A Detail-Free and Efficient Auction for Budget Constrained Bidders

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Abstract

Consider an auction for a divisible good where bidders have private budgets. Recent work by Dobzinski, Lavi, and Nisan (2012) shows there is no individually rational dominant strategy mechanism that implements a Pareto efficient outcome and satisfies weak budget balance when bidders have private budgets.

My main result shows that when bidders have full-support beliefs over their rivals’ types, a clinching auction played by proxy-bidders implements a Pareto efficient outcome. The auction is not dominant strategy implementable, but it can be solved using two rounds of iterative deletion of weakly dominated strategies. The predictions do not require that bidders share a common prior and they place no restrictions on higher-order beliefs. The results are also extended to the sale of an indivisible good.

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1 Introduction

The Vickrey-Clarke-Groves mechanism is celebrated as a major achievement in the theory of mechanism design. However, the practical applicability of the VCG mechanism is limited when there is a difference between a bidder’s willingness to pay for goods and her ability to pay. In particular, the VCG mechanism loses its desired efficiency and incentive properties when bidders are budget constrained.

Prior work on auctions with budgets provided alternatives to VCG that are efficient, under the restriction that budgets are public.\textsuperscript{1} The public budget restriction is useful from a theoretical perspective, because it allows us to model bidders as having one dimensional types. However, when bidders’ budgets are private, Dobzinski, Lavi, and Nisan (2012) (hereafter, DLN) show that efficient auction design is incompatible with dominant strategy incentive compatibility and weak budget balance. The contribution of this paper is to show that efficient auction design is possible, even when bidders have private budgets and multidimensional private information. Specifically, I show that a clinching auction played by proxy bidders is efficient and solvable using two rounds of iterative deletion of weakly dominated strategies. My only additional assumption is that bidders have full-support beliefs over other bidders’ values and budgets. The mechanism is detail-free and places no restrictions on bidders sharing a common prior.

The question of efficient auction design with budgets is practically relevant. This is because the presence of budgets plays a crucial role in determining bid behavior\textsuperscript{2} and budgets are a prominent feature in many well-studied auction markets. In online ad auctions used by Yahoo! and Google, Dobzinski, Lavi, and Nisan (2012) argue that budgets are more important than values when people determine bids. Salant (1997) shows that consultants considered bidders’ budgets and values when determining bids for spectrum rights. In a similar example, Rothkopf (2007) recalls his consulting experience to argue that budgets limit the usefulness of the Vickrey auction. He advised a firm that valued a spectrum license at $85 million, yet was only able to finance a $65 million bid. Maskin (2000) claims that budgets should be a consideration for governments who are selling publicly-owned assets. He says that budgets are especially important in developing economies, because bidders are less likely to have easy access to well-functioning credit markets. In most of these cases, it is natural to assume that firms have private information on their liquidity and access to financial markets. Thus bidders’ budgets are private, and bidders have beliefs over their rivals’ budgets.

\textsuperscript{1}See Maskin (2000), and Dobzinski, Lavi, and Nisan (2012).

\textsuperscript{2}For one example, Che and Gale (1996) shows that revenue equivalence fails when bidders have budgets. Many other examples are discussed in the related literature section.
In this paper, I design an efficient and detail-free auction when bidders have private values and private budgets. While the setting I study is similar to the setting in DLN, I get contrasting results. There are two reasons for this. First, DLN’s impossibility result only applies to dominant strategy implementable mechanisms. The proxy clinching auction that I study is not dominant strategy implementable. Instead, I show overreporting values and underreporting budgets are dominated, and truthful reporting is the unique prediction of two rounds of iterative elimination of dominated strategies.

Second, DLN’s impossibility theorem places no restrictions on bidder beliefs. For the case of public budgets, DLN show that a clinching auction is efficient and dominant strategy implementable. However, when budgets are private, bidders may have an incentive to overreport their budgets. This incentive to overreport budgets disappears when we place a mild full-support condition on bidders’ beliefs over their rivals’ types. Specifically, I show that if a bidder believes that her rivals play undominated strategies, and she has full-support beliefs, then truthful reporting is her unique undominated best reply. The bidder does not overreport her budget because she believes there is a positive probability of paying an amount that exceeds her actual budget if she overreports. Thus, truthful reporting is the unique prediction of two rounds of iterative elimination when we assume bidders have full-support beliefs on their rivals’ types. As a robustness check, I show that we obtain similar results when we relax the assumption that bidders have hard budgets. In particular, I consider the case where bidders have continuous utility functions and get high (but finite) disutility from spending an amount that exceeds their budget.

The assumption of full-support beliefs is general enough to nest cases where bidder types are i.i.d., correlated, or do not satisfy a common prior assumption. The assumption is similar to the full-support assumption used by Ausubel (2004). There are no restrictions on bidders’ higher-order beliefs. Thus, the proxy clinching auction is detail-free, and robust in the sense of Wilson (1987).

I extend my results to a setting with an indivisible good by using randomization. In the indivisible good setting, the proxy clinching auction sells probabilities of winning the indivisible good like a divisible good in net supply one. Just as in the divisible goods case, the proxy clinching auction implements an (ex-post) Pareto efficient allocation when the good is indivisible.

The rest of the paper proceeds as follows. The remainder of the introduction relates my work to the literature on auctions with budgets. Section 2 formalizes the auction setting, describes the proxy clinching auction, and lists some basic properties of the mechanism. Section 3 provides a motivating example. Section 4 characterizes bid behavior in the auction. Section 5 discusses efficiency. Section 6 extends the results to the indivisible good setting.
Section 7 concludes.

Related Literature

Budgets are practically relevant in many auction settings. Indeed, the presence of budgets influences bid behavior. For example, the presence of budgets has been shown to change revenue and efficiency rankings of standard auction formats.  

Since the presence of budgets influences how bidders behave in auctions, it will similarly influence how sellers would like to design auctions. Much of the work on auction design with budgets fits into one of two categories: the design of revenue maximizing auctions and the design of efficient auctions. The earlier work on auction design is restricted to cases where bidders have public budgets. On revenue, Laffont and Roberts (1996) construct a revenue maximizing auction; and on efficiency, Maskin (2000) presents an efficient auction.

The restriction to public budgets is useful, because it allows us to study bidders with one-dimensional private information. However, the public budget restriction is often violated in practice. For example, in Google’s auctions for television ad space, they allow bidders to input a budget and willingness to pay for a block of advertisements (see Nisan et al. (2009)). Indeed, DLN argue that budgets are more salient to bidders than their valuations. For another example, consider auctions for spectrum licenses. Firms may have incomplete information on their rivals’ cash holdings and access to credit. Thus, auction design with private budgets is practically relevant.

More recent work has studied auction design with private budgets. Che and Gale (2000), Pai and Vohra (2014), and Baisa (2014) study revenue maximizing auctions in this setting. However, this paper focuses on efficiency. I consider a seller who sells a divisible good to bidders with private values and private budgets. This setting is closest to the settings studied by Borgs et al. (2005); DLN; and Hafalir, Ravi, and Sayedi (2012). DLN show that there is no efficient auction that is dominant strategy implementable, satisfies weak budget balance, and is individually rational. The contribution of this paper is to show that efficient auction design is possible if (1) bidders have full-support beliefs over their rivals’ types, and (2) we weaken our solution concept to iterative elimination of dominated strategies.

Other papers have proposed alternatives to VCG when bidders have budgets. Hafalir, Ravi, and Sayedi (2012) propose a Vickrey auction with budgets. Just as in this paper, Hafalir, Ravi, and Sayedi (2012) obtain bounds on bid behavior by eliminating dominated strategies. However, their mechanism does not necessarily implement a Pareto efficient allocation, and it is not solvable using iterative elimination. A precise characterization of bid behavior would likely require that bidders share a common prior.

Bhattacharya et al. (2010) consider a similar setting to that of DLN and Hafalir, Ravi, and Sayedi (2012). Like this paper, they study efficient auction design and focus on clinching auctions. However, they exogenously assume that bidders are unable to overreport their budgets. With this restriction, they achieve a similar result; a clinching auction is efficient, dominant strategy implementable, and satisfies weak budget balance. The contribution of this paper is to show that the clinching auction is efficient, even without an exogenously restricting bidder reports. In particular, bidders have an endogenous incentive to not overreport their budgets when they have full-support beliefs over their rivals’ types. Bhattacharya et al. (2010) also considers using randomized payments to induce bidders to truthfully report their budgets. A second contribution of this paper is to show that a deterministic mechanism can also induce bidders to truthfully report their budgets. In addition, deterministic mechanisms have the advantage of being practically implementable, as randomized payments are rarely seen in practice, and can lead to violations ex-post individual rationality.

Most recently, Che, Gale, and Kim (2013) study the related question of designing an efficient mechanism for a unit mass of budget constrained agents. However, their focus is on a case where there is a $2 \times 2$ type space.

## 2 Model

Much of my model description closely follows DLN. I later discuss how my results extend to the sale of an indivisible good.

A seller owns a divisible good that is in net supply one. There are $N$ bidders, where bidder $i$ has value $v_i$ for each unit she wins, and budget $b_i$. I call $\theta_i = (v_i, b_i)$ bidder $i$’s payoff type and assume that $\theta_i \in \Theta$, where $\Theta := \{\theta \in [0, 1]^2 | 0 \leq b \leq v \leq 1\}$. Thus, if $v_i = b_i$, then

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4The setting is isomorphic to one where $\Theta := \{(v, \bar{v})^2 | 0 \leq \bar{v} \leq b \leq v \leq \bar{v} < \infty \}$. We assume $[v, \bar{v}] = [0, 1]$
bidder $i$’s willingness to pay for one unit of the good equals her ability to pay. If $v_i > b_i$, then bidder $i$ faces a binding budget constraint. For ease of notation, I say $u_i(x_i, p_i)$ is the utility of bidder $i$ with payoff type $\theta_i$, when she wins $x_i \in [0, 1]$ units and pays $p_i$, where

$$u_i(x_i, p_i) = \begin{cases} v_i x_i - p_i & \text{if } p_i \leq b_i, \\ -\infty & \text{if } p_i > b_i. \end{cases}$$

Bidder $i$’s payoff type $\theta_i$ is private information.

An outcome describes payments and the allocation of the good.

**Definition 1.** (Outcomes)

A (feasible) outcome $(x, p) \in [0, 1]^N \times \mathbb{R}^N$ is a vector of allocated quantities $x_1, \ldots, x_N$ and a vector of payments $p_1, \ldots, p_N$ with the property that $\sum_{i=1}^N x_i \leq 1$.

The proxy clinching auction implements feasible outcomes that are (ex-post) individually rational, satisfy weak budget balance, and are Pareto efficient. An outcome is individually rational if all bidders receive non-negative payoffs.

**Definition 2.** (Individual rationality)

An outcome $(x, p) \in \mathbb{R}_+^{2N}$ is individually rational if $u_i(x_i, p_i) \geq 0 \ \forall i = 1, \ldots, N$.

Like DLN, I am interested in studying the implementation of outcomes that satisfy weak budget balance. DLN refer to this as a no positive transfers condition. Weak budget balance is an individual rationality constraint on the auctioneer that avoids trivializing the efficient implementation problem. If we do not impose a budget balance restriction, the auctioneer can pay all bidders a large amount and then hold a second price auction amongst the (now unconstrained) bidders. This is efficient and dominant strategy implementable, but violates budget balance.

**Definition 3.** (Weak budget balance)

An outcome $(x, p) \in \mathbb{R}_+^{2N}$ satisfies the weak budget balance if $\sum p_i \geq 0$.

Lastly, I study the implementation of (ex-post) Pareto efficient outcomes. My definition of Pareto efficiency is the same as the definition used by DLN.

**Definition 4.** (Pareto efficient)

An outcome $\{(x_i, p_i)\}$ is Pareto efficient if there does not exist a different outcome $\{(x_i', p_i')\}$ that makes all players better off, $x_i' v_i - p_i' \geq x_i v_i - p_i \ \forall i$ and gives weakly greater revenue $\sum_i p_i' \geq \sum p_i$, where at least one of the inequalities holds with a strict inequality.

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For simplicity. We could also allow for $b_i > v_i$; but such bidders have the same incentives as a bidder with $b_i = v_i$. 

Bidder $i$ has beliefs about the distribution of her rivals’ values and budgets. I assume that bidder $i$’s first-order beliefs satisfy a full-support assumption. Roughly speaking, this states that any realization of opponent payoff types is possible with some probability.

**Assumption 1.** (Full-support beliefs)

*Bidder $i$ has a full-support prior if for any $\theta_i \in \Theta$ and any subset $A \subset \Theta^{N-1}$ where $A$ has strictly positive Lebesgue measure $\mu(A) > 0$, then $F_i(A|\theta_i) > 0$.*

This is satisfied in models where bidder types are *i.i.d.* draws from some commonly known distribution that has full-support. However, I do not require that bidders share a common prior, nor do I require that it is commonly known that all bidder’s beliefs satisfy this condition.

We could go further, and explicitly model bidder $i$’s higher-order beliefs about other bidders’ beliefs, or her beliefs about other bidders’ beliefs about other bidders’ beliefs, and so on. However, explicitly modeling higher-order beliefs can be intractable, and is unnecessary for the purpose of this paper. Indeed, we do not even need to explicitly model bidder $i$’s beliefs over her opponents’ beliefs, or even assume that bidder belief types and payoff types satisfy a common prior.

DLN do not explicitly model bidders’ beliefs. In their examples, bidders have an incentive to overreport their budgets. I show that when bidders have full-support beliefs, bidders no longer have the incentive to overreport budgets.

### 2.1 Description of the mechanism

The proxy clinching auction is a direct revelation mechanism where bidders report a payoff type (value and budget) to the auctioneer. Proxy bidders then play a clinching auction, like that described by Ausubel (2004). However, I adapt the clinching mechanism to include budgets, and I allow bidders’ demands to change depending upon the amount of money they spend.

Thus, for bidder $i$, a pure strategy $a_i$ is a mapping from her type to her report, $a_i : \Theta \to \Theta$. Given the profile of reported bidder types $(\theta_1, \ldots, \theta_N)$ the proxy clinching auction determines the number of units bidder $i$ wins $Q_i : \Theta^N \to [0,1]$ and her payment $P_i : \Theta^N \to [0,1]$.

If all bidders report zero demand, the good is split equally among all bidders. That is, if the profile of reported types $(\theta_1, \ldots, \theta_N)$, is such that $b_i = 0 \ \forall i$, then $Q_i = \frac{1}{M} \mathbb{1}_{\theta_i > 0}$ and $P_i = 0 \ \forall i$, where $M$ is the number of bidders who report strictly positive values. If only one bidder $i$ reports positive demand, then $i$ wins all units for a price of zero. That is, if $b_i > 0$, and $b_j = 0 \ \forall j \neq i$, then $Q_i = 1$, $P_i = 0$ and $Q_j = P_j = 0 \ \forall j \neq i$. In each case, we say the auction terminates at time 0.
The non-trivial cases occur when at least two bidders report non-zero demands. That is, there exists bidders \( i, j \) such that \( i \neq j \) and \( b_i, b_j > 0 \). The proxy clinching auction starts at time 0. Time continuously increases until the auction terminates. The time \( t \) represents the marginal price of additional units at time \( t \).

More formally, let \( q_i(t) \in [0, 1] \) be the number of units clinched by bidder \( i \) at time \( t \). Similarly, \( p_i(t) \in \mathbb{R}_+ \) is the total amount that bidder \( i \) has committed to pay at time \( t \). I construct both \( q_i \) and \( p_i \) to be non-decreasing. At time \( t = 0 \), we set \( q_i(0) = p_i(0) = 0 \) \( \forall i \).

For convenience, denote \( p_i^-(t) := \lim_{t' \to t^-} p_i(t') \) and \( q_i^-(t) := \lim_{t' \to t^-} q_i(t') \).

Bidders continuously report their demands for additional units as time (i.e. the marginal price of additional units) increases. At time \( t \), bidder \( i \)'s demand for additional units is

\[
d_i(t) = \begin{cases} 
\min\{1 - q_i^-(t), \frac{b_i - p_i^-(t)}{t} \} & \text{if } b_i \geq p_i^-(t) \text{ and } v_i > t. \\
0 & \text{else}
\end{cases}
\]

Bidder \( i \) demands the maximal number of additional units she can afford if the marginal price of additional units is below her value. She demands no additional units if the marginal price of units exceeds her value, or if she has spent in excess of her budget. Let \( z_i(t) \) be the total number of units bidder \( i \) demands at time \( t \), including the units she has already clinched up to time \( t \), \( z_i(t) = q_i^-(t) + d_i(t) \). I refer to \( z_i(t) \) as the total demand of bidder \( i \). Thus, the total demand of bidder \( i \) at time \( t \) includes the units bidder \( i \) has already clinched up to time \( t \).

Each bidder \( i \) faces a residual demand curve \( s_i \) that is a function of other (proxy) bidders’ reported demands and the quantity they have clinched,

\[
s_i(t) = \begin{cases} 
1 - \sum_{j \neq i} z_j(t) & \text{if } 1 \geq \sum_{j \neq i} z_j(t), \\
0 & \text{else}
\end{cases}
\]

If at time \( t, d_i(t') > 0 \ \forall t' < t \), then

\[
q_i(t) = \min\{\sup_{t' \leq t} s_i(t'), d_i(t) + \sup_{t < t'} s_i(t)\}.
\]

That is, the supply curve determines the quantity that bidder \( i \) clinches, but we add the additional restrictions that (1) the quantity bidder \( i \) clinches is non-decreasing, and that (2) at time \( t \), bidder \( i \) never clinches any more additional units than she demands. Thus, the amount bidder \( i \) has clinched at time \( t \) can never exceed her total demand for units at time \( t \), \( z_i(t) \).

If we reach some time \( t \) where \( d_i(t) = 0 \), then bidder \( i \) does not clinch any additional units. In particular, if we define \( t^* = \sup\{t | d_i(t') > 0 \ \forall t' \leq t\} \) as the first time bidder \( i \) has
zero demand for additional units, then bidder $i$ clinches no additional units following time $t^*$.

$$q_i(t) = q_i(t^*) \text{ if } t > t^*.$$  

Therefore, having zero demand is equivalent to dropping out of the auction.

Bidder $i$ pays $t$ per unit for any additional units that she clinches at time $t$. Thus,

$$p_i(t) = q_i(t) t - \int_0^t q_i(s) ds.$$  

The auction terminates at time $\tau$, where $\tau$ is the first time the quantity of unclinched units up to $\tau$ (weakly) exceeds the demands for additional units, \(1 - \sum_{i=1}^{N} q_i^- (t) \geq \sum_{i=1}^{N} d_i(t)\). Or equivalently, time $\tau$ is the first time when all of the bidders total demands falls below the supply of units, \(1 \geq \sum_{i=1}^{N} z_i(t)\). The terminating time $\tau$ is a function of the full profile of bidder reports; I suppress notation for succinctness.

If \(1 = \sum_{i=1}^{N} z_i(\tau)\) at time $\tau$, then each bidder wins $Q_i = z_i(\tau)$ units and pays $P_i = p_i^-(\tau) + \tau d_i(\tau)$.

If \(1 > \sum_{i=1}^{N} z_i(\tau)\), we use a rationing rule. Note that \(\lim_{t' \to \tau^-} \sum_{i=1}^{N} z_i(t') \geq 1\). If not, the auction would terminate at a time earlier than $\tau$. Let $H := \lim_{t' \to \tau^-} \sum_{i=1}^{N} z_i(t')$ and $L := 1 > \sum_{i=1}^{N} z_i(\tau)$. By construction, $H \geq 1 \geq L$. Thus, in this case where $H > L$ and we use the rationing rule, bidder $i$ wins $Q_i$ units, where

$$Q_i := \frac{1 - L}{H - L} \left( \lim_{t \to \tau^-} z_i(t) \right) + \frac{H - 1}{H - L} z_i(\tau).$$

She pays $P_i$, where

$$P_i = p_i^-(\tau) + \tau \left( Q_i - q_i^- (\tau) \right).$$

If $z_i$ is left continuous at $\tau$, the above expression simplifies to say that bidder $i$ wins $z_i(\tau)$ units and pays $p_i^-(\tau) + \tau d_i(\tau)$. When $z_i$ has a left discontinuity at $\tau$, bidder $i$ wins between $\lim_{t \to \tau^-} z_i(t)$ and $z_i(\tau)$ units. The precise number of units is a weighted sum of the two quantities. The weights are chosen to ensure feasibility. Bidder $i$ pays $\tau$ for any additional units won at time $\tau$. Note that by construction, $\sum_{i=1}^{N} Q_i = 1$.

### 2.2 Basic properties of the proxy clinching auction

Lemma 1 below summarizes seven properties that follow from the construction of the proxy-clinching auction.

**Lemma 1.** (Properties of the proxy clinching auction)
Consider any profile of bidder reports \((\theta_1, \ldots, \theta_N)\) in the proxy clinching auction. Suppose that at least two bidders report positive demands \(b_i, b_j > 0\), for some \(i, j \in \{1, \ldots, N\}\). Then,

1. \(\tau > 0\).
2. \(z_i(t)\) is non-increasing in \(t\) over \((0, \tau)\).
3. \(s_i\) is non-decreasing in \(t\), and \(q_i(t) = s_i(t) \forall t \in (0, \tau)\).
4. \(\tau \leq 1\).
5. If \(d_i(t) = 0\) for some \(t < \tau\), then \(Q_i = P_i = 0\).
6. If \(t \in (0, \tau)\), then \(p_i(t) < b_i\); and \(P_i \in [0, b_i] \forall i\).

The first point states that if at least two people state positive demands, then the proxy clinching auction will not terminate at time 0. The second point states that a bidder’s total demand for units weakly decreases as time increases. This is intuitive as time represents the marginal price of additional units. A direct implication of this point is that the number of units bidder \(i\) has clinched at time \(t\) is the number of units that are neither clinched nor demanded by her opponents. The fourth point states that the auction terminates before the marginal price of additional units strictly exceeds all bidders’ values. The fifth point states that if a bidder reports that she has zero demand before the auction terminates, then she drops out of the auction. That is, she wins no units and pays nothing. The sixth point states that the proxy clinching auction never requires bidders to make a payment that exceeds their stated budget.

3 A Motivating Example

This example has three purposes. The first is to show how the proxy clinching auction maps reported types to outcomes. The second is to illustrate DLN’s impossibility result. The final purpose is to provide intuition for my main result: that the proxy clinching auction implements a Pareto efficient allocation when bidders have full-support beliefs.

There are two bidders, 1 and 2. Bidder 1 has type \(\theta_1 = (\frac{3}{4}, \frac{2}{3})\) and bidder 2 has type \(\theta_2 = (1, \frac{1}{2})\). Thus, both bidders have a willingness to pay that exceeds their ability to pay - they are budget constrained.

I describe an outcome of the clinching auction in this setting. Recall that at any time \(t\), the number of units clinched by a bidder \(i\) is the number of units that are not clinched or demanded by her opponents. Suppose both bidders report their types truthfully. In order
to study how units are allocated during the auction, we look consider the auction in three different stages.

1. When $t$ is sufficiently low, both bidders demand all units of the good and neither bidder clinches any units.

2. When $t$ is in an intermediate range, bidder 2 no longer demands all of the units of the good. This is because when bidder 2 is budget constrained, and she does not have the ability to pay for all units of the good when the price is sufficiently high. However, since bidder 1 has a greater budget, she still demands all units of the good. Thus, bidder 1 clinches any units of the good that are not demanded by bidder 2.

3. When $t$ becomes sufficiently large, bidder 1’s demand for units declines. This is because bidder 1 is also budget constrained. Bidder 1 is no longer able to pay for all units of the good when the marginal price of units is sufficiently large. Thus, both bidder 1 and bidder 2 clinch units. The number of units clinched by bidder 1 is the number of units that are not demanded or clinched by bidder 2. Similarly, the number of units clinched by bidder 2 is the number of units that are not demanded or clinched by bidder 1.

In the first stage, both bidders demand all of the available units of the good. At time $t \leq \frac{1}{2}$, both bidders have values and budgets that exceed the marginal price of units. Thus, both bidders demand one unit and,

$$q_i(t) = \max \{1 - z_j(t), 0\} = 0, \forall t \leq \frac{1}{2} \ i \neq j, \ i = 1, 2. \ \Rightarrow \ p_1(t) = p_2(t) = 0 \ \forall t \leq \frac{1}{2}.$$

When $t \in \left(\frac{1}{2}, 0.698\right]$ we are in the second stage described above. If the price of additional units is above $\frac{1}{2}$, bidder 1 no longer demands the entire quantity of the good. This is because bidder 1 is limited by her (reported) budget of $\frac{1}{2}$. Thus, she can not afford to buy all of the remaining units when the marginal price of a unit is above $\frac{1}{2}$. However, when the price is slightly above $\frac{1}{2}$, bidder 2 still demands all available units. This is because both her budget and value both exceed $\frac{1}{2}$. Thus, once the marginal price of additional units exceeds $\frac{1}{2}$, bidder 1 reduces her stated demands, and bidder 2 clinches any units that are not demanded by bidder 1. At the same time, bidder 1 does not clinch any units because bidder 2 demands all remaining unclinched units, $z_2(t) = 1$, units when $t$ is close to $\frac{1}{2}$. Thus,

$$z_1(t) = q_1(t) + d_1(t) = 0 + \frac{b_1 - p_1(t)}{t} = \frac{1}{2} \frac{t}{t}.$$

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And,

\[ q_1(t) = 1 - z_1(t) = 1 - \frac{1}{2t}, \quad p_1(t) = tq_1(t) - \int_0^t q_1(s)ds = \frac{1}{2} \left( \ln(t) - \ln\left(\frac{1}{2}\right) \right). \]

This continues until bidder 1’s total demand (the number of units she has clinched, plus the number of additional units she demands) drops below one. This occurs when the marginal price of additional units is approximately .698.

When the marginal price of additional units exceeds .698, then both bidder 1 and bidder 2 have total demands that are less than one. Thus, the number of units clinched by bidder 1 is the number of units that are not demanded or clinched by bidder 2. Or equivalently,

\[ q_i(t) = 1 - z_j(t) = 1 - q_j^-(t) - \frac{b_j - p_j^-(t)}{t} = 1 - \frac{b_j + \int_0^t q_j(t)}{t} \text{ where } i = 1, 2, \ i \neq j. \]

This continues until the marginal price of additional units equals \( \frac{3}{4} \). At that time, bidder 1 no longer demands any additional units of the good. Bidder 1 drops out, and the auction terminates. Bidder 2 wins all unclinched units at a marginal price of \( \frac{3}{4} \) per unit. When the auction terminates bidder 1 wins \( Q_1 = .331 \) units and she pays \( P_1 = .201 \). Bidder 2 wins \( Q_2 = .669 \) units and pays \( P_2 = \frac{1}{2} \). The following figures display \( q_1(t) \) and \( p_2(t) \) between 0 and \( \frac{3}{4} \).

However, conditional upon knowing her rival’s reported type, bidder 1 has a profitable deviation. To see this, suppose that bidder 1 reports her type to be \( \tilde{\theta}_1 = (\frac{3}{4}, \frac{3}{4}) \). In addition,
suppose bidder 2 reports her type truthfully. Thus, for any time \( t < \frac{3}{4} \), bidder 1 demands all available units of the good. That is, for any time \( t < \frac{3}{4} \), we have that \( z_1(t) = 1 \), and thus \( q_2(t) = 1 - z_1(t) = 0 \). This implies

\[
q_1(t) = 1 - z_2(t) = \begin{cases} 
0 & \text{if } t < \frac{1}{2} \\
1 - \frac{1}{2^t} & \text{if } t \geq \frac{1}{2}
\end{cases}
\]

and \( p_1(t) = tq_1(t) - \int_0^t q_1(s)ds \). The auction ends at time \( t = .75 \), when bidder 1 no longer demands any additional units. Thus, bidder 2 wins all remaining units at a price of .75 per units and, \( Q_1 = \frac{1}{3}, \ Q_2 = \frac{2}{3}, \ P_1 = .202, \) and \( P_2 = .5 \). By overreporting her budget, bidder increases her payoff because

\[
u_1(\frac{1}{3}, 0.202) > u_1(0.331, 0.201).
\]

The incentive to overreport budgets follows from the construction of the mechanism and the presence of budgets. If bidder 1 reports her true (lower) budget, bidder 2 is able to clinch units at an early time, when the price of units is low. This means, bidder 2’s total demand for units is higher, because bidder 2 is able to clinch units at a lower price and use less of her budget to acquire the same number of units. Because bidder 2 has a higher total demand, this means that the residual demand left to bidder 1 is smaller. The residual demand is the quantity that bidder 1 clinches. Thus, bidder 1 must wait longer to clinch the same number of units. That is, bidder 1 pays a higher marginal price to acquire the same number of units.

This is the motivation behind DLN’s impossibility theorem. They show that when bidders budgets are public, the clinching auction is the unique efficient mechanism, but with private budgets, bidders have an incentive to overreport their budget. Thus, there is no mechanism that is dominant strategy incentive compatible and Pareto efficient when budgets are private.

Yet, in the above example, bidder 1 has an incentive to overreport because she knows that her payment will not exceed her budget. Suppose instead, that bidder 2 reported that her type was \( (\frac{8}{11}, \frac{8}{11}) \). If bidder 1 reported her type to be \( (\frac{3}{4}, \frac{3}{4}) \), now neither bidder is (reportedly) budget constrained. In this case, the outcome of the proxy clinching auction is equivalent to the outcome of the second price auction. This means bidder 1 wins all units at a price of \( \frac{8}{11} \). This exceeds bidder 1’s (actual) budget of \( \frac{2}{3} \). Therefore, bidder 1 would have be better off if she instead truthfully reported her type.

This example provides the intuition behind my main result. With incomplete information, a bidder may not know what type her opponent will report. Propositions ??? and ?? provide bounds on the reports bidder 1’s opponents will make. In particular, underreporting budgets
is a dominated strategy. Then, if bidder 1 overreports her budget, full support beliefs imply their is a positive probability her payment will exceed her actual budget. Thus, bidder 1 gets a greater expected payoff from truthfully reporting her budget if she has full support beliefs, and she assumes her opponents play undominated strategies. That is, truthful reporting is the unique prediction of two rounds of iterative elimination of weakly dominated strategies. This is formalized in Proposition ????.

As a robustness check, I consider a setting where bidders do not have hard budget constraints. Instead, I assume bidders get high disutility from spending an amount in excess of their budget. I show that any misreport gives bidders a lower expected payoff than truthful reporting if (1) all bidders play undominated strategies, (2) bidders have full support beliefs, and (3) the disutility of paying spending money in excess of your budget is sufficiently large. This is described in Lemma ????.

4 Bid Behavior in the proxy-clinching auction

I study a bidder’s incentive to report her payoff type in the proxy clinching auction. Consider bidder $i$ with payoff type $\theta_i = (v_i, b_i) \in \Theta$. I show that any report $(v, b) \in \Theta$ where bidder $i$ overreports her values (reports $v > v_i$) and/or underreports her budget (reports $b < b_i$) is weakly dominated. In addition, if bidder $i$ truthfully reports her budget, underreporting her value is weakly dominated. I use these bounds on bidder reports to show that truthful reporting is the unique best reply to any undominated strategy when bidders have full-support first order beliefs.

Proposition 1 shows that if bidder $i$ has type $\theta_i = (v_i, b_i)$, then reporting $(v, b) \in \Theta$ where $v > v_i$ is weakly dominated by reporting $(v_i, \min\{v_i, b\})$. That is, holding your reported budget fixed, you get a weakly lower payoff by overreporting your value. 5 The intuition mirrors the intuition for why bidders do not overreport values in the second price auction. If bidder $i$ reports value $v > v_i$, then any additional units she wins after time $v_i$ decrease her payoff. Bidder $i$ would do better by reporting value $v_i$ and only winning the units she had clinched prior to time $v_i$, when the marginal price of units is less than her value.

**Proposition 1.** (Overreporting values is weakly dominated)

*Overreporting values is a weakly dominated strategy. Specifically, if bidder $i$ has type $(v_i, b_i)$, then reporting type $(v, b) \in \Theta$, where $v > v_i$, is weakly dominated by reporting $(v_i, \min\{v_i, b\})$ if $b > 0$.*

5Subject to the constraint the reported budget is less than the reported value.
Proposition 1 gives a useful upper bound on bidders’ reported values. Similarly, Proposition 2 gives a lower bound on bidders’ reported budgets. Specifically, Proposition 2 shows that simultaneously underreporting values and budgets is weakly dominated by truthful reporting. Thus, with Propositions 1 and 2, we can say that overreporting values and/or underreporting budgets is weakly dominated.

**Proposition 2.** (Underreporting both values and budgets is weakly dominated)

If bidder \( i \) has type \((v_i, b_i)\), then reporting type \((v, b) \in \Theta, \) where \( v \leq v_i \) and \( b < \min\{v, b_i\} \), is weakly dominated by reporting \((v_i, b_i)\).

The proof of Proposition 2 is broken into three Lemmas (2-4).

Lemmas 2 and 3 compare her payoff when she reports \((v, b)\) with bidder \( i \)'s payoff when she reports \((v, \min\{b_i, v\})\), where \( b < \min\{b_i, v\} \). That is, fixing bidder \( i \)'s reported value, I compare her payoff from reporting a high budget with her payoff from reporting a lower budget. I impose the constraint that her reported value \( v \) is less than or equal to her true value and her reported budget never exceeds her actual budget.

The superscript \( \ell \) denotes when bidder \( i \) reports value \( v \) and the lower budget \( b \). The superscript \( h \) denotes when bidder \( i \) reports value \( v \) and a higher budget equal to \( \min\{v, b_i\} \), where \( b < \min\{v, b_i\} \). Lemma 2 shows that given the reports of bidder \( j \neq i \), bidder \( i \) clinches more units at time \( t \) by reporting the higher budget than she does by reporting the lower budget. Or equivalently, \( q^h_i(t) \geq q^\ell_i(t) \).

**Lemma 2.** (Bidders clinch more objects by reporting a higher budget)

For any profile of reports of bidders \( j \neq i \). If \( t \in (0, \min\{\tau^\ell, \tau^h\}) \), then \( q^h_i(t) \geq q^\ell_i(t) \).

The intuition for the Lemma is straightforward. At any time \( t \), if bidder \( i \) reports the higher budget, she clinches more units than she would if she were to report the lower budget. By reporting the higher budget, bidder \( i \) has weakly greater demand for units. When she reports the weakly greater demand, she clinches a weakly greater number of units at any time \( t \).

I use Lemma 2 to show that reporting \((v, \min\{b_i, v\})\) weakly dominates reporting \((v, b)\) for any \( b < \min\{v, b_i\} \), when \( v \leq v_i \). In other words, simultaneously underreporting values and budgets is weakly dominated by a report where you underreport your value but truthfully report your budget (subject to the constraint that the reported budget does not exceed the reported value).

To prove this, I use Lemma 2 to show that when bidder \( i \) reports a higher budget, she wins a weakly greater number units, \( Q^h_i \geq Q^\ell_i \). In addition, I show that bidder \( i \) pays a lower price to win her first \( Q^\ell_i \) units by reporting the higher budget. This is because
}\(q_h^i(t) \geq q_h^i(t) \forall t \in (0, \min\{\tau^e, \tau^h\})\). That is, bidder  
win her first \(Q^e_i\) units at an earlier time when she reports the high budget versus the low budget. Thus, her marginal price of each unit is lower when she reports the higher budget. These results combine to show that reporting \((v, \min\{b_i, v\})\) weakly dominates reporting \((v, b)\) for any \(b < \min\{v, b_i\}\), when \(v \leq v_i\). This argument is proven formally in Lemma 3.

**Lemma 3.**
If bidder \(i\) has type \((v_i, b_i)\), then reporting \((v, b) \in \Theta\), where \(v \leq v_i\) and \(b < \min\{v, b_i\}\), is weakly dominated by bidding \((v, \min\{v, b_i\})\).

To finish the proof of Proposition 2, I show that reporting \((v, \min\{v, b_i\})\) is weakly dominated by reporting \((v_i, b_i)\) if \(v < v_i\). That is, bidder \(i\) gets a weakly greater payoff from truthful reporting than she does from any report where she underreports her value and truthfully reports her budget (subject to the constraint that her reported budget does not exceed her reported value). This is stated formally in Lemma 4.

The intuition behind the proof mirrors the standard argument used to show that underreporting values is weakly dominated in a second price auction. If she reports a value \(v\) less than her true value \(v_i\), then at any time \(t \in (v, v_i)\) she demands no additional units. However, bidder \(i\) increases her payoff by clinching additional units that are being sold at a marginal price \(v < t < v_i\). In addition, bidder \(i\) never exceeds her actual budget by reporting the higher value, as \(P_i \leq b_i\) under truthful reporting.

**Lemma 4.**
If bidder \(i\) has type \((v_i, b_i)\), then reporting \((v, \min\{v, b_i\})\), where \(v < v_i\), is weakly dominated by bidding \((v_i, b_i)\).

Lemmas 3 and 4 combine to prove Proposition 2. For any profile of bids and valuations reported by bidders \(j \neq i, \theta_{-j}\), Lemma 3 states that reporting \((v, b) \in \Theta\) where \(v \leq v_i\) and \(b < \min\{v, b_i\}\) gives a weakly lower payoff than reporting \((v, \min\{v, b_i\})\). In addition, Lemma 4 shows that reporting \((v, b_i) \in \Theta\), where \(v < v_i\), gives a weakly lower payoff than reporting truthfully \((v_i, b_i)\). Thus, truthful reporting gives a weakly greater payoff than reporting \((v, b) \neq (v_i, b_i)\) where \(v \leq v_i\) and \(b \leq b_i\).

I describe \(U(\theta_i) \subset \Theta\) as the set of all undominated strategies in the proxy clinching auction when bidder \(i\) has type \(\theta_i\). Thus far, our results have shown that \(a \in U(\theta_i)\) only if

\[a = (v, b) \text{ where } (v, b) = (v_i, b_i) \text{ or } v < v_i \text{ and } b_i < b \leq v \leq v_i.\] (1)

That is, overreporting values and underreporting budgets are weakly dominated. In addition, if bidder \(i\) reports her budget truthfully, underreporting her value is also weakly dominated.
When bidders have full-support first-order beliefs, the bounds on bid behavior described in equation (1) imply that truthful reporting is bidder $i$’s unique best reply to any undominated strategy profile by bidders $j \neq i$. This is because if bidder $i$ overreports her budget, full-support first-order beliefs imply that there is a positive probability that her payment exceeds her budget. Specifically, full-support first-order beliefs imply that if bidder $i$ reports her type to be $(v,b_i)$ where $b > b_i$ and $b_i < v \leq v_i$, then there is a positive probability that she pays $P_i \in (b_i,v)$ if and her opponents play undominated strategies. This is proved in Proposition 3.

**Proposition 3.** (Truthful reporting is the unique undominated best reply)

*Truthful reporting is the unique undominated best reply to any undominated strategy profile of bidders $j \neq i$.*

Proposition 3 implies that two rounds of iterative elimination of weakly dominated strategies predict that bidders report their types truthfully. Thus, for any realization of bidder types $(\theta_1, \ldots, \theta_N)$, there is a unique outcome implied by of iterative deletion of weakly dominated strategies.

Thus, if bidders report their types truthfully, we can say that if the auction ends at time $\tau$, all bidders receive their total demand for units at time $\tau$.

**Corollary 1.** (Properties of the allocation)

*Given any profile of bidder types $(\theta_1, \ldots, \theta_N)$, if bidder $i$ reports her type truthfully, and the*
auction terminates at time $\tau$, then

$$Q_i = \begin{cases} 
0 & \text{if } \tau > v_i \\
[0, \lim_{t \to \tau^-} z_i(t)] & \text{if } \tau = v_i \\
z_i(\tau) & \text{if } v_i > \tau 
\end{cases}$$

This follows directly from our prior results. If $\tau > v_i$, then exists $t \in (0, \tau)$ such that $d_i(t) = 0$. Lemma 1 then implies that $Q_i = P_i = 0$. If $\tau > v_i$, then

$$z_i(t) = q_i^-(t) + d_i(t) = \min\{1, \frac{b_i + \int_0^\tau q_i(s) ds}{\tau}\} \forall t \in [0, \tau].$$

Since $z_i(\tau) = \lim_{t \to \tau^-} z_i(t)$, then $Q_i = z_i(\tau)$ by construction. If $\tau = v_i$, then $z_i(\tau)$ is discontinuous at $v_i$. Thus, bidder $i$ wins some amount between $0$ and $\lim_{t \to v_i^-} z_i(t)$.

4.1 A Robustness Check

The results in the prior section exploit the fact that bidders get infinite disutility from exceeding their budget. Bidders do not overreport their budget because any overreport gives a positive probability of obtaining infinite disutility when bidders have full-support beliefs.

As a robustness check, I show that we obtain similar results when bidders have continuous utility functions, and receive high (but finite) marginal disutility from spending in excess of their budget. In particular, I compare the payoff of truthful reporting with the payoff of any undominated strategy. I show that if a bidder believes her opponents play undominated strategies, truthful reporting gives a higher expected than the undominated strategy, as long as the marginal disutility of spending over the budget is sufficiently high.

Specifically, assume that bidder $i$, with type $\theta_i = (v_i, b_i)$, has preferences

$$u(x, -p, \theta_i) = \begin{cases} 
xv - p & \text{if } p \leq b_i \\
xv - b_i - \varphi(p - b_i) & \text{if } p > b_i 
\end{cases},$$

where $\varphi > 1$. A bidder’s marginal disutility of spending a dollar over her budget is $\varphi > 1$.

I show that implications of Propositions 1 and 2 hold when we assume bidders have continuous utility functions ($\varphi$ is finite). That is, overreporting values and underreporting budgets are weakly dominated. The intuition for both results mimics the intuition used when we assume bidders receive $-\infty$ utility from exceeding their budget.

To show that the implications of Proposition 2 hold, the proof is again similar to the
proof that bidders do not overreport values in the second price auction. Holding a bidder's reported budget fixed, if a bidder overreports her value, there is a possibility that she wins additional units at a price per unit that exceeds her value. This is avoided if she reports her value truthfully.

The implications of Proposition 2 hold, because the proof of Proposition 2 is unchanged when we assume $\varphi$ is finite. In particular, Proposition 2 states that simultaneously underreporting budgets and values is weakly dominated by truthful reporting. This result is unchanged when $\varphi$ is finite, because a bidder’s payment does not exceed her budget if she reports her type truthfully or if she simultaneously underreports her value and budget. In the proof, we never consider a case where bidder $i$ pays an amount that exceeds her budget $b_i$, thus the proof does not depend on bidder $i$’s utility if she spends more than her actual budget.

Thus, the set of undominated strategies is unchanged when $\varphi$ is finite. The result is stated formally in Corollary 2.

**Corollary 2.** If bidder $i$ has type $\theta_i$, then the report $(v, b) \in U(\theta_i)$ only if

$$(v, b) \text{ is such that } (v, b) = (v_i, b_i) \text{ or } b_i < b \leq v \leq v_i.$$ 

Thus, if bidder $i$ reports the undominated strategy $(v, b) \neq (v_i, b_i)$, then it is necessarily the case that bidder $i$ overreports her budget. However, if (1) bidder $i$ believes her rivals’ play undominated strategies, and (2) bidder $i$ receives sufficiently large disutility from spending in excess of her budget ($\varphi$ is large); then truthful reporting gives greater expected utility than reporting $(v, b)$. This is stated formally in Proposition 4. The intuition for this result mimics the intuition used to prove Proposition 3. If bidder $i$ overreports her budget, then full-support beliefs imply there is a positive probability that she pays an amount that exceeds her actual budget. When $\varphi$ is sufficiently large, the disutility associated with exceeding the budget is greater than any possible gain that could occur from misreporting. Thus, the expected payoff from reporting $(v, b)$ is negative. At the same time, truthful reporting guarantees that bidder $i$ receives a non-negative payoff. This implies that the report $(v, b)$ is eliminated in the second round of iterative elimination of weakly dominated strategies, when $\varphi$ is sufficiently large.

**Proposition 4.** Consider any report $(v, b) \neq (v_i, b_i)$, where $(v, b) \in U(\theta_i)$. Suppose all bidders play undominated strategies and bidder $i$ has full support first order beliefs. The expected utility of truthful is strictly greater than the expected utility of reporting $(v, b)$.

Proposition 4 serves as a robustness check on our prior results. In particular, our predictions on bid behavior are not driven by the fact that we model bidders as having discontinuous utility functions.
5 Efficiency

Given any profile of bidder types \((\theta_1, \ldots, \theta_N)\), Proposition 3 shows when bidders have full-support beliefs, two rounds of iterative deletion imply a unique outcome in the proxy clinching auction. I use the first welfare theorem to show that the outcome of the proxy clinching auction is Pareto efficient.

Suppose that bidders have types \((\theta_1, \ldots, \theta_N)\), where \(b_i, b_j > 0\) for at least two bidders \(i, j\) where \(j \neq i\). In addition, suppose that the outcome of the proxy clinching auction is such that bidder \(i\) wins \(Q_i\) units and pays \(P_i\) and the auction terminates at time \(\tau\). Note that \(P_i = Q_i \tau - \int_0^\tau q_i(s)ds\).

The outcome of the proxy clinching auction is a Walrasian equilibrium of a two commodity endowment economy with \(N + 1\) agents. The two commodities being traded are money and the divisible good. The \(N+1\) agents begin with endowments that are identical to the outcome of the proxy-clinching auction. Specifically, each bidder \(i\) is endowed with \(Q_i\) units of the good and \(b_i - P_i\) units of money. Bidder \(i\) has preferences

\[
U_i(q, m) = \begin{cases} 
qv_i + m & \text{if } q \in [0, 1] \\
v_i + m & \text{if } q > 1 
\end{cases}
\]

Note that \(U_i(q, b_i - P_i) = u_i(q, b_i)\). The auctioneer is agent 0. She is endowed with no units of the good and \(\sum P_i\) units of money. Her preferences are \(U_0(q, m) = m\). I show that the endowment economy has a Walrasian equilibrium with no trade, where price of money is 1 and the price of the good is \(\tau\). Thus, the initial endowment of goods is Pareto efficient, and so is the outcome of the proxy clinching auction.

**Proposition 5.** (The proxy clinching auction is efficient)

*For any profile of bidder types \((\theta_1, \ldots, \theta_N)\), the proxy clinching auction implements a Pareto efficient outcome.*

The intuition behind the proof is straightforward. The auction terminates at time \(\tau\). Prior to time \(\tau\), bidder \(i\) may have already clinched some units at a price lower than \(\tau\). This discount is equal to \(\int_0^\tau q_i(s)ds\). Thus, it is as though bidder \(i\) has \(b_i + \int_0^\tau q_i(s)ds\) to spend. Bidder \(i\)'s Marshallian demand \(M_i\) for units of the good when she has wealth \(b_i + \int_0^\tau q_i(s)ds\) and the price of the good is \(\tau\) is

\[
M_i = \begin{cases} 
0 & \text{if } \tau > v_i \\
[0, \min\{1, \frac{b_i + \int_0^\tau q_i(s)}{\tau}\}] & \text{if } \tau = v_i \\
\min\{1, \frac{b_i + \int_0^\tau q_i(s)}{\tau}\} & \text{if } \tau < v_i 
\end{cases}
\]
This is the quantity that bidder $i$ wins in the proxy clinching auction. Thus, the initial endowments are a Walrassian equilibrium, and the outcome of the proxy clinching auction is a Walrassian equilibrium. This is proved in the appendix.

6 Indivisible Goods

The results of the prior sections can also be used to study a setting where the auctioneer sells a single indivisible good. The auctioneer uses the proxy clinching auction to sell probabilities of winning the single indivisible good as though it was a divisible good in net supply one. Thus, the proxy clinching auction determines each bidder’s probability of winning the good and her payment. Bidders’ payments have an all-pay structure. That is, a bidder pays the same amount whether or not she wins the good. Bidder payments are increasing in the bidder’s probability of winning.

6.1 The Proxy-Clinching Auction with Indivisible Goods

The proxy clinching auction is adapted to this setting by interpreting quantities as probabilities. In particular, consider the proxy clinching auction in a divisible good setting where bidders report types $(\theta_i, \theta_{-i})$. Suppose that bidder $i$ wins quantity $Q_i$ and pays $P_i$. Then, in the indivisible good setting, if bidders report types $(\theta_i, \theta_{-i})$ in the proxy clinching auction, bidder $i$ wins the good with probability $Q_i$, and she pays $P_i$, independent of whether she wins the good. Thus, bidder $i$ has expected utility $U_i$, where

$$u_i(Q_i, P_i) = \begin{cases} v_i Q_i - P_i & \text{if } P_i \leq b_i, \\ -\infty & \text{if } P_i > b_i. \end{cases}$$

This is the same as the utility of a bidder with type $(v_i, b_i)$ who wins $Q_i$ units and pays $P_i$ in the divisible good setting. Thus, a bidder’s incentive to report her type is the same as it is in the proxy clinching auction for divisible goods.

**Corollary 3.** (Truthful reporting is the unique undominated best response)

*If a bidder has full-support first-order beliefs, truthful reporting is the only strategy that survives two rounds of iterative deletion of weakly dominated strategies.*
6.2 Efficiency

I modify my earlier definition of an outcome to include randomization and indivisibility. Let \( \mathcal{A} \) be the set of all feasible assignments, where

\[
\mathcal{A} := \{ a | a \in \{0, 1\}^N \text{ and } \sum_{i=1}^{N} a_i \leq 1 \},
\]

where \( a_i = 1 \) if bidder \( i \) is given the object. A (deterministic) outcome \( \phi \) specifies both transfers and a feasible assignment: \( \phi \in \mathcal{A} \times \mathbb{R}^{N} \). I define \( \Phi := \mathcal{A} \times \mathbb{R}^{N} \) as the set of all (deterministic) outcomes. Thus, a (probabilistic) outcome \( \alpha \) is an element of \( \Delta(\Phi) \).

An outcome \( \alpha \) is (ex-post) Pareto efficient if, conditional on bidder types \( (\theta_1, \ldots, \theta_N) \), there is no other outcome that gives weakly greater expected revenues and makes all bidders weakly better off in expectation. That is, conditional upon knowing all bidders private information, we cannot increase one bidder’s expected utility without necessarily decreasing revenue or lowering another bidder’s expected utility. Pareto efficiency is defined formally in Definition 5. This notion of ex-post Pareto efficiency is the same as the one used by Holmstrom and Myerson (1983). The notation \( E_{\alpha}[v_i, b_i] \) denotes the expected utility of a bidder \( i \) with valuation \( v_i \) and budget \( b_i \) in outcome \( \alpha \in \Delta(\Phi) \). Similarly, \( E_{\alpha}[P] \) denotes the expected total payments collected by the auctioneer in outcome \( \alpha \).

**Definition 5.** (Pareto efficient)

An outcome \( \alpha \in \Delta(\Phi) \), is Pareto efficient if \( \nexists \alpha' \in \Delta(\Phi) \) such that

\[
E_{\alpha'}[v_i, b_i] \geq E_{\alpha}[v_i, b_i] \quad \forall i = 1, \ldots, N,
\]

and

\[
E_{\alpha'}[P] \geq E_{\alpha}[P],
\]

where at least one of the above holds with a strict inequality.

**Corollary 4.** (The proxy clinching auction is efficient)

For any profile of bidder types \( (\theta_1, \ldots, \theta_N) \in \Theta^N \), the outcome of the proxy clinching auction is Pareto efficient.

The proof of this Corollary follows from our prior results and it is shown below.

Let \( \mathcal{A} \subset \Delta(\Phi) \) be the set of all outcomes where bidders pay a constant amount to the auctioneer. That is, the set of all outcomes with all-pay payment schemes. Note that the outcome of the proxy clinching auction is an element of \( \mathcal{A} \), by construction.
Fix a profile of bidder types \((\theta_1, \ldots, \theta_N)\). Consider some outcome \(\alpha \in \Delta(\Phi)\) where bidder \(i\) wins with probability \(q_i\) and pays \(p_i\) in expectation. Thus, the auctioneer is paid \(\sum_{i=1}^{N} p_i\) in expectation. Then, there is a corresponding outcome \(\bar{\alpha} \in \mathcal{A}\), where bidder \(i\) wins with probability \(q_i\) and pays \(p_i\) with certainty. This holds because

\[
\mathbb{E}_{\bar{\alpha}}[v_i, b_i] = q_i v_i - p_i \geq \mathbb{E}_{\alpha}[v_i, b_i] \text{ if } b_i \geq p_i,
\]

and

\[
\mathbb{E}_{\bar{\alpha}}[v_i, b_i] = \mathbb{E}_{\alpha}[v_i, b_i] = -\infty \text{ if } b_i < p_i.
\]

The auctioneer collects equal expected payments under \(\alpha\) and \(\bar{\alpha}\). Thus, outcome \(\bar{\alpha}\) makes all bidders weakly better off and gives equal revenue. Thus, if outcome \(\bar{\alpha}\) is Pareto dominated by some outcome \(\beta \in \Delta(\Phi)\), then there exists a corresponding outcome \(\tilde{\beta} \in \mathcal{A}\) that also Pareto dominates outcome \(\alpha\).

Suppose that the outcome of the proxy clinching auction is \(\eta \in \mathcal{A}\). A direct implication of Proposition 5 is that for any profile of bidder types \((\theta_1, \ldots, \theta_N) \in \Theta^N\), the outcome of the proxy clinching auction \(\eta\) is never Pareto dominated by some outcome \(\beta \in \Delta(\Phi)\). Thus, \(\nexists \tilde{\eta} \in \mathcal{A}\) such that

\[
\mathbb{E}_{\tilde{\eta}}[v_i, b_i] \geq \mathbb{E}_{\eta}[v_i, b_i], \forall i
\]

and

\[
\mathbb{E}_{\tilde{\eta}}[P] \geq \mathbb{E}_{\eta}[P],
\]

where at least one of the above holds with a strict inequality.

Because there does not exist a \(\tilde{\eta} \in \mathcal{A}\) that Pareto dominates \(\eta\), there is no outcome \(\gamma \in \Delta(\Phi)\) that Pareto dominates \(\eta\). This holds because if there was an outcome \(\gamma\) that did Pareto dominate \(\eta\), there would exist a corresponding \(\tilde{\gamma} \in \mathcal{A}\) that also Pareto dominates \(\eta\). However, since there is no \(\tilde{\gamma} \in \mathcal{A}\) that exists, it follows that the outcome of the proxy clinching auction \(\eta\) is Pareto efficient.

### 7 Conclusion

Prior work has shown that a proxy clinching auction is efficient when budgets are public, but with private budgets, the efficient auction is not dominant strategy implementable. In this paper, we show that when bidders have full-support first order-beliefs, the proxy clinching auction is efficient and solvable using two rounds of iterative elimination of dominated strategies. The full-support assumption appears reasonable when we study a setting without complete information, and imposing this minimal structure on beliefs is sufficient for
characterizing bid behavior. The mechanism I propose is detail-free and does not impose restrictions on bidders’ higher-order beliefs. Indeed, I do not even require that bidders share a common prior.

There are a number of related questions that follow from this work. First, the model studied here could be extended to a case where bidders do not have constant marginal values for additional units. The intuition for not overreporting budgets should still hold with a full-support belief on opponents’ types. Second, one could move from a private value setting, to an interdependent value setting. This paper extends Ausubel’s (2004) clinching auction to the setting where bidders have budgets. With interdependent values, we could similarly adapt Perry and Reny’s (2005) ascending auction to include budget constraints. Finally, it may be useful to study how full-support beliefs can be used to characterize behavior in other mechanism design environments where budgets are relevant features.
References


Appendix

Proof of Lemma 1

Part 1. By assumption, there exists at least two bidders $i, j$ with $b_i, b_j > 0$. Let $\delta = \min\{b_i, b_j\}$.

At any time $t < \delta$, I show that $z_i(t) = 1$. Note that, $p_i^-(t) \leq t q_i^-(t) \leq t < b_i$ where the second inequality holds because $q_i(t) \in [0, 1]$. If $q_i^-(t) = 1$ this holds trivially, and if $q_i^-(t) < 1$, then $p_i(t) \leq t q_i(t) \leq t < b_i$ and $d_i(t) = 1 - q_i^-(t)$, because $\frac{b_i - p_i^-(t)}{t} \geq \frac{b_i - t q_i^-(t)}{t} > 1 - q_i^-(t) = d_i(t)$. Thus, $z_i(t) = q_i^-(t) + d_i(t) = 1$. Similarly, $z_j(t) = 1 \forall t < \delta$.

Thus, $\sum_{i=1}^{N} z_i(t) \geq 2 > 1$ for all $t < \delta$. This implies the auction does not terminate until at or after time $\delta$, and $\tau \geq \delta > 0$.

Part 2. I consider three cases. First, suppose that at time $t$, $d_i(t) = 0$. If $d_i(t) = 0$, then $p_i^-(t) \geq b_i$ and/or $t \geq v_i$. For all $t' > t$, $q_i^-(t') = q_i^-(t)$ by construction. Similarly, $p_i^-(t') = p_i(t') \forall t' > t$. Thus, $d_i(t') = 0 \forall t' > t$, and $z_i(t') = q_i^-(t') = z_i(t')$.

Second, suppose that at time $t$, $d_i(t) = 1 - q_i^-(t) > 0$. Then $z_i(t) = 1 \geq z_i(t') \forall t \in (t, \tau)$, since $z_i(t') \leq 1 \forall t' \in (0, \tau)$ by construction.

Finally, suppose that at time $t$, $d_i(t) \in (0, 1 - q_i^-(t))$. Thus, $d_i(t) = \frac{b_i - p_i^-(t)}{t}$. First, I show that at any time $t' > t$, $p_i^-(t') \leq b_i$. I use proof by contradiction. If $\exists t'$ such that $p_i^-(t') > b_i$, then, $\exists \tilde{t}$ is such that $t < \tilde{t} \implies p_i^-(t) \leq b_i$, and $t > \tilde{t} \implies p_i^-(t) > b_i$, because $p_i$ is non-decreasing. At time $\tilde{t}$, $d_i(\tilde{t}) \leq \frac{b_i - p_i^-(\tilde{t})}{\tilde{t}}$. In addition, $q_i(\tilde{t}) \leq q_i^-(\tilde{t}) + d_i(\tilde{t}) \implies p_i(\tilde{t}) \leq p_i^-(\tilde{t}) + \tilde{t} d_i(\tilde{t}) \leq b_i$. Recall, we define $t^*$ as the last time where bidder $i$ demands additional units $t^* = \sup\{t | d_i(t') \geq 0 \forall t' \leq t\}$. Since $d_i(t) = 0 \forall t > \tilde{t}$, then $t^* \leq \tilde{t}$. Thus $q_i(t^*) = q_i(t') \forall t' > t^*$ by construction. It follows that $p_i(t*) = p_i(t') \forall t > t^*$, and $p_i(t*) \leq b_i$. This contradicts the assumption that there exists a time $t'$ where $p_i(t') > b_i$.

Thus, if $t' > t$, then $p_i^-(t') \geq t(q_i^-(t') - q_i^-(t)) + p_i^-(t)$. This implies,

$$d_i(t') \leq \frac{b_i - p_i^-(t')}{t'} \leq \frac{b_i - p_i^-(t')}{t'} \leq \frac{b_i - (t(q_i^-(t') - q_i^-(t)) + p_i^-(t))}{t'}$$

and,

$$d_i(t') \leq \frac{b_i - (t(q_i^-(t') - q_i^-(t)) + p_i^-(t))}{t'} \leq \frac{b_i - p_i(t)}{t} + q_i^-(t) - q_i^-(t') = z_i(t) - q_i^-(t').$$

Thus,

$$d_i(t') \leq z_i(t) - q_i^-(t') \implies d_i(t') + q_i^-(t') = z_i(t') \leq z_i(t).$$
Part 3. First, I show that \( q_i(t) \leq s_i(t) \). Recall that \( s_i(t) = \max \{1 - \sum_{j \neq i} z_j(t), 0\} \). Since \( z_j(t) \) is non-increasing in \( t \), for all \( j \), then \( s_i(t) \) is non-decreasing. Thus, if \( d_i(t') > 0 \ \forall t' < t \), then \( q_i(t) = \min \{s_i(t), \lim_{t' \to t^-} s_i(t') + d_i(t)\} \) and \( q_i(t) \leq s_i(t) \). If \( d_i(t') = 0 \) for some \( t' < t \), then \( q_i(t) = q_i(t^*) \), where \( t^* \) is defined as \( t^* = \sup \{t | d_i(t') > 0 \ \forall t' < t\} \). Thus, \( q_i(t) = q_i(t^*) \leq s_i(t) \) since \( s_i(t) \) is non-decreasing. Thus, \( q_i(t) \leq s_i(t) \).

The remainder of the proof proceeds by contradiction. Suppose there exists a time \( t \in (0, \tau) \) such that \( q_i(t) < s_i(t) \).

I consider two cases. First, suppose \( t \) is such that \( d_i(t') > 0 \ \forall t' < t \), then \( q_i(t) = \min \{s_i(t), \lim_{t' \to t^-} s_i(t') + d_i(t)\} \implies q_i(t) = \lim_{t' \to t^-} s_i(t') \). Note that \( q_i(t) = \lim_{t' \to t^-} s_i(t') + d_i(t) < s_i(t) \) because \( q_i(t) = \min \{s_i(t), \lim_{t' \to t^-} s_i(t') + d_i(t)\} < s_i(t) \). Thus, \( q_i(t) = z_i(t) < s_i(t) \). Since \( s_i(t) > q_i(t) \geq 0 \), then \( s_i(t) = 1 - \sum_{j \neq i} z_j(t) \) and

\[ z_i(t) < s_i(t) \implies \sum_{i=1}^{N} z_i(t) < 1. \]

This contradicts the assumption that \( t \in (0, \tau) \), because the above condition requires that the auction terminate by period \( t \) at the latest.

Second, suppose \( t \) is such that \( d_i(t') = 0 \) for some \( t' < t \), then \( q_i(t) = q_i^-(t) = q_i(t') \) and \( p_i(t') = p_i(t') \). Thus, \( d_i(t) = 0 \) and

\[ z_i(t) = q_i(t) < s_i(t) \implies \sum_{i=1}^{N} z_i(t) < 1. \]

This also contradicts the assumption that \( t \in (0, \tau) \).

Part 4. Suppose that \( \tau > 1 \). Then, for any time \( t \in (1, \tau) \), \( \sum_{i=1}^{N} z_i(t) > 1 \). Yet, \( d_i(t) = 0 \ \forall i, t > 1 \). Thus, \( q_i(1) = z_i(t) = q_i^-(t) \ \forall t \in (1, \tau) \). Let \( M \) be the set of all bidders \( i \) such that \( q_i(1) > 0 \). If \( i \in M \), then

\[ z_i(t) = q_i(t) = s_i(t) = 1 - \sum_{j \neq i} z_j(t) > 0, \ \forall t \in (1, \tau). \]

Thus, \( 1 = \sum_{i=1}^{N} z_i(t) \), which contradicts that \( \sum_{i=1}^{N} z_i(t) > 1 \ \forall t \in (1, \tau) \).
Part 5. I prove this by contradiction. Suppose that \( d_i(t) = 0 \) for some \( t \in (0, \tau) \), yet \( Q_i > 0 \). Note that \( d_i(t) = 0 \) implies \( q_i(t') = q_i(t) \), \( p_i(t') = p_i(t) \), and \( d_i(t') = d_i(t) = 0 \ \forall \tau > t' > t \). Thus, \( Q_i > 0 \implies Q_i = z_i(t') = q_i(t') = q_i(t) > 0 \ \forall t' \in (t, \tau) \). Thus,

\[
z_i(t') = q_i(t') = 1 - \sum_{j \neq i} z_j(t') \ \forall t' \in (t, \tau) \implies \sum_{i=1}^N z_i(t') = 1 \ \forall t' \in (t, \tau).
\]

Thus, the auction terminates by time \( t' \) or earlier \((\tau \leq t')\). This contradicts our assumption that the auction ends at time \( \tau > t' \).

Part 6. I prove this by contradiction. First, I show that \( p_i(t) < b_i \ \forall t \in (0, \tau) \). Suppose that \( t \in (0, \tau) \), yet \( p_i(t) \geq b_i \) for some \( i \). Then \( d_i(t) = 0 \). We have already shown that if \( d_i(t) = 0 \) for some \( t < \tau \), then \( Q_i = P_i = 0 \). However, \( P_i \geq p_i(t) \geq b_i > 0 \), where the second inequality follows because \( P_i \geq p_i(t) \ \forall t \in (0, \tau) \). Thus, \( 0 = P_i > 0 \) and we have a contradiction.

Next, I show that \( P_i \leq b_i \). Note that \( p_i(t) + td_i(t) \leq b_i \ \forall t \in (0, \tau) \). This holds because \( p_i(t) = q_i(t) - \int_0^t q_i(s)ds \) and \( d_i(t) \leq \frac{b - p_i(t)}{t} \). Thus \( p_i(t) + td_i(t) \leq b_i - p_i(t) + p_i(t) = b_i \). Finally, the construction of the proxy clinching auction implies, \( P_i \leq \lim_{t \to -\tau} p_i(t) + td_i(t) \leq b_i \).

Proof of Proposition 1. I use an \( h \) superscript for variables when bidder \( i \) reports \((v, b)\), where \( v > v_i \); and an \( \ell \) superscript when bidder \( i \) reports \((v, \min\{v_i, b\})\).

For any \( t \leq v_i \), \( z_i^\ell(t) = z_i^h(t) = \min\{1, q_i(t) + \frac{b - p_i(t)}{t} \} = \min\{1, \frac{b + \int_0^t q_i(s)ds}{t} \} \). Thus, \( s_i^h(t) = s_i^\ell(t) \ \forall t \leq v_i \). Therefore, if \( \tau^\ell < v_i \), then \( \tau^h = \tau^\ell \) and bidder \( i \) receives an equal payoff in each case, \( u_i(Q_i^\ell, -P_i^\ell) = u_i(Q_i^h, -P_i^h) \).

If \( \tau^\ell = v_i \), then \( \tau^h \geq v_i \). First, suppose \( \tau^h = v_i \). Thus, \( Q_i^\ell \in [z_i^\ell(v_i), \lim_{t \to v_i^-} z_i^\ell(t)] \), and \( Q_i^h = z_i^h(v_i) \) because \( z_i^\ell \) is left discontinuous at \( v_i \) and \( z_i^h \) is left continuous at \( v_i \). In addition, \( s_i^h(t) = s_i^\ell(t) \ \forall t \leq v_i \) implies that \( q_i^h(t) = q_i^\ell(t) \ \forall t \in (0, v_i) \). Thus, \( Q_i^h = z_i^h(v_i) = \lim_{t \to v_i^-} z_i^h(v_i) = \lim_{t \to v_i^-} z_i^\ell(t) = Q_i^\ell \), and \( P_i^h = P_i^\ell + v_i(Q_i^h - Q_i^\ell) \). Thus, \( P_i^\ell > b_i \) only if \( P_i^h > b_i \). If \( P_i^\ell > b_i \), then \( u_i(Q_i^\ell, -P_i^\ell) = u_i(Q_i^h, -P_i^h) = -\infty \). If \( P_i^h > b_i \geq P_i^\ell \), then \( u_i(Q_i^\ell, -P_i^\ell) > u_i(Q_i^h, -P_i^h) = -\infty \). Finally, if \( b_i \geq P_i^h \geq P_i^\ell \), then

\[
u_i(Q_i^h, -P_i^h) = Q_i^h v_i - (P_i^h + v_i(Q_i^h - Q_i^\ell)) = Q_i^\ell v_i - P_i^\ell = u_i(Q_i^\ell, -P_i^\ell).
\]

Next, suppose that \( \tau^\ell = v_i < \tau^h \). Since, \( s_i^h(t) = s_i^\ell(t) \ \forall t \leq v_i \) and \( Q_i^\ell \leq s_i^\ell(v_i) \), then
\( q^h_i(v_i) \geq Q^h_i \) and \( p^h_i(v_i) = \mathcal{P}^h_i + v_i(Q^h_i - Q^f_i) \). In addition, \( Q^h_i \geq q^h_i(v_i) \geq Q^f_i \) and \( \mathcal{P}^h_i \geq p^h_i(v_i) + v_i(Q^h_i - q^h_i(v_i)) = \mathcal{P}^h_i + v_i(Q^h_i - Q^f_i) \). Thus, \( \mathcal{P}^h_i \geq \mathcal{P}^f_i \). If \( \mathcal{P}^h_i > b_i \), then \( u_i(Q^h_i, -\mathcal{P}^h_i) = u_i(Q^h_i, -\mathcal{P}^f_i) = -\infty \). If \( \mathcal{P}^h_i > b_i \geq \mathcal{P}^f_i \), then \( u_i(Q^h_i, -\mathcal{P}^f_i) > u_i(Q^h_i, -\mathcal{P}^h_i) = -\infty \). Finally, if \( b_i \geq \mathcal{P}^h_i \geq \mathcal{P}^f_i \), then

\[
u_i(Q^h_i, -\mathcal{P}^h_i) \leq Q^h_i v_i - (\mathcal{P}^h_i + v_i(Q^h_i - Q^f_i)) = Q^h_i v_i - \mathcal{P}^h_i = u_i(Q^h_i, -\mathcal{P}^f_i).
\]

Finally, suppose \( \tau^f, \tau^h > v_i \). Then, \( \exists t \in (0, \tau^f) \) such that \( d^f_i(t) = 0 \). Thus, \( Q^f_i = \mathcal{P}^f_i = 0 \) by Lemma 1 and \( u_i(Q^f_i, -\mathcal{P}^f_i) = 0 \). Also, recall that \( q^f_i(t) = q^h_i(t) = 0 \ \forall t < v_i \). Thus, \( \mathcal{P}^h_i = Q^h_i \tau^h - \int_0^{\tau^h} q^h_i(s)ds = Q^h_i \tau^h - \int_{v_i}^{\tau^h} q^h_i(s)ds \geq Q^h_i v_i \). Thus,

\[
u_i(Q^h_i, -\mathcal{P}^h_i) \leq Q^h_i v_i - \mathcal{P}^h_i \leq Q^h_i v_i - Q^h_i v_i = 0 = u_i(Q^h_i, -\mathcal{P}^f_i).
\]

**Proof of Lemma 2.** I use the superscript \( h \) to denote variables when bidder \( i \) reports \((v, b_h)\), and superscript \( f \) for variables when bidder \( i \) reports type \((v, b_f)\), where \( b_h > b_f \). I use notation \( z_{-i}(t) : = \sum_{j \neq i} z_j(t) \), and similarly, \( q_{-i}(t) = \sum_{j \neq i} q_j(t) \).

The proof is by contradiction. Suppose there exists a time \( t \in (0, \min\{\tau^h, \tau^f\}) \) where

\[ q^f_i(t) > q^h_i(t) \geq 0. \]

Then

\[ q^h_i(t) = 1 - z^f_{-i}(t) > \max\{0, 1 - z^h_{-i}(t)\} = q^f_i(t) \implies z^h_{-i}(t) > z^f_{-i}(t). \]

Note that \( z^f_{-i}(t) < 1 \). Thus, \( z^h_{-i}(t) < 1 \ \forall j \neq i \). Then,

\[
\sum_{j \neq i} b_j + \int_0^t q^h_j(s)ds = \sum_{j \neq i} b_j + \int_0^t q^f_j(s)ds \leq z^h_{-i}(t) > z^f_{-i}(t) = \sum_{j \neq i} b_j + \int_0^t q^f_j(s)ds \leq z^h_{-i}(t).
\]

Thus,

\[
\sum_{j \neq i} \int_0^t q^h_j(s)ds > \sum_{j \neq i} \int_0^t q^f_j(s)ds.
\]

Note, \( q^h_j(s) > 0 \) only if \( s < v_j \), by Lemma 1. Thus, we can rewrite the above condition as

\[
\int_0^t (q^h_{-i}(s) - q^f_{-i}(s))ds > 0.
\]
That is,
\[
q^\ell_{i}(t) - q^{h}_{i}(t) > 0 \implies \int_{0}^{t} (q^{h}_{n}(s) - q^{\ell}_{n}(s)) \, ds > 0.
\]

Let \( \tilde{t} \) be defined as the infimum of all times where the above integral condition holds.
\[
\tilde{t} := \inf \{ t | \int_{0}^{t} (q^{h}_{n}(s) - q^{\ell}_{n}(s)) \, ds > 0 \}.
\]

Thus, \( \forall t \leq \tilde{t} \), \( q^{h}_{i}(t) \geq q^{\ell}_{i}(t) \). Note also that \( \int_{0}^{t} (q^{h}_{n}(s) - q^{\ell}_{n}(s)) \, ds \) is continuous in \( t \). Thus, \( \int_{0}^{\tilde{t}} (q^{h}_{n}(s) - q^{\ell}_{n}(s)) \, ds = 0 \). Combining the definition of \( \tilde{t} \) with continuity, we get \( \forall \epsilon > 0 \), \( \exists \delta > 0 \), such that \( t' \in (\tilde{t}, \tilde{t} + \delta) \implies \int_{0}^{t'} (q^{h}_{n}(s) - q^{\ell}_{n}(s)) \, ds = 0 < \int_{0}^{t'} (q^{h}_{n}(s) - q^{\ell}_{n}(s)) \, ds < \epsilon \).

This implies \( \exists j \) such that,
\[
q^{h}_{j}(t') > q^{\ell}_{j}(t') \geq 0.
\]

Let \( M \) be the set of all bidders \( j \neq i \) such that \( q^{h}_{j}(t') > q^{\ell}_{j}(t') \). Then, if \( j \in M \),
\[
q^{h}_{j}(t') = 1 - z^{h}_{-j}(t') > \max \{ 0, 1 - z^{\ell}_{-j}(t') \} = q^{\ell}_{j}(t').
\]

This implies, \( z^{\ell}_{-j}(t') - z^{h}_{-j}(t') > 0 \). Or equivalently,
\[
z^{\ell}_{-j}(t') - z^{h}_{-j}(t') = \sum_{n \neq j} b^{h}_{n} + \int_{0}^{t'} q^{h}_{n}(s) \, ds - \sum_{n \neq j} b^{\ell}_{n} + \int_{0}^{t'} q^{\ell}_{n}(s) \, ds \mathbb{I}_{v_{n} < t'} > 0.
\]

\[
\implies \int_{0}^{t'} (q_{-j}(s) - q_{-j}(s)) \, ds > b_{n} - b_{\ell}.
\]

Thus,
\[
\sum_{j \in M} \int_{0}^{t'} (q_{-j}(s) - q_{-j}(s)) > \#M (b_{n} - b_{\ell}).
\]

We can rewrite the lefthand side as,
\[
(\#M - 1) \sum_{j = 1}^{N} \int_{0}^{t'} (q^{\ell}_{j}(s) - q^{h}_{j}(s)) \, ds + \sum_{j \notin M} \int_{0}^{t'} (q^{\ell}_{j}(s) - q^{h}_{j}(s)) \, ds.
\]

Looking at the first term, \( \forall \epsilon > 0 \),
\[
(\#M - 1) \sum_{j = 1}^{N} \int_{0}^{t'} (q^{\ell}_{j}(s) - q^{h}_{j}(s)) \, ds \leq \epsilon
\]

when \( t' \) is sufficiently close to \( \tilde{t} \), because \( \int_{0}^{\tilde{t}} (q_{-i}(s) - q^{h}_{-i}(s)) = 0 \) and \( \int_{0}^{\tilde{t}} (q^{\ell}_{i}(s) - q^{h}_{i}(s)) \leq 0 \).
Proof of Lemma 3. I use the superscript $h$ to denote variables when bidder $i$ reports $(v, \min\{v, b_i\}) \in \Theta$, and superscript $\ell$ for variables when bidder $i$ reports type $(v, b_\ell) \in \Theta$. ,
We assume $b_h := \min\{v, b_i\} > b_t$. I consider three cases.

**Case 1:** $\tau^\ell \geq \tau^h$ and $\tau^h \neq v$.

Note that $z_i^h$ is continuous at $t = \tau^h$, as

$$z_i^h(t) = \begin{cases} 
\min\{1, \frac{b_h + \int_0^t q_i^h(s)ds}{t}\} & \text{if } t < v \\
0 & \text{if } t \geq v.
\end{cases}$$

Thus, $Q_i^h = z_i^h(\tau^h) \geq z_i^\ell(\tau^h) \geq Q_i^\ell$ where the first inequality holds because $\int_0^{\tau^h} q_i^h(s)ds \geq \int_0^{\tau^h} q_i^\ell(s)ds$ and the second holds because $z_i^\ell(t)$ is declining in $t$ and $\tau^\ell \geq \tau^h$. Note that

$$\mathcal{P}_i^\ell = p_i^\ell(\tau^h) + (\mathcal{P}_i^\ell - p_i^\ell(\tau^h)) = \tau^h Q_i^\ell - \int_0^{\tau^h} q_i^\ell(s)ds - \int_\tau^{\tau^h} q_i^\ell(s)ds.$$ 

Since $\tau^\ell \geq \tau^h$, then

$$\mathcal{P}_i^\ell \geq Q_i^\ell \tau^h - \int_0^{\tau^h} q_i^\ell(s)ds.$$ 

Thus, if $\tau^h \leq v_i$,

$$u_i(Q_i^h, -\mathcal{P}_i^h) - u_i(Q_i^\ell, -\mathcal{P}_i^\ell) \geq (Q_i^h - Q_i^\ell) (v_i - \tau^h) + \int_0^{\tau^h} (q_i^h(s) - q_i^\ell(s))ds \geq 0.$$ 

If $\tau^h > v_i$, then $Q_i^h = Q_i^\ell = \mathcal{P}_i^h = \mathcal{P}_i^\ell = 0$ because $d_i^h(t) = d_i^\ell(t) = 0$ when $t \in (v_i, \tau^h)$.

**Case 2:** $\tau^\ell \geq \tau^h = v$.

If $\tau^\ell > v$, then $\exists t$ such that $d_i^\ell(t) = 0$. Lemma 1 shows that this implies $Q_i^\ell = \mathcal{P}_i^\ell = 0$. In addition, $\mathcal{P}_i^h \leq \tau^h Q_i^h \leq v_i Q_i^h$. Therefore,

$$u_i(Q_i^h, -\mathcal{P}_i^h) = Q_i^hv_i - \mathcal{P}_i^h \geq 0 = u_i(Q_i^\ell, -\mathcal{P}_i^\ell).$$

If $\tau^h = \tau^\ell = v$, then $z_i^h(t)$ is discontinuous at $t = \tau^h = \tau^\ell := \tau$. For all $t < \tau$, we have that $z_i^h(t) \geq z_i^\ell(t)$ because $z_i^h(t) = \min\{1, \frac{b_h + \int_0^t q_i^h(s)ds}{t}\} \geq \min\{1, \frac{b + \int_0^t q_i^\ell(s)ds}{t}\}$ as $b > b_i$ and $q_i^h(t) \geq q_i^\ell(t)$. At time $\tau$, $z_i^h(\tau) = q_i^{-h}(\tau) \geq q_i^{-\ell}(\tau)$. Thus, $Q_i^h \geq Q_i^\ell$ by construction. In addition,

$$\mathcal{P}_i^h = \tau Q_i^h - \int_0^\tau q_i^h(s)ds \leq \mathcal{P}_i^\ell + \tau(Q_i^h - Q_i^\ell) = \tau Q_i^h - \int_0^\tau q_i^\ell(s)ds,$$

since $\int_0^\tau q_i^h(s)ds \geq \int_0^\tau q_i^\ell(s)ds$. Thus, $v_i \geq v = \tau$ implies,

$$u_i(Q_i^h, -\mathcal{P}_i^h) = Q_i^hv_i - \mathcal{P}_i^h \geq Q_i^hv_i - (\mathcal{P}_i^\ell + v_i (Q_i^h - Q_i^\ell)) = u_i(Q_i^\ell, \mathcal{P}_i^\ell).$$
Case 3: \( \tau^h > \tau^\ell \).

If \( \tau^\ell > v \), then \( \exists t < \tau^\ell < \tau^h \) such that \( d^h_i(t) = d^\ell_i(t) = 0 \). Lemma 1 shows that this implies \( Q^\ell_i = Q^h_i = P^\ell_i = P^h_i = 0 \).

For the case where \( \nu \geq \tau^\ell \), note that \( z^h_i(t) \geq z^\ell_i(t) \forall t \in (0, \tau^\ell) \). Thus, at time \( \tau^\ell \) we have that

\[
\sum_{i=1}^{N} z^h_i(\tau^\ell) > 1 \geq \sum_{i=1}^{N} z^\ell_i(\tau^\ell).
\]

Recall, \( q^{-h}_i(t) \geq q^{-\ell}_i(t) \geq \forall t \in [0, \tau^\ell] \). This implies,

\[
s^h_i(\tau^\ell) \geq s^\ell_i(\tau^\ell) > 0.
\]

Thus,

\[
q^h_i(\tau) = \max\{0, 1 - \sum_{j \neq i} z^h_j(\tau)\}.
\]

Note that \( Q^\ell_i \leq \max\{0, 1 - \sum_{j \neq i} z_j(\tau^\ell)\} \). Thus, \( q^h_i(\tau^\ell) \geq Q^\ell_i \). Recall that \( P^\ell_i = \tau^\ell Q^\ell_i - \int_0^{\tau^\ell} q^\ell_i(s) ds \) and \( P^h_i = \tau^h q^h_i(\tau^h) - \int_0^{\tau^h} q^h_i(s) ds \). First suppose that \( \nu' \geq \tau^\ell \). Then,

\[
u_i(q^h_i(\tau^\ell), -p^h_i(\tau^\ell)) - u_i(Q^h_i, -P^h_i) = (q^h_i(\tau^\ell) - Q^h_i) (v_i - \tau^\ell) + \int_0^{\tau^\ell} (q^h_i(s) - q^\ell_i(s)) ds \geq 0.
\]

Note that \( Q^h_i \geq q^h_i(\tau^\ell) \) and \( P^h_i \leq p^h_i(\tau^\ell) + \nu(Q^h_i - q^h_i(\tau^\ell)) \leq p^h_i(\tau^\ell) + \nu(Q^h_i - q^\ell_i(\tau^\ell)) \). Thus,

\[
u_i(Q^h_i, -P^h_i) \geq \nu_i(q^h_i(\tau^\ell), -p^h_i(\tau^\ell)) \geq \nu_i(Q^\ell_i, -P^\ell_i).
\]

Proof of Lemma 4. I use the superscript \( h \) to denote variables when bidder \( i \) reports \( (v_i, b_i) \in \Theta \), and superscript \( \ell \) for variables when bidder \( i \) reports type \( (v, \min\{b_i, v\}) \in \Theta \). We assume \( v < v_i \). First, I show if \( \tau^\ell < v_i \), then \( \tau^h = \tau^\ell \). To do this, I show that \( z^h_i(t) = z^\ell_i(t) \forall t < v \). If \( v < b_i \), then

\[
z^\ell_i(t) = \max\{\frac{v + \int_0^t q^\ell_i(s) ds}{t}, 1\} = 1 = \max\{\frac{b_i + \int_0^t q^h_i(s) ds}{t}, 1\} = z^h_i(t) \text{ if } t < v.
\]

In addition, if \( v \geq b_i \), then

\[
z^\ell_i(t) = \max\{\frac{b_i + \int_0^t q^\ell_i(s) ds}{t}, 1\} \text{ if } t < v.
\]
This is the same function form as \( z^h_i(t) \) if \( t < v < v_i \). Thus there is no difference in bidder \( i \) reported preferences by reporting \((v, b_i)\) or \((v_i, b_i)\) when the time is \( t < v \). Thus,

\[
\sum_{j=1}^{N} z^h_j(t) = \sum_{j=1}^{N} z^f_j(t) \text{ if } t < v.
\]

This implies, \( \tau^h = \tau^f = \inf \{ t : \sum_{j=1}^{N} z^h_j(t) \leq 1 \} \).

Next, I show that, \( \tau^f \geq v \implies \tau^h \geq v \). This is because \( z^h_i(t) = z^f_i(t) \) \( \forall j = 1, \ldots, N \) and \( t < v \). Thus, \( \tau^f \geq v \implies \sum_{i=1}^{N} z^h_i(t) = \sum_{i=1}^{N} z^f_i(t) > 1 \forall t < v \). Or equivalently \( \tau^h \geq v \).

This allows me to break the remainder of the proof into three cases.

**Case 1** \( \tau^f < v \).

There is no difference in the outcome of the auction prior to time \( v \) because, \( z^h_i(t) = z^f_i(t) \) \( \forall t < v \). Since the auction terminates at time \( \tau^f < v \) under either report, then \( Q^h_i = Q^f_i \) and \( P^h_i = P^f_i \).

**Case 2** \( \tau^f = v, \tau^h \leq v_i \).

When bidder \( i \) reports the lower valuation, \( z^f_i(t) \) is discontinuous at \( \tau^f \). Thus, she wins \( Q^f_i \in [z^f_i(\tau^f), \lim_{t \to -\tau^f} z^f_i(t)] \) units and pays \( P^f_i = Q_i \tau^f - \int_{0}^{\tau^f} q^f_i(s) ds = P^f_i = Q_i \tau^f - \int_{0}^{\tau^f} q^h_i(s) ds, \) where the final equality holds as \( q^h_i(s) = q^f_i(s) \) \( \forall s \leq \tau^f \).

If \( \tau^h = \tau^f = v \), then \( Q^h_i = z^h_i(v) \), because \( z^h_i \) is continuous at \( v \). Thus, \( Q^h_i \geq Q^f_i \) because \( z^h_i(t) = \lim_{t \to -v} z^h_i(t) = \lim_{t \to -v} z^f_i(t) \geq Q^f_i \) where the final inequality follows from the construction of the mechanism. In addition \( P^h_i = P^f_i + v(Q^h_i - Q^f_i) < P^f_i + v_i(Q^h_i - Q^f_i) \).

If \( \tau^h > \tau^f = v \), then I first show \( q^h_i(\tau^f) \geq Q^f_i \). This holds because at time \( \tau^f \) we have that

\[
\sum_{i=1}^{N} z^h_i(\tau^f) > 1 \geq \sum_{i=1}^{N} z^f_i(\tau^f).
\]

Recall, \( z^h_i(t) = z^f_i(t) \) \( \forall t \in [0, \tau^f] \). This implies,

\[
q^h_i(\tau^f) = s^h_i(\tau^f) = s^f_i(\tau^f) > 0.
\]

Note that \( Q^f_i \leq s^f_i(\tau^f) = \max\{0, 1 - \sum_{j \neq i} z_j(\tau^f)\} \). Thus,

\[
Q^f_i \leq s^f_i(\tau^f) = s^h_i(\tau^f) = q^h_i(\tau^f).
\]

This implies \( Q^h_i \geq q^h_i(\tau^f) \geq Q^f_i \), and

\[
P^h_i \leq P^f_i + v(q^h_i(\tau^f) - Q^f_i) + \tau^h(Q^h_i - q^h_i(\tau^f)) \leq P^f_i + v_i(Q^h_i - Q^f_i).
\]
Thus, when $\tau^e = v$, $\tau^h \leq v_i$, then $Q^h_i \geq Q^e_i$, and

$$P^h_i \leq P^e_i + v_i(Q^h_i - Q^e_i).$$

Note also that $P^h_i, P^e_i \leq b_i$ by Lemma 1. Thus,

$$u_i(Q^h_i, -P^h_i) = Q^h_i v_i - P^h_i \geq Q^e_i v_i - P^e_i = u_i(Q^e_i, -P^e_i).$$

**Case 3** $\tau^e > v_i$.

Since $\tau^e > v$, $\exists t \in (0, \tau^e)$ such that $d_i^e(t) = 0$. Lemma 1 then implies that $Q^e_i = P^e_i = 0$.

If bidder $i$ reports the high type, then she wins $Q^h_i$ units and pays $P^h_i \leq Q^h_i v_i$, where the inequality follows from the construction of the auction. Since Lemma 1 implies $P^h_i \leq b_i$, then

$$u_i(Q^h_i, -P^h_i) \geq 0 = u_i(Q^e_i, -P^e_i).$$

**Proof of Proposition 3.** We assume bidder $i$’s opponents play undominated strategies. That is bidders $j \neq i$ play strategy profile $a_{-i} = (a_1 \ldots, a_{i-1}, a_{i+1}, \ldots, a_N)$, where $a_j : \Theta \to \Delta(\Theta)$, and $(v, b) \in \text{supp}(a_j(\theta_j))$ implies $0 \leq b_j \leq b \leq v \leq v_j \leq 1$.

Let $\mathcal{U}(\theta_i)$ be the set of undominated reports conditional on having type $\theta_i$. Our prior results show that $(v, b) \in \mathcal{U}(\theta_i)$ only if either $0 \leq b_i < b \leq v \leq v_i \leq 1$, or $b_i = b$ and $v = v_i$. I want to show that

$$(v_i, b_i) \in \text{arg } \max_{(v, b) \in \mathcal{U}(\theta_i)} \mathbb{E}_{F_i, a_{-i}} (U_i(Q_i, -P_i)|\theta_i),$$

and if $(v', b') \in \mathcal{U}(\theta_i)$ and $(v', b') \neq (v_i, b_i)$, then

$$(v', b') \notin \text{arg } \max_{(v, b) \in \mathcal{U}(\theta_i)} \mathbb{E}_{F_i, a_{-i}} (U_i(Q_i, -P_i)|\theta_i).$$

First, note that the payoff from truthful reporting is non-negative. If bidder $i$ reports her type truthfully, $P_i \leq b_i$ and $P_i \leq Q_i v_i$. Thus

$$\max_{(v, b) \in \mathcal{U}(\theta_i)} \mathbb{E}_{F_i, a_{-i}} (U_i(Q_i, -P_i)|\theta_i) \geq 0 \forall F_i.$$  

Second, note that for any realization of the proxy-clinching auction, bidder $i$’s utility is bounded by $v_i \leq 1$ for any report $(v, b)$. This is because $Q_i v_i \leq 1$ and $P_i \geq 0$.

If bidder $i$ reports $(v, b) \neq (v_i, b_i)$, then $(v, b) \in \mathcal{U}(\theta_i) \implies 0 \leq b_i < b \leq v \leq v_i \leq 1.
Let $T_{-i} = \{ \theta_{-i} | b > \max_{j \neq i} v_j \text{ and } \exists j \text{ s.t. } b_j > b_i \}$. Note that this set has positive measure, $\mu(T_{-i}) > 0$. Full-support beliefs imply that, bidder $i$ believes there is a positive probability that $\theta_{-i} \in T_{-i}$. That is, $F_i(T_{-i}|\theta_i) > 0$. If $\theta_{-i} \in T_{-i}$, then $a_{-i}(\theta_{-i})$ is such that all reported values $v'_j$ have $\min\{b_j, v'_j\} \leq v'_j \leq \max_{j \neq i} v_j < \delta$.

Thus, $\sum_{j \neq i} z_j(b_i + \epsilon) \geq 1$ where $\epsilon > 0$ is sufficiently small, because $\exists j$ with type $\theta_j$ such that $v_j, b_j > b_i$ and $z_j(b_i + \epsilon) = 1$. Thus $\tau > b_i + \epsilon$, and $q_i(b_i + \epsilon) = 0$. Yet, $\tau \leq \max_{j \neq i} v_j$, because if $\tau > \max_{j \neq i} v_j$, then $\exists t \in (\max_{j \neq i} v_j, \tau)$ s.t. $\sum_{j \neq i} z(t) = 0$. Since $z_i(t) = 1 \forall t \leq \tau \leq \max_{j \neq i} v_j$, then $Q_i = 1$. In addition $q_i(t) = 0$ when $t = b_i + \epsilon$. Thus $P_i > b_i + \epsilon$.

Thus bidder $i$ will receive a payoff $-\infty$ payoff if she reports $(v, b)$ and $\theta_{-i} \in T_{-i}$. Since this occurs with positive probability and the payoff from participating in the auction is bounded at one, we find that bidder $i$ receives a negative expected payoff from reporting $(v, b) \neq (v_i, b_i)$. Thus reporting $(v, b)$ is never a best reply, and reporting $(v_i, b_i)$ gives a greater expected payoff than any undominated strategy and is a unique best reply.

**Proof of Corollary 2.** First, I show that reporting $(v_i, \min\{b, v_i\})$ weakly dominates reporting $(v, b)$, when $v > v_i$. The proof is nearly identical to the proof of Proposition 1.

I use an $h$ superscript for variables when bidder $i$ reports $(v, b)$, where $v > v_i$; and an $\ell$ superscript when bidder $i$ reports $(v_i, \min\{b, v_i\})$. For any $t \leq v_i$, $z_i^\ell(t) = z_i^h(t) = \min\{1, q_i^\ell(t) + \frac{b - z_i^\ell(t)}{t} \} = \min\{1, \frac{b + z_i^\ell(t)}{t} \}$. Thus, $s_j^h(t) = s_j^\ell(t) \forall t \leq v_i$. Therefore, if $\tau^\ell < v_i$, then $\tau^h = \tau^\ell$ and bidder $i$ receives an equal payoff in each case, $u_i(Q_i^\ell, -P_i^\ell) = u_i(Q_i^h, -P_i^h)$.

If $\tau^\ell = v_i$, then $\tau^h \geq v_i$. First, suppose $\tau^h = v_i$. Thus, $Q_i^\ell \in [z_i^\ell(v_i), \lim_{t\to v_i^-} z_i^\ell(t)]$, and $Q_i^h = z_i^h(v_i)$ because $z_i^\ell$ is left discontinuous at $v_i$ and $z_i^h$ is left continuous at $v_i$. In addition, $s_j^h(t) = s_j^\ell(t) \forall t \leq v_i$ implies that $q_i^h(t) = q_i^\ell(t) \forall t \in (0, v_i)$. Thus, $Q_i^h = z_i^h(v_i) = \lim_{t\to v_i^-} Q_i^h(t) \geq Q_i^\ell$, and $P_i^h = P_i^\ell + v_i(Q_i^h - Q_i^\ell) \implies P_i^h \geq P_i^\ell$. Thus, if $P_i^\ell \leq b_i$, then

$$u_i(Q_i^\ell, -P_i^\ell) = Q_i^\ell v_i - P_i^\ell = Q_i^h v_i - (P_i^\ell + v_i(Q_i^h - Q_i^\ell)) \geq u_i(Q_i^h, -P_i^h).$$

If $P_i^\ell > b_i$ then

$$u_i(Q_i^\ell, -P_i^\ell) = Q_i^\ell v_i - b_i - \varphi(P_i^\ell - b_i) = Q_i^h v_i - b_i - (P_i^h - P_i^\ell) - \varphi(P_i^\ell - b_i) \geq Q_i^h v_i - b_i - \varphi(P_i^h - b_i),$$

and

$$u_i(Q_i^\ell, -P_i^\ell) \geq Q_i^h v_i - b_i - \varphi(P_i^h - b_i) = u_i(Q_i^h, -P_i^h).$$
Next, suppose that $\tau^e = v_i < \tau^h$. Since, $s^e_i(t) = s^f_i(t) \forall t \leq v_i$ and $Q^e_i \leq Q^f_i$, then $q^h_i(v_i) \geq Q^e_i$ and $P^h_i(v_i) = P^e_i + v_i(Q^h_i - Q^e_i)$. In addition, $Q^h_i \geq q^h_i(v_i) \geq Q^e_i$ and $P^h_i \geq\ P^h_i(v_i) + v_i(Q^h_i - q^h_i(v_i)) = P^e_i + v_i(Q^h_i - Q^e_i)$. Thus, if $P^e_i \leq b_i$, then

$$u_i(Q^e_i, -P^e_i) = Q^e_i v_i - P^e_i \geq Q^h_i v_i - (P^e_i + v_i(Q^h_i - Q^e_i)) \geq u_i(Q^h_i, -P^h_i).$$

If $P^e_i > b_i$ then

$$u_i(Q^e_i, -P^e_i) = Q^e_i v_i - b_i - \varphi(P^e_i - b_i) \geq Q^h_i v_i - b_i - (P^h_i - P^e_i) - \varphi(P^h_i - b_i) \geq Q^h_i v_i - b_i - \varphi(P^h_i - b_i),$$

and

$$u_i(Q^e_i, -P^e_i) \geq Q^h_i v_i - b_i - \varphi(P^h_i - b_i) = u_i(Q^h_i, -P^h_i).$$

Finally, suppose $\tau^e, \tau^h > v_i$. Then, $\exists t \in (0, \tau^e)$ such that $d^e_i(t) = 0$. Thus, $Q^e_i = P^e_i = 0$ by Lemma 1 and $u_i(Q^e_i, -P^e_i) = 0$. Also, recall that $q^e_i(t) = q^h_i(t) = 0 \forall t < v_i$. Thus, $P^h_i = Q^h_i \tau^h - \int_{v_i}^{\tau^h} q^h_i(s)ds = Q^h_i \tau^h - \int_{v_i}^{\tau^h} q^h_i(s)ds \geq Q^h_i v_i$. Thus,

$$u_i(Q^h_i, -P^h_i) \leq Q^h_i v_i - P^h_i \leq Q^h_i v_i - Q^h_i v_i = 0 = u_i(Q^e_i, -P^e_i).$$

Thus, reporting $(v, b)$ weakly dominates reporting $(v, \min\{b, v_i\})$, when $v > v_i$. In addition, we can invoke Proposition 2, because the proof of Proposition 2 does not change if $\varphi$ is finite or infinity. Thus, if bidders have utility functions of the form

$$u(x, -p, \theta_i) = \begin{cases} xv - p & \text{if } p \leq b_i \\ xv - b_i - \varphi(p - b_i) & \text{if } p > b_i \end{cases},$$

then $U(\theta_i)$ is such that $a \in U(\theta_i)$ only if

$$a = (v, b) \text{ where } (v, b) = (v_i, b_i) \text{ or } b_i < b \leq v \leq v_i.$$
\( \tau \leq \max_{j \neq i} v_j \), then \( Q_i = 1 \). In addition \( q_i(t) = 0 \) when \( t = b_i + \epsilon \). Thus \( \mathcal{P}_i > b_i + \epsilon \). This occurs with probability \( F_i(T_i|\theta_i) \).

Recall that bidder \( i \)'s utility is bounded. The highest payoff bidder \( i \) can receive is when she wins 1 unit and pays 0. Thus, her utility is bounded by \( v_i \). Thus, bidder \( i \)'s expected utility from reporting type \((v, b)\) is bounded by

\[
F_i(T_i|\theta_i) (v_i - b_i - \varphi(\epsilon)) + (1 - F_i(T_i|\theta_i)) v_i.
\]

The above quantity is less than 0 when \( \varphi \) is sufficiently large. Thus, reporting \((v_i, b_i)\) gives strictly greater expected utility than reporting \((v, b)\) where \( 0 \leq b_i < b \leq v_i \), when \( \varphi \) is sufficiently large.

**Proof of Proposition 5.** Consider an endowment economy with two commodities and \( N + 1 \) agents. The two commodities are the good and money. Agent 0 (the auctioneer) has utility over units of the commodity \( q \) and money \( m \),

\[
U_0(q, m) = m.
\]

Agent 0 is endowed with 0 units of the commodity and \( \sum \mathcal{P}_i \) units of money. Agent \( i \in 1, \ldots, N \) has preferences

\[
U_i(q, m) = \begin{cases} 
  v_i + m - b_i & \text{if } q > 1, \\
  qv_i + m - b_i & \text{if } q \in [0, 1]. 
\end{cases}
\]

Agent \( i \) is endowed with \( Q_i \) units of the good and \( b_i - \mathcal{P}_i \) units of money.

This endowment economy has a Walrasian equilibrium where agents do not trade and the market clearing prices are \( \tau \) for the good and 1 for money.

To show this, we must find each agent’s Marshallian demands given her endowment when the price of the good is \( \tau \) and the price of money is 1. This requires studying the outcome of the proxy clinching auction when bidders report types truthfully.

Consider a bidder \( i \) with valuation \( v_i < \tau \). In the proxy clinching auction bidders reports her type truthfully, the auction ends at time \( \tau \). Thus, there exists a time \( t \in (v_i, \tau) \) such that \( d_i(t) = 0 \). Thus, \( Q_i = \mathcal{P}_i = 0 \) by Lemma 1. In the general equilibrium economy, this means that bidder \( i \) is endowed with 0 units of the good and \( b_i \) units of money. At prices \( \tau \) (for the good) and 1 (for money), she consumes only money, and consumes \( b_i \) units. Thus, she does not trade any money.
Next, consider a bidder with valuation $v_i = \tau$. In the proxy clinching auction bidder $i$ wins $Q_i$ units and pays $P_i = p_i^{-}(\tau) + \tau (Q_i - q_i^{-}(\tau)) \leq b_i$. In the general equilibrium economy, this means that bidder $i$ is endowed with $Q_i$ units of the good and $b_i - P_i$ units of money. When prices are $\tau = v_i$ and 1, she demands any combination of money $m$ and the good $q$ such that

$$m + \tau q = (b_i - P_i) + \tau Q_i.$$  

Thus, she is indifferent between trading and not trading either commodity at these prices.

Finally, consider a bidder with valuation $v_i > \tau$. In the proxy clinching auctions, where all bidders report types truthfully, bidder $i$ wins $Q_i = z_i(\tau) = \min\{1, \frac{b_i + \int_{0}^{\tau} q(s)ds}{\tau}\}$ units, since $z_i(t)$ is continuous at $t = \tau$. She pays $P_i \leq b_i$. Note that $P_i = \tau Q_i - \int_{0}^{\tau} q_i(s)ds$. Thus, $Q_i = \min\{1, \frac{b_i + \int_{0}^{\tau} q(s)ds}{\tau}\} = \min\{1, \frac{\tau Q_i + b_i - P_i}{\tau}\}$.

In the general equilibrium economy, bidder $i$ is endowed with $Q_i$ units and $b_i - P_i$ units of money. When $v_i > \tau$, she demands as much of the good that she can afford, up to a quantity of 1. Thus, she demands $\min\{1, \frac{\tau Q_i + b_i - P_i}{\tau}\}$ units of the good. Recalling that $Q_i = \min\{1, \frac{\tau Q_i + b_i - P_i}{\tau}\}$, this means she demands $Q_i$ units of the good. Her remaining wealth is spent to consume money. Thus, if $Q_i < 1$, then $Q_i = \min\{1, \frac{b_i + \tau Q_i - P_i}{\tau}\} = \frac{b_i + \tau Q_i - P_i}{\tau} \implies b_i = P_i$. That is, agent $i$ demands no money, and spends all of her budget on consuming the good. If $Q_i = 1$, then agent $i$ buys one unit of the good and, $\frac{\tau Q_i + b_i - P_i}{\tau} \geq 1 \implies b_i \geq P_i$, and the bidder demands one unit of the good and $b_i - P_i$ units of money.

Thus, there is a Walrasian equilibrium where the price of money is 1 and the price of the good is $\tau$. The first welfare theorem implies that the Walrasian equilibrium is Pareto efficient. Thus, $\mathbb{P}\{q_i, m_i\}_{i=0}^{N}$ s.t. $\sum_{i=0}^{N} q_i \leq 1$ and

$$U_i(q_i, b_i - P_i) \geq U_i(q_i, m_i) \forall i = 1, \ldots, N$$

and

$$U_0(0, \sum_{i=1}^{N} P_i) = \sum P_i \geq U_0(q_0, m_0) = m_0.$$  

Noting that $u_0(q, m) = U_0(q, m)$ and $u_i(q, m) = U_i(q, m - b_i) \forall i = 1, \ldots, N$, we have that the outcome of the proxy clinching is Pareto efficient.