is a strategy profile \( b \) in which this class of equilibria only contains one strategy profile, that would be the equilibrium in which \( b_i = v_i(t_i, t_i) \) for every \( i = 1, 2 \). In the next proposition we prove that this strategy profile is not in dominant strategies.

**Proposition 5.2** Suppose that there is a \( y \in T_j \) such that \( \frac{\partial v_i(t_i, t_j)}{\partial y_j}(y) > 0 \), where \( j \neq i \). Then, \( b_i = v_i(t_i, t_i) \) is not a dominant strategy for bidder \( i = 1, 2 \).

**Proof.** We prove that \( b_1 = v_1(t_1, t_1) \) is not a dominant strategy for bidder 1. First, we show that there is a set of types \( I \in T_1 \) with \( \mu_1(I) > 0 \), such that for every \( t_1 \in I \) bidding according \( b_1(t_1) = v_1(t_1, t_1) \) has strictly negative payoff for bidder 1. By assumption we conclude that there is a set \( B \in T_2 \) with \( \mu_2(B) > 0 \) such that the map \( v_1(\tilde{t}, \cdot) : T_2 \rightarrow [\underline{a}, \overline{a}] \) is strictly increasing on \( B \). Therefore, for every \( t_2 \in B \setminus \{\tilde{t}\} \) we have

\[
v_1(t, t_2) < v_1(t, \tilde{t}),
\]

also for every \( t_2 \in T_2 \) we know that

\[
v_1(t, t_2) \leq v_1(t, \tilde{t}).
\]

Therefore,

\[
\int_{T_2} v_1(t, t_2) \mu_2(dt_2) < v_1(t, \tilde{t}) = b_1(t).
\]

Let \( y(t_1) = \int_{T_2} v_1(t_1, t_2) \mu_2(dt_2) \). Choose any \( \tilde{b} \in (y(\tilde{t}), b_1(\tilde{t})) \). Note that the maps \( y \) and \( b_1 \) are strictly increasing, hence invertible. Therefore, \( b_1^{-1}(\tilde{b}) < \tilde{t} < y^{-1}(\tilde{b}) \). Let \( I = (b_1^{-1}(\tilde{b}), \tilde{t}) \), which is a nonempty interval of \( T_1 \) with \( \mu_1(I) > 0 \). One can easily verify that for every \( t_1 \in I \) we have \( y(t_1) < \tilde{b} < b_1(t_1) \).

Suppose bidder 2 bids \( \tilde{b} \) for every \( t_2 \in T_2 \) and bidder 1 bids according to \( b_1(t_1) \). If \( t_1 \in I \), then bidder 1 wins the auction, since \( \tilde{b} < b_1(t_1) \) and his expected payoff for every \( t_1 \in I \) is

\[
\int_{T_2} \left( v_1(t_1, t_2) - \tilde{b} \right) \mu_2(dt_2) = y(t_1) - \tilde{b} < 0.
\]

In the case that \( t_1 \notin I \), the value of the map \( y(t_1) \) strictly decreases when compared to the case \( t_1 \in I \). Consequently, even if bidder 1 wins the auction \( y(t_1) - \tilde{b} \) remains
strictly negative. Hence, we have

\[
\int_{b_1(t_1) > \tilde{b}_2} \int (v_1(t_1, t_2) - \tilde{b}_2) \mu_2(dt_2)\mu_1(dt_1) < 0.
\]

Now, let \( \tilde{b}_1(t_1) = a \) and \( \tilde{b}_2(t_2) = \tilde{b}_2 \). Moreover, let \( \gamma_1, \tilde{\gamma}_1 \), and \( \tilde{\gamma}_2 \) be the induced distributional strategies respectively by \( b_1, \tilde{b}_1, \) and \( \tilde{b}_2 \). Then, we have

\[
\Pi_1(\gamma_1, \tilde{\gamma}_2) < \Pi_1(\tilde{\gamma}_1, \tilde{\gamma}_2),
\]

in which \( \Pi_1(\tilde{\gamma}_1, \tilde{\gamma}_2) = 0 \) as bidder 1 always loses the auction by bidding \( \tilde{b}_1(t_1) = a \).

This shows that \( b_1 = v_1(t_1, t_1) \) is not a dominant strategy for every bidder 1. The argument for bidder 2 is similar.

Notice that in the case when \( \phi(t) = t \), for every \( t \in [\underline{t}, \overline{t}] \), the distributional perfect BNE in which each player \( i \) bids \( b_i(t_i) = v_i(t_i, t_i) \) is a symmetric BNE. Moreover, if we assume that

\[
\frac{\partial v_i}{\partial t_i}(t_1, t_2) > \frac{\partial v_j}{\partial t_i}(t_1, t_2),
\]

for \( j \neq i \), then \( (b_1, b_2) \), in which \( b_i = v_i(t_i, t_i) \) for every bidder \( i \), is an efficient equilibrium.\(^{14}\) According to Krishna [16], this condition is called the single crossing condition and ensures that the bidder with higher type has higher ex post valuation. Another interesting case of Proposition 5.1 is when \( \Lambda \) is a common value auction, i.e. when \( v_1(t_1, t_2) = v_2(t_1, t_2) \). In this case, the condition \( v_i(t, \phi(t)) = v_i(\phi(t), t) \), for \( i = 1, 2 \), holds for every increasing bijection \( \phi \). Therefore, in common value auctions for every increasing bijection \( \phi \), the strategy profile \( (b_1, b_2) \) is distributional perfect.

In private value Vickrey auctions, each bidder has a dominant strategy that forms a distributional perfect BNE and leads to an efficient outcome. This observation motivates us to study the relation between distributional perfection and efficiency in our setting:

**Remark.** In the auction \( \Lambda \) with single crossing condition, if \( \phi(t) \geq \phi^{-1}(t) \), for every \( t \in [\underline{t}, \overline{t}] \), then the strategy profile \( (b_1, b_2) \) is efficient. The argument is as follows: Suppose that \( x, y \) are the actual types that bidders 1 and 2 receive respectively and \( x > y \). Then, the single crossing condition implies that bidder 1’s ex post valuation is strictly larger than bidder 2’s, that is \( v_1(x, y) > v_2(x, y) \). Now, if \( \phi(y) \geq \phi^{-1}(y) \), then we have

\[
b_1(x) = v_1(x, \phi(x)) > v_1(y, \phi(y)) \geq v_1(y, \phi^{-1}(y)) = v_2(\phi^{-1}(y), y) = b_2(y).
\]

\(^{14}\)An outcome in any auction is called efficient if the winner is the one with the highest ex post valuation.
This means that bidder 1 who had a higher ex post valuation for the object, wins the auction and is awarded the object.

Birulin in [10] on page 678 introduces the following class of discontinuous asymmetric strategy profiles for every \( s_1, s_2 \in \left[ t, \bar{T} \right] \) with \( s_1 < s_2 \):

\[
\hat{b}_1(t_1) = \begin{cases} 
  v_1(s_2, t_1) & \text{if } t_1 \in [s_1, s_2] \\
  v_1(t_1, t_1) & \text{otherwise},
\end{cases}
\]

\[
\hat{b}_2(t_2) = \begin{cases} 
  v_2(t_2, s_1) & \text{if } t_2 \in [s_1, s_2] \\
  v_2(t_2, t_2) & \text{otherwise}.
\end{cases}
\]

He shows that if we assume the single crossing condition for the auction \( \Lambda \), then \((\hat{b}_1, \hat{b}_2)\) is an ex post equilibrium in undominated strategies, moreover this equilibrium allocates the object inefficiently. In Proposition 5.3 we prove that distributional perfection at least rules out a subclass of these equilibria.

**Proposition 5.3** Suppose the auction \( \Lambda \) satisfies the single crossing condition such that one of the followings holds:

1. \( v_2(s_2, s_1) < v_2(l, s_2) \),
2. \( v_1(s_2, s_1) > v_1(s_1, T) \).

Then, the ex post equilibrium \((\hat{b}_1, \hat{b}_2)\) is not distributional perfect.

**Proof.** First we discuss the case \( v_2(s_2, s_1) < v_2(l, s_2) \). Suppose by way of contradiction that \((\hat{b}_1, \hat{b}_2)\) is distributional perfect. Let \( \hat{\gamma}_i \) be the distributional strategy induced by \( \hat{b}_i \) and \((\gamma^k)_{k=1}^\infty\) be a sequence of completely mixed distributional strategies for each bidder \( i = 1, 2 \) that satisfies condition (i) and (ii) of Definition 3.1. Therefore, for bidder 2 and every \( k \), there is a distributional strategy \( \tilde{\gamma}_2^k \in BR_2(\gamma^k) \) such that \( \rho^w(\hat{\gamma}_2^k, \tilde{\gamma}_2^k) \to 0 \) as \( k \to \infty \). Hence, by triangle inequality we conclude that \( \rho^w(\hat{\gamma}_2^k, \gamma_2) \to 0 \) as \( k \to \infty \). For every \( k \), let \( \beta_1^k, \beta_2^k \), and \( \tilde{\beta}_2^k \) be a corresponding behavioral strategy to \( \gamma_1^k, \gamma_2^k \), and \( \tilde{\gamma}_2^k \) respectively. Also, let \( \beta^k = (\beta_1^k, \beta_2^k) \). Then, by Lemma 6.1 we have \( \tilde{\beta}_2^k(t_2, \cdot) \in BR_2(t_2, \beta^k) \) for \( \mu_2 \)-a.e. \( t_2 \in T_2 \). By redefining \( \tilde{\beta}_2^k \), without loss of generality, we can assume that \( \tilde{\beta}_2^k(t_2, \cdot) \in BR_2(t_2, \beta^k) \) for every \( t_2 \in T_2 \).

\(^{15}\)Our presentation of the bid functions are slightly different from Birulin [10]. He presents the bid functions in terms of the opponents’ valuations. We do this for each bidder in terms of his own valuations.
As \( v_2(s_2, s_1) < v_2(t, s_2) \), then there is a \( \delta > 0 \) such that for every \( t_2 \in (s_2 - \delta, s_2] \), we have \( v_2(t_2, s_1) < v_2(t, t_2) \). Let bidder 2’s type be \( t_2 \in (s_2 - \delta, s_2] \). It is obvious that bidder 2’s valuation is at least \( v_2(t, t_2) \). Since for every \( k \), \( \gamma^k_1 \) is completely mixed, for every nonempty open set \( B \subseteq A_1 \), there is a type \( t_1 \in T_1 \) such that \( \beta^k_1(t_1, B) > 0 \). Therefore, bidder 1 might bid any bid \( a_1 \in A_1 \) according to \( \beta^k_1 \). If \( a_1 < v_2(t, t_2) \), then bidding \( v_2(t, t_2) \) is better than bidding \( p_2 < v_2(t, t_2) \), since it increases the chance of winning. If \( a_1 \geq v_2(t, t_2) \), then bidder 2 is indifferent between bidding any bid \( p_2 \leq v_2(t, t_2) \). Therefore, bidder 2’s best response against \( \beta^k_1 \), for every \( k \), can not be any bid below \( v_2(t, t_2) \). Moreover, for every \( t_2 \in (s_2 - \delta, s_2] \), we have \( \hat{b}_2(t_2) = v_2(t_2, s_1) < v_2(t, t_2) \). Hence, if bidder 2 is given a type \( t_2 \in (s_2 - \delta, s_2] \), then any of his best responses against \( \beta^k_1 \), for every \( k \), is far from \( \hat{b}_2(t_2) \). In particular, as for every \( t_2 \in T_2 \) we have \( \overline{\beta}^k_2(t_2, \cdot) \in BR_2(t_2, \beta^k) \), for every \( k \), then for every \( t_2 \in (s_2 - \delta, s_2] \) we have that \( \rho^w(\hat{b}_2(t_2), \overline{\beta}^k_2(t_2, \cdot)) \) does not converge to zero when \( k \to \infty \). Notice that \( \hat{b}_2(t_2) \) is interpreted as a behavioral strategy that assigns probability 1 on \( \hat{b}_2(t_2) \) for every \( t_2 \in T_2 \).

Now, we show that there is \( \varepsilon > 0 \) such that \( \rho^w(\overline{\gamma}^k_2, \tilde{\gamma}_2) > \varepsilon \) for every \( k \). One can find \( \varepsilon > 0 \) and \( \lambda > 0 \) such that if for every \( t_2 \in (s_2 - \delta, s_2] \) we define

\[
B_{t_2} = \left( \hat{b}_2(t_2) - \lambda, \hat{b}_2(t_2) + \lambda \right),
\]

then we have

1. \( s_2 - \delta + \varepsilon < s_2 - \varepsilon \),

2. \( \mu_2(s_2 - \delta + \varepsilon, s_2 - \varepsilon) > \varepsilon \),

3. \( \overline{\beta}^k_2(t_2, (B_{t_2})^\varepsilon) = 0 \).

Now, define \( B = \{(t_2, a_2) \mid t_2 \in (s_2 - \delta + \varepsilon, s_2 - \varepsilon], a_2 \in B_{t_2}\} \). Clearly, \( B \) is measurable. As \( \hat{b}_2(t_2)(B_{t_2}) = 1 \) and \( \overline{\beta}^k_2(t_2, (B_{t_2})^\varepsilon) = 0 \), we have

\[
\tilde{\gamma}_2(B) = \int_{s_2 - \delta + \varepsilon}^{s_2 - \varepsilon} \hat{b}_2(t_2)(B_{t_2}) \mu_2(dt_2) = \mu_2(s_2 - \delta + \varepsilon, s_2 - \varepsilon),
\]

and

\[
\overline{\gamma}^k_2(B^\varepsilon) = \int_{s_2 - \delta}^{s_2} \overline{\beta}^k_2(t_2, (B_{t_2})^\varepsilon) \mu_2(dt_2) = 0.
\]
Consequently,

\[ \gamma_2(B) > \gamma_2^k(B^c) + \varepsilon, \]

then \( \rho^w(\gamma_2^k, \gamma_2) > \varepsilon \) for every \( k \). This contradicts with the assumption \( \rho^w(\gamma_2^k, \gamma_2) \rightarrow 0 \) as \( k \rightarrow \infty \). Therefore, \((\hat{b}_1, \hat{b}_2)\) is not distributional perfect.

With an analogous argument for the case \( v_1(s_2, s_1) > v_1(s_1, t) \), the proof of the proposition is complete.

Conditions (1) and (2) in the above proposition are not very restrictive. For example, all auctions in which either \( s_1 = t \) or \( s_2 = \bar{t} \), satisfy conditions (1) and (2) respectively. As an extreme case of this type of equilibria, we see that the BNE in which one bidder always bids \( \bar{a} \) and the other bidder always bids \( a \) is not distributional perfect. This kind of equilibrium is called wolf and sheep equilibrium and is considered as a less intuitive one.

In the auction \( \Lambda \), we change the condition (3) to be \( v_i \) is strictly increasing at \( t_i \) and decreasing at \( t_j \) for \( j \neq i \) and we call it \((3')\). Let \( \Lambda' \) be the auction that satisfies conditions (1), (2), \((3')\), and single crossing condition. In this auction \( v_1(t, \bar{t}) = v_2(\bar{t}, t) = a \) and \( v_1(\bar{t}, t) = v_2(t, \bar{t}) = \bar{a} \). Moreover, the class of BNEs corresponding to \( \phi \) in Proposition 5.1 reduces to the single equilibrium where \( \phi(t) = t \), that is \( b_1(t_1) = v_1(t_1, t_1) \) and \( b_2(t_2) = v_2(t_2, t_2) \). In the next proposition we show that this equilibrium is distributional perfect. To prove perfection in the following proposition we encounter more difficulties as compare to the proposition 5.1. The reason is that the bid functions of the bidders are not surjective which together with the possibility of making mistakes from the opponent, give the opportunity to the bidder to bid strictly higher than \( b_i(1) = v_i(1, 1) \) when he is from high type. The similar argument is valid if the bidder is from low type, i.e. he can bid strictly less than \( b_i(0) = v_i(0, 0) \).

**Proposition 5.4** In the auction \( \Lambda' \), the strategy profile \((b_1, b_2)\) in which \( b_1(t_1) = v_1(t_1, t_1) \) and \( b_2(t_2) = v_2(t_2, t_2) \) is a distributional perfect BNE.

**Proof.** Let \( w(t) = v_1(t, t) = v_2(t, t) \), then the function \( w : [t, \bar{t}] \rightarrow [a, \bar{a}] \) is strictly increasing but not surjective. It is easy to check that \((b_1, b_2)\) is a BNE. Now, we prove that it is distributional perfect. Let \( \gamma_1 \) be the distributional strategy induced by \( b_1(t_1) \) and for every \( k \in N \) define \( \gamma_1^k \) and \( \beta_1^k \) as in the proposition 5.1. Let \( p_2 \) be a pure behavioral
strategy for bidder 2 and \( \sigma_2 \) be the induced distributional strategy by \( p_2 \). Then, the expected payoff of bidder 2 choosing \( \sigma_2 \) against \( \gamma_1^k \) conditioning on winning is:

\[
\Pi_2(\gamma_1^k, \sigma_2) = \int_{T_2} \Pi_2^b(\beta_1^k, p_2 \mid t_2) \mu_2(dt_2),
\]

To maximize \( \Pi_2(\gamma_1^k, \sigma_2) \), we maximize \( \Pi_2^b(\beta_1^k, p_2 \mid t_2) \) in the following three cases:

1. \( a \leq p_2 \leq w(t) \):

\[
\Pi_2^b(\beta_1^k, p_2 \mid t_2) = 0 + \varepsilon_k \int_{p_2 > a_1} (v_2(t_1, t_2) - a_1) \, d\eta_1
\]

2. \( w(t) \leq p_2 \leq w(T) \):

\[
\Pi_2^b(\beta_1^k, p_2 \mid t_2) = (1 - \varepsilon_k) \int_{t}^{w^{-1}(p_2)} (v_2(t_1, t_2) - w(t_1)) \mu_1(dt_1) + \varepsilon_k \int_{p_2 > a_1} (v_2(t_1, t_2) - a_1) \, d\eta_1
\]

3. \( w(T) \leq p_2 \leq \bar{a} \):

\[
\Pi_2^b(\beta_1^k, p_2 \mid t_2) = (1 - \varepsilon_k) \int_{t}^{T} (v_2(t_1, t_2) - w(t_1)) \mu_1(dt_1) + \varepsilon_k \int_{p_2 > a_1} (v_2(t_1, t_2) - a_1) \, d\eta_1
\]

Let \( p_{21}^k \), \( p_{22}^k \) and \( p_{23}^k \) be the bid functions for bidder 2 that maximizes \( \Pi_2^b(\beta_1^k, p_2 \mid t_2) \) in the cases 1, 2 and 3 respectively for every \( t_2 \). With a similar argument as in the proposition 5.1, we can see that \( p_{22}^k \) converges to \( b_2(t_2) = v_2(t_2, t_2) = w(t_2) \), as \( k \to \infty \).

Moreover, let

\[
M_k^1 = \{ t_2 \in T_2 \mid \Pi_2^b(\beta_1^k, p_{21}^k \mid t_2) \geq \Pi_2^b(\beta_1^k, p_{22}^k \mid t_2) \},
\]

for every \( k \in \mathbb{N} \). It is obvious that \( \mu_2(M_k^1) \to 0 \) when \( k \to \infty \). Also, let

\[
M_k^2 = \{ t_2 \in T_2 \mid \Pi_2^b(\beta_1^k, p_{23}^k \mid t_2) \geq \Pi_2^b(\beta_1^k, p_{22}^k \mid t_2) \}.
\]

Notice that \( v_2(t_1, t_2) - w(t_1) \geq 0 \) for every \( t_1 \leq t_2 \) and negative otherwise, also we know that \( w^{-1}(p_{22}^k) \to t_2 \), as \( k \to \infty \). Hence, for \( \mu_2 \)-a.e. \( t_2 \in T_2 \) and enough large \( k \),

\[
\Pi_2^b(\beta_1^k, p_{23}^k \mid t_2) \leq \Pi_2^b(\beta_1^k, p_{22}^k \mid t_2).
\]

This means that \( \mu_2(M_k^2) \to 0 \) when \( k \to \infty \). Therefore, for \( \mu_2 \)-a.e. \( t_2 \in T_2 \), \( \Pi_2^b(\beta_1^k, p_2 \mid t_2) \) is maximized when \( p_2 = b_2(t_2) = v_2(t_2, t_2) \) as \( k \to \infty \). By a similar argument for bidder 2 the proof is complete.
In the auction $\Lambda'$, there are many equilibria similar to $(\hat{b}_1, \hat{b}_2)$ in auction $\Lambda$ that can be ruled out by distributional perfection. The examples of such equilibria can be found in Bajoori et al. [5].

6 Appendix

Lemma 6.1 Let $\beta = (\beta_1, \ldots, \beta_n)$ be a behavioral strategy profile and $\gamma = (\gamma_1, \ldots, \gamma_n)$ be the induced distributional strategy profile. Assume that the set of the best responses of player $i$ against behavioral strategy profile $\beta_{-i}$ is not empty. Then, if $\gamma_i \in BR_i(\gamma)$, then $\beta_i(t_i, \cdot) \in BR_i(t_i, \beta)$ for $\mu_i$-a.e. $t_i \in T_i$.

Proof. Suppose for every player $i$ that $\gamma_i \in BR_i(\gamma)$. Then, we have

$$\Pi_i(\gamma_i, \gamma_{-i}) \geq \Pi_i(\tilde{\gamma}_i, \gamma_{-i}),$$

for every player $i$’s distributional strategy $\tilde{\gamma}_i$. Therefore, for every player $i$’s behavioral strategy $\tilde{\beta}_i$ we have

$$\int_{T_i} \Pi_i^b(\beta_i, \beta_{-i} \mid t_i) \mu_i(dt_i) \geq \int_{T_i} \Pi_i^b(\tilde{\beta}_i, \beta_{-i} \mid t_i) \mu_i(dt_i).$$

(1)

Moreover, by assumption, for every player $i$ there is a behavioral strategy $\zeta_i$ such that $\zeta_i(t_i, \cdot) \in BR_i(t_i, \beta)$ for every $t_i \in T_i$, which means having type $t_i \in T_i$

$$\Pi_i^b(\zeta_i, \beta_{-i} \mid t_i) \geq \Pi_i^b(a_i, \beta_{-i} \mid t_i),$$

for every $a_i \in A_i$. To prove that $\beta_i(t_i, \cdot)$ for $\mu_i$-a.e. $t_i \in T_i$ is a best response for every player $i$ against $\beta_{-i}$, we show that $\Pi_i^b(\zeta_i, \beta_{-i} \mid t_i) = \Pi_i^b(\beta_i, \beta_{-i} \mid t_i)$ for $\mu_i$-a.e. $t_i \in T_i$. Suppose the opposite. First, let

$$S_i = \left\{ t_i \in T_i \mid \Pi_i^b(\zeta_i, \beta_{-i} \mid t_i) < \Pi_i^b(\beta_i, \beta_{-i} \mid t_i) \right\}.$$

Clearly, $S_i$ is $\mu_i$-measurable. Suppose $\mu_i(S_i) > 0$. Notice that given $t_i \in T_i$, for $\beta_i(t_i, \cdot)$-a.e. $a_i \in A_i$ we have

$$\Pi_i^b(\beta_i, \beta_{-i} \mid t_i) = \Pi_i^b(a_i, \beta_{-i} \mid t_i).$$

Then, given $t_i \in S_i$, for $\beta_i(t_i, \cdot)$-a.e. $a_i \in A_i$ we conclude that

$$\Pi_i^b(\zeta_i, \beta_{-i} \mid t_i) < \Pi_i^b(a_i, \beta_{-i} \mid t_i),$$
which is a contradiction with optimality of $\zeta_i$ for every type. Second, let

$$S_i = \left\{ t_i \in T_i \mid \Pi^b_i(\zeta_i, \beta_{-i} \mid t_i) > \Pi^b_i(\beta_i, \beta_{-i} \mid t_i) \right\}$$

and suppose $\mu_i(S_i) > 0$. Therefore, we have

$$\int_{S_i} \Pi^b_i(\zeta_i, \beta_{-i} \mid t_i) \mu_i(dt_i) > \int_{S_i} \Pi^b_i(\beta_i, \beta_{-i} \mid t_i) \mu_i(dt_i).$$

Since $\zeta_i(t_i, \cdot)$ is a best response for every type $t_i \in T_i$ against $\beta_{-i}$, for every $a_i \in A_i$ we have

$$\Pi^b_i(\zeta_i, \beta_{-i} \mid t_i) \geq \Pi^b_i(a_i, \beta_{-i} \mid t_i),$$

thus

$$\int_{T_i \setminus S_i} \Pi^b_i(\zeta_i, \beta_{-i} \mid t_i) \mu_i(dt_i) \geq \int_{T_i \setminus S_i} \Pi^b_i(\beta_i, \beta_{-i} \mid t_i) \mu_i(dt_i).$$

Consequently,

$$\int_{T_i} \Pi^b_i(\zeta_i, \beta_{-i} \mid t_i) \mu_i(dt_i) > \int_{T_i} \Pi^b_i(\beta_i, \beta_{-i} \mid t_i) \mu_i(dt_i).$$

This is a contradiction with (1). Overall, we showed that if there is a subset of $T_i$ on which the expected payoff of player $i$, playing according to $\zeta_i$ is different from playing according to $\beta_i$, the measure of that subset is zero. In other words, for every $i$ there exists a set $S_i \in T_i$ with $\mu_i(S_i) = 0$ such that for every $t_i \in T_i \setminus S_i$ we have $\beta_i(t_i, \cdot) \in \text{BR}_i(t_i, \beta)$. This completes the proof.

In the lemma 6.2, we use the following metrics: Take $t, s \in T$ and $a, b \in A$. Define

$$d_T(t, s) = \left( \sum_{i=1}^{n} d^2_{T_i}(t_i, s_i) \right)^{\frac{1}{2}}, \quad d_A(a, b) = \left( \sum_{i=1}^{n} d^2_{A_i}(a_i, b_i) \right)^{\frac{1}{2}}.$$  

Moreover, let $d_{T \times A}((t, a), (s, b)) = (d_T(t, s) + d_A(a, b))^\frac{1}{2}$.

**Lemma 6.2** In Bayesian game $\Gamma$, for every player $i$ and every $\varepsilon > 0$, there is a set $K \in T$ with $\mu(K^c) < \varepsilon$ such that $\pi_i$ is continuous on $K \times A$ for every $i$.

**Proof.** Take $\varepsilon > 0$ and $a \in A$. First we prove that there is a set $K \in T$ with $\mu(K^c) < \varepsilon$ such that the function $\pi_i(\cdot, a) : K \rightarrow \mathbb{R}$ is continuous. As $A$ is separable, it has a countable dense subset, say $B$. Hence, there exists a sequence $(a_k)_{k=1}^\infty$ in $B$ such
that $a_k \neq a$, for every $k$, and $d_A(a_k, a) \to 0$ as $k \to \infty$. According to Lusin’s Theorem for the collection of the functions $\{\pi_i(\cdot, a_k) \mid k = 1, 2, \ldots\}$ there is a compact set $E \in \mathcal{T}$ with $\mu(E^c) < \frac{\varepsilon}{2}$ such that $\pi_i(\cdot, a_k)$ is continuous on $E$ for every $k \in \mathbb{N}$. Moreover, by equicontinuity of payoffs there is a set $F \in \mathcal{T}$ with $\mu(F^c) < \frac{\varepsilon}{2}$ such that the collection of the functions $\{\pi_i(t, \cdot) : A \to \mathbb{R} \mid t \in F\}$ is equicontinuous. Let $K = E \cap F$. It is clear that $\mu(K^c) < \varepsilon$. Moreover, because $\pi_i(t, \cdot)$ is continuous in actions for every $t \in K$, the sequence of the functions $(\pi_i(\cdot, a_k))_{k=1}^\infty$ converges pointwise to $\pi_i(\cdot, a)$ as $k \to \infty$. It is easy to check that equicontinuity of payoffs and the fact that $d_A(a_k, a) \to 0$ as $k \to \infty$, implies that the sequence of functions $(\pi_i(\cdot, a_k))_{k=1}^\infty$ uniformly converges to $\pi_i(\cdot, a)$ on $K$ as $k \to \infty$. Hence, $\pi_i(\cdot, a) : K \to \mathbb{R}$ is continuous too.

Now, we prove that $\pi_i$ is continuous at $(t, a) \in K \times A$. We know that $\{\pi_i(t, \cdot) \mid t \in K\}$ is equicontinuous, then there is a $\delta_1 > 0$ such that if $d_A(a, b) < \delta_1$, we have $|\pi_i(t, a) - \pi_i(t, b)| < \frac{\varepsilon}{2}$ for every $t \in K$. Also, we know $\pi_i(\cdot, a)$ is continuous at $t \in K$, so there is a $\delta_2 > 0$ such that if $d_T(t, s) < \delta_2$, we have $|\pi_i(t, a) - \pi_i(s, a)| < \frac{\varepsilon}{2}$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then, for every $(s, b) \in K \times A$ with $d_{T \times A}((t, a), (s, b)) < \delta$ we have $d_A(a, b) < \delta_1$ and $d_T(s, t) < \delta_2$. Consequently, we have

$$|\pi_i(t, a) - \pi_i(s, b)| < |\pi_i(t, a) - \pi_i(s, a)| + |\pi_i(s, a) - \pi_i(s, b)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$  

This proves that $\pi_i$ is continuous on $K \times A$.  

Lemma 6.3 In Bayesian game $\Gamma$, for every player $i$ the function $\Pi_i : \Delta \to \mathbb{R}$ is continuous with respect to the weak metric on $\Delta$.

Proof. To prove that $\Pi_i$ is continuous at a distributional strategy profile $\gamma$, let $(\gamma^k)_{k=1}^\infty$ be a sequence of distributional strategy profiles converging to $\gamma$ with respect to the weak metric as $k \to \infty$. Now, by Lemma 6.2 we have the following:

For every $i$ and every $\delta > 0$, there are a continuous and bounded function $v_\delta : T \times A \to \mathbb{R}$ and a set $K \in \mathcal{T}$ such that $\mu(K^c) < \delta$, and $v_\delta = \pi_i$ on $K \times A$.

As $\mu(K^c) < \delta$, we have $\gamma^k(K^c \times A) < \delta$, for every $k$. Similarly, $\gamma(K^c \times A) < \delta$. Moreover, because $v_\delta$ and $\pi_i$ are bounded, there is an $M > 0$ such that $|\pi_i - v_\delta| \leq M$. Hence, for every $k$ we have

$$\int_{T \times A} |\pi_i - v_\delta| f(t) d\gamma^k = \int_{K^c \times A} |\pi_i - v_\delta| f(t) d\gamma^k \leq M \cdot \delta.$$


Similarly,
\[\int_{T \times A} |\pi_i - v_\delta| f(t) \, d\gamma = \int_{K^e \times A} |\pi_i - v_\delta| f(t) \, d\gamma \leq M \cdot \delta.\]

Furthermore, as \(f\) is \(\hat{\mu}\)-integrable, there is a sequence \((f_\ell)_{\ell=1}^\infty\) of bounded and continuous functions such that \(\int_T |f(t) - f_\ell(t)|\hat{\mu}(dt) \to 0\), as \(\ell \to \infty\). Now, we have
\[
|\Pi_i(\gamma^k) - \Pi_i(\gamma)| = |\int \pi_i f(t) d\gamma^k - \int \pi_i f(t) d\gamma| \\
\leq \int |\pi_i - v_\delta| f(t) d\gamma^k + \int |f(t) - f_\ell(t)|v_\delta d\gamma^k \\
+ \int v_\delta f_\ell(t) d\gamma^k - \int v_\delta f(t) d\gamma \\
+ \int |f(t) - f_\ell(t)|v_\delta d\gamma + \int |\pi_i - v_\delta| f(t) d\gamma \\
\leq M \delta + \int |f(t) - f_\ell(t)|v_\delta d\gamma^k \\
+ \int v_\delta f_\ell d\gamma^k - \int v_\delta f_\ell d\gamma \\
+ \int |f(t) - f_\ell(t)|v_\delta d\gamma + M \delta.
\]

Hence, \(|\Pi_i(\gamma^k) - \Pi_i(\gamma)|\) converges to zero when \(k \to \infty\), \(\ell \to \infty\), and \(\delta \to 0\). This proves that \(\Pi_i\) is continuous at \(\gamma\).

References


