Competitive Screening and Search*

Daniel Garrett
Toulouse School of Economics
daniel.garrett@tse-fr.eu

Renato Gomes
Toulouse School of Economics
renato.gomes@tse-fr.eu

Lucas Maestri
Toulouse School of Economics
lucas.maestri@tse-fr.eu

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Abstract

We study the steady-state equilibrium of a matching market with adverse selection in which privately informed agents are matched with new principals over time. Agents can only accept offers from principals with whom they are matched. Equilibrium offers are menus which screen agent information. While principals are ex-ante identical, the menus offered in equilibrium are dispersed. We derive the equilibrium distribution over menus. Menus are ordered so that "more generous" menus leave more surplus uniformly over agent types. More generous contracts are found to be more efficient.

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*For helpful comments and suggestions...

1 Introduction

Economists have long understood the role played by nonlinear pricing in sorting consumers according to their unobserved preference characteristics. In a setting where a single firm enjoys full market power, the seminal contributions of Mussa and Rosen (1978), Maskin and Riley (1984) and Goldman, Leland and Sibley (1984) show that the optimal nonlinear price schedules distort quality provision (relative to efficiency) for all types except at the top. The contributions of Stole (1995) and Rochet and Stole (2002) extend this result to a duopolistic setting where firms are horizontally differentiated (a la Hotelling). As horizontal differentiation vanishes, these papers recover the Bertrand Paradox: The equilibrium quality provision is efficient for all types, and firms enjoy zero profits. Interestingly, this result implies that in the limit of perfect competition, equilibrium outcomes are independent of whether there is symmetric or asymmetric information about customers' types.[1]

In this paper, we study competition in nonlinear price schedules by perfectly homogenous firms that face asymmetric information about customers' (vertical) types. We resolve the Bertrand Paradox by introducing search frictions, according to which customers are unaware of the existence of a firm unless they receive an advertising message describing the nonlinear schedule offered by the firm. Upon receiving new advertising messages, matched customers may switch to another firm that offers a better deal. In the spirit of Mortensen (1998) and Burdett and Mortensen (1998), we consider stationary competitive equilibria in which the market coverage is constant and there is free entry. We index the degree of ex-ante competition (or contestability) in the market by varying the cost of advertising faced by firms.

Our analysis reveals that equilibrium outcomes involve dispersion in nonlinear price schedules. The firms' choice of schedules balances retention, profit, and efficiency considerations, where retention reflects both the probability of the offer's initial acceptance as well as the rate at which the relationship terminates. The analysis of this three-sided trade-off illuminates the interplay between competition and incentive constraints (and their associated distortions relative to efficiency). For each customer type, firms trade off rent extraction and retention. Across customer types, firms trade off efficiency and rent extraction (as dictated by incentive constraints). Increasing the indirect utility of customers with high types relaxes the low types' incentive constraints, and we show that this results in an interesting order property of equilibrium menus. Firms that offer schedules that deliver high indirect utility to high-type customers also deliver high indirect utilities to low types. Related to this, schedules that deliver higher indirect utilities to all types lead to smaller distortions in low-types' qualities.

In equilibrium, firms post nonlinear price schedules from a range of menus which are ordered as described above. At the bottom of this range lies the Mussa-Rosen schedule. For this schedule, both the probability of inducing a buyer to accept and commence the relationship and the expected

[1] See also Wilson (1993) and Armstrong and Vickers (2001) on this point.

duration of the relationship are at their shortest, i.e. expected retention is at its lowest. The support of the distribution of equilibrium offers is bounded above by the efficient (or Bertrand) schedule, which displays the highest retention but extracts zero rents. The equilibrium support of schedules depends on the cost of advertising. While the bottom of the support is always the Mussa-Rosen schedule, the top is decreasing in the advertising cost. Firms are indifferent between offering any two menus from the support, an equilibrium requirement which in turn determines the possible distributions over offers (we will see that this distribution is often uniquely determined).

Our equilibrium characterization delivers some interesting comparative statics results. First, as advertising costs increase, and the market becomes less contestable, firms assign higher probability to the schedules which are closer to the Mussa-Rosen schedules (where this proximity holds whether we consider the customers' indirect utilites or the offered quantitities). As advertising costs approach some upper bound (proportional to the monopolistic profit level of Mussa and Rosen), the distribution over contracts converges to a degenerate distribution centered at the Mussa-Rosen schedule. In contrast, as advertising costs approach zero, and the market becomes perfectly contestable, the distribution over contracts converges to a degenerate distribution centered at the Bertrand schedule (i.e., the Bertrand Paradox obtains). Thus, by varying the level of the advertising costs, we are able to capture the entire spectrum of competitive intensity, from perfect competition to monopoly.

1.1 Related Literature

Our analysis captures important aspects of markets where customers frequently purchase the same good, and firms constantly face the threat of losing long-time customers to competitors who offer better deals. In turn, the contribution of Inderst (2001) provides insight in the polar opposite case, where customers buy the good only once, but may delay the purchase in the hope of finding better offers. Inderst's main result establishes that, as customers become sufficiently patient, and competition is fiercer, the set of equilibria does not depend on whether customers' types are known by principals (confirming in a different setup the insights from Stole (1995), Armstrong and Vickers (2001), and Rochet and Stole (2002)).

Another strand of literature studies competition in auction design. In settings without search or matching frictions, McAfee (1993), Peters (1997), and Peters and Severinov (1997) show auction markets become competitive as the number of competitors increase. As a result, principals post second-price auctions with reserve prices equal to their costs, inducing efficient allocations. Apart from focusing on price discrimination, our work also differs from the studies above by also characterizing competing mechanism outcomes when the competition is not extreme.

The works of Burdett, Shi and Wright (2001) and Eeckhout and Kircher (2010a, 2010b) consider models where principals post prices (rather than mechanisms) and agents engage in directed search (i.e., choose which principal to contract with after observing the prices posted by all principals).

This literature focuses on the coordination frictions (as opposed to search frictions) that arise in unmeditated matching markets, and ignores the mechanism design issues that lie at the core of our work.

2 Model and Preliminaries

A dynamic economy is populated by a unit-mass continuum of customers who derive utility from consuming a vertically differentiated non-durable good. If a customer with valuation per quality θ purchases a unit of the good with quality q at a price x, his flow utility is

$$u(q, x, \theta) \equiv \theta \cdot q - x.$$

Otherwise, his flow utility is zero. Customers are heterogeneous in their (time-invariant) valuations per quality: the valuation of each customer is an iid draw from a discrete distribution with support $\{\theta_L, \theta_H\}$, where $\Delta\theta \equiv \theta_H - \theta_L > 0$, and associated probabilities p_L and p_H.

A continuum of firms compete by posting *menus* of contracts with different combinations of quality and price. The flow profit of a firm who sells a unit of the good with quality q at a price x is

$$x - \varphi(q),$$

where $\varphi(q)$ is the cost to the firm of providing quality q. Otherwise, the flow profit is zero. We assume that $\varphi(\cdot)$ is twice continuously differentiable, strictly increasing and strictly convex.

The contracts in each menu are *stationary* in the sense that prices and qualities are not permitted to vary across the life of the contract. Assuming that firms offer stationary contracts is realistic in many markets where (a) firms are bound to treat all customers identically (e.g., because of arbitrage or reselling), and/or (b) competition authorities restrict the use of dynamic payment rules such as those induced by loyalty programs. In Section 5, we offer further discussion of this assumption.

Each firm can serve a finite number of customers, but the precise amount will play no role. To simplify the exposition, let us assume that each firm has a unit capacity.

By the Revelation Principle we may assume that the firms post menus with two price-quality pairs: $\mathcal{M} \equiv ((q_L, x_L), (q_H, x_H)) \subset (\mathbb{R}_+ \times \mathbb{R})^2$, where (q_k, x_k) is the contract designed for the type $k \in \{L, H\}$.[2] Furthermore, every menu has to satisfy the incentive compatibility constraint, for each type $k \in \{L, H\}$:

$$IC_k : \qquad u(q_k, x_k, \theta_k) \geq \max_{\hat{k} \in \{L,H\}} \theta_k \cdot q_{\hat{k}} - x_{\hat{k}},$$

as well as the individual rationality constraint (IR) $u(q_k, x_k, \theta_k) \geq 0$. A menu $\mathcal{M}$ that satisfies the IC and IR constraints is said to be *implementable*. The set of implementable menus is denoted by $\mathbb{I}$.

[2] The restriction to deterministic mechanisms comes with no loss of generality. As the analysis that follows will make clear, firms will always prefer to offer deterministic mechanisms in equilibrium.

Firms compete for customers under the presence of matching frictions. Following the influential work of Burdett and Mortensen (1998), we study matching frictions in an undirected search framework, where customers are not aware of the existence of a firm unless they receive an advertising message describing the menu offered by that firm.

At every point in time, firms have the option to send an advertising message describing its menu of contracts at a flow cost c. Customers receive advertising messages according to independent Poisson processes, which arrival rate is given by the *matching function* $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$. The arrival rate $\lambda(v)$ depends on the mass of firms who are advertising, v. The function $\lambda(v)$ satisfies $\lambda(0) = 0$, is strictly increasing, weakly concave, and onto all of $\mathbb{R}_+$. The case of strict concavity captures the possibility that the advertising technology displays rivalry (or negative externalities) across ads, i.e., as more firms advertise, the rate at which a given ad reaches a customer, $\left(\frac{\lambda(v)}{v}\right)$, decreases. We assume that $\lambda_0 \equiv \lim_{v \searrow 0} \frac{\lambda(v)}{v}$ exists and is finite, and interpret this to be the rate at which a given ad reaches a customer in case $v = 0$ (in particular, consider a firm that chooses to advertise when no others are doing so).

Customers must respond to offers as soon as they arrive, and there is no recall. If a customer decides to purchase the good from a firm, we say that he is *matched* to that firm. A match continues provided that the firm continues to operate, and provided the customer continues to purchase from the firm. If the customer ceases to purchase, then the match is dissolved and cannot be resumed.

Customers abandon their present match with a firm when they receive a new advertising message describing a more attractive offer. Note that the rate at which each customer receives offers is equal to $\lambda(v)$, *irrespective* of whether the customer is matched or not to a firm. Since firms commit to stationary contracts, the firm with which a customer is presently matched cannot respond to new offers, for instance by cutting prices or increasing quality.

Matches may also end because firms may receive an exogenous shock that removes them from the market. Firms receive such "bankruptcy" shocks according to (independent) Poisson processes with arrival rate γ.

As will be described in detail shortly, our solution concept captures the "steady-state" outcome of this matching economy. Under this equilibrium concept, the strategy of each firm consists in committing to a menu of stationary contracts, and choosing whether to advertise at each moment in time. We describe the firm's choice of a menu by means of a (possibly degenerate) distribution over implementable mechanisms $\tilde{F}$ with support $\text{supp}(\tilde{F}) \subseteq \mathbb{I}$. For each customer type $k \in \{L, H\}$, let us denote by F_k the (possibly degenerate) marginal distribution of flow indirect utilities induced by the distribution of mechanisms $\tilde{F}$.

Under our "steady-state" solution concept, the expected present discounted value of a unit of capacity, denoted by Π, is constant in time. Because customers voluntarily destroy matches only when better offers arrive, the expected present value for a firm of matching with a k-type customer under the contract (q, x), denoted by $J_k(q, x)$, solves the following continuous-time Bellman equation:

$$\underbrace{r \cdot J_k(q,x)}_{\text{instantaneous return}} = \underbrace{x - \varphi(q)}_{\text{flow profit}} - \underbrace{\lambda(v) \cdot (1 - F_k(u(q,x,\theta_k))) \cdot [J_k(q,x) - \Pi]}_{\text{defection loss}} - \underbrace{\gamma \cdot J_k(q,x)}_{\text{bankruptcy loss}} \tag{1}$$

where r is the time discount rate. Intuitively, the instantenous return from matching with a k-type customer under the contract (q, x) equals the flow profit, discounted by the expected instantaneous loss from having the customer defecting to some other firm, and the expected instantaneous loss from receiving a bankruptcy shock.

Denote by μ the mass of workers matched with firms in the steady-state equilibrium of this economy (which will be defined formally shortly). We call μ the *steady-state market coverage* (for short, market coverage). For the market coverage to be constant, the flow out of the set of matched customers, $\gamma \cdot \mu$, has to equal the flow into the set of matched customers, $\lambda(v) \cdot (1 - \mu)$. This leads to the *market coverage condition*:

$$\frac{\mu}{1-\mu} = \frac{\lambda(v)}{\gamma}. \tag{2}$$

Denote by $G_k(u)$ the marginal distribution of indirect flow utilities received by k-type customers who are matched to *some* firm at the steady-state equilibrium of this economy. Because customers only destroy matches voluntarily when better offers arrive, the flow into the set of k-type customers who are matched and obtain an indirect flow utility of no more than u is given by $\lambda(v) \cdot (1-\mu) \cdot F_k(u)$. In turn, the flow out of the set of k-type customers who obtain an indirect flow utility of no more than u is given by $[\gamma + \lambda(v) \cdot (1 - F_k(u))] \cdot G_k(u) \cdot \mu$. This implies that

$$G_k(u) = \frac{\lambda(v) \cdot (1-\mu) \cdot F_k(u)}{[\gamma + \lambda(v) \cdot (1 - F_k(u))] \cdot \mu} = \frac{F_k(u) \cdot \gamma}{\gamma + \lambda(v) \cdot (1 - F_k(u))}, \tag{3}$$

where the second equality follows from the market coverage condition (2).

Equipped with the notation developed above, we can now derive the expected present discounted value of a unit of capacity, Π. As firms choose mechanisms $\mathcal{M} = \{(q_L, x_L), (q_H, x_H)\}$ to maximize their expected present discounted profits, in equilibrium, Π has to satisfy the following Bellman equation:

$$r \cdot \Pi = -c + \max_{\mathcal{M} \in \mathbb{I}} \left\{ \frac{\lambda(v)}{v} \cdot \sum_{k=H,L} p_k \cdot [(1-\mu) + \mu \cdot G_k(u(q_k, x_k, \theta_k))] \cdot (J_k(q_k, x_k) - \Pi) \right\}, \tag{4}$$

where $c > 0$ is the flow cost of advertising. The right-hand side of the equation above describes the instantaneous return from one unit of spare capacity. This has two components: the first is the flow cost of advertising. The second is the expected instantaneous gain induced by an optimal mechanism, where the terms $p_k \cdot [(1-\mu) + \mu \cdot G_k(u(q_k, x_k, \theta_k))]$ describe the probability of forming a match with a worker of type k conditional on a type-k worker receiving an advertising message.

We describe the steady-state outcome of this economy by means of a distribution over mechanisms $\tilde{F}$, a mass of advertising firms v, and a market coverage μ such that (i) firms choose their mechanisms and advertising decisions to maximize profits, (ii) competition among firms drives economic profits to zero, and (iii) the mass of firms advertising and the market coverage ratio are stationary. This is formalized in the next definition.

Definition 1 *[**Competitive Stationary Equilibrium**] A competitive stationary equilibrium (CSE) is described by a distribution over mechanisms $\tilde{F}$, mass of advertising firms v, and a market coverage μ such that:*

1. *(profit-maximization) the expected present discounted value of a unit of capacity is maximal at every mechanism offered in equilibrium, i.e., condition (4) holds for all $\mathcal{M} \in supp\,(\tilde{F}) \subseteq \mathbb{I}$,*

2. *(free entry) the expected present discounted value of a unit of capacity is zero, $\Pi = 0$, and unmatched firms are indifferent between advertising or leaving the market,*

3. *(stationarity) the mass of firms advertising v and the market coverage ratio μ satisfy the market coverage condition (2).*

2.1 Incentive Compatibility

A key step in our analysis is to formulate the firms' maximization problem in terms of the flow of indirect utilities offered to customers. To this end, denote by

$$q_k^* \equiv \arg\max_q \; \theta_k \cdot q - \varphi(q),$$

the efficient quality for type-k customers, and let $S_k^* \equiv \theta_k \cdot q_k^* - \varphi(q_k^*)$ be the social surplus associated with the efficient quality provision. The next lemma uses the incentive constraints and the optimality of equilibrium contracts to map flow indirect utilities into flow quality levels.

Lemma 1 *Consider a menu $\mathcal{M} = \{(q_L, x_L), (q_H, x_H)\}$ in the support of the equilibrium distribution over mechanisms, $\tilde{F}$, and let $u_k \equiv u(q_k, x_k, \theta_k)$. Then, for all $k \in \{L, H\}$,*

$$q_k = \mathbf{1}_k(u_H - u_L) \cdot \frac{u_H - u_L}{\triangle\theta} + (1 - \mathbf{1}_k(u_H - u_L)) \cdot q_k^*, \tag{5}$$

where $\mathbf{1}_H(z)$ is an indicator function that equals one if and only if $z \geq q_H^* \cdot \triangle\theta$, and $\mathbf{1}_L(z)$ is an indicator function that equals one if and only if $z \leq q_L^* \cdot \triangle\theta$.

The result above is standard in adverse selection models. Consider some menu $\mathcal{M} \in \text{supp}\,(\tilde{F})$ offered in equilibrium. If the IC_k constraint does not bind under $\mathcal{M}$, then profit-maximization by firms implies that the quality provision to the other type of customer (i.e., type $-k$) is efficient under

$\mathcal{M}$. However, if the IC_k constraint does bind under $\mathcal{M}$, then the quality to customers of type $-k$ is chosen to make type-k customers indifferent between either contract. These facts are summarized in equation (5).

In light of Lemma 1, we can describe each menu in the support of $\tilde{F}$ in terms of the flow indirect utilities induced by $\mathcal{M}$. Accordingly, we shall write $\mathcal{M} = (u_L, u_H)$ to describe the menu $\mathcal{M} = ((q_L, x_L), (q_H, x_H))$, where the map between q's and u's follows from equation (5). In a similar fashion, for convenience, we will more often refer to the marginal distribution of flow indirect utilities, F_k, rather than to the distribution over mechanisms $\tilde{F}$.

For each menu $\mathcal{M} = (u_L, u_H)$ offered in equilibrium, let

$$S_k(u_L, u_H) \equiv \theta_k \cdot q_k(u_L, u_H) - \varphi(q_k(u_L, u_H)) \tag{6}$$

be the social surplus induced by $\mathcal{M}$ for each customer type, where the quality levels $q_k(u_L, u_H)$ are computed according to (5). We can then write the profit derived by the firm from the contract to type-k customers under the menu $\mathcal{M} = (u_L, u_H)$ as $S_k(u_L, u_H) - u_k$.

Two natural benchmarks play an important role in the analysis that follows. The first one is the static monopolistic (or Mussa-Rosen) solution. Under this benchmark, the quality provided to low types, denote it q_L^M, is implicitly defined by:

$$\varphi'(q_L^M) = \theta_L - \frac{p_H}{p_L} \cdot \Delta\theta,$$

where the right-hand side of the equation above is assumed to be strictly positive (i.e., the Mussa-Rosen contract involves underprovision of quality, but not exclusion). In turn, quality provision for high types is efficient: $q_H^M = q_H^*$. Finally, recall that, in the monopolistic solution, the indirect utility left to low types is zero, $u_L^M = 0$ (as the IR is binding), and the indirect utility left to high types is $u_H^M = q_L^M \cdot \triangle\theta$, as the IC_H is binding. We refer to $\mathcal{M}^M \equiv \left(\left(q_L^M, \theta_L \cdot q_L^M\right), \left(q_H^*, \theta_H \cdot q_H^* - u_H^M\right)\right)$ as the *monopolist* (or *Mussa-Rosen*) *menu*.

The second benchmark is the competitive (or Bertrand) solution. Under this benchmark, quality provision is efficient to both types, and firms derive zero profits from each contract in the menu. We refer to $\mathcal{M}^* \equiv ((q_L^*, \varphi(q_L^*)), (q_H^*, \varphi(q_H^*)))$ as the *competitive* (or *Bertrand*) *menu*.

We can now proceed to characterizing the equilibrium of the economy. In order to ensure the existence of a non-trivial equilibrium, we require that the cost of advertising is not too high relative to its effectiveness in reaching customers (otherwise, the unique equilibrium involves no firms advertising). The condition for this depends on the profit that can be made by advertising the Mussa-Rosen menu. In particular, we assume from hereon that

$$0 < c < \frac{\lambda_0}{\gamma + r} \cdot \pi^M,$$

where $\pi^M = p_L \left(\theta_L q_L^M - \varphi\left(q_L^M\right)\right) + p_H \left(\theta_H q_H^* - q_L^M \varphi\left(q_H^*\right) - \Delta\theta\right)$ is the expected flow profit from offering the Mussa-Rosen menu when the distribution of types is the one in the population.

3 Screening and Retention

This paper studies competition among firms that face asymmetric information regarding customers' types. The presence of search frictions, according to which a customer can only purchase from a firm after receiving an advertising message from that firm, introduces an important distinction between ex-ante and ex-post competition. While firms will generally be able to extract positive rents from customers *after* a match is realized, there is perfect competition *ex-ante* (i.e., before advertising takes place). This feature is captured by the equilibrium requirement that the ex-ante value of a unit of capacity is zero: $\Pi = 0$. The next lemma, which follows from arguments similar to those in Mortensen (1998), exploit this property to offer a convenient formulation of the firms' optimal screening problem.

Lemma 2 *A menu $\mathcal{M} = (u_L, u_H)$ is offered in equilibrium only if*

$$\mathcal{M} \in \arg\max_{(u_L,u_H)} \left\{ \sum_{k=H,L} p_k \cdot \Phi_k(u_k) \cdot (S_k(u_L, u_H) - u_k) \right\} \qquad s.t. \qquad u_H \geq u_L \geq 0, \tag{7}$$

where the retention functions, $\Phi_k(\cdot)$, are defined as

$$\Phi_k(u) \equiv \frac{\gamma}{\gamma + \lambda(v) \cdot (1 - F_k(u))} \cdot \frac{\gamma + r}{\gamma + r + \lambda(v) \cdot (1 - F_k(u))}.$$

The *retention level* $\Phi_k(u)$ captures the expected discounted duration of a match with a type-k customer who obtains flow indirect utility u. Retention levels have a one-to-one relation to the equilibrium distributions of flow payoffs, $F_k(u)$. As one should expect, the smaller is u, the larger is the mass of firms offering contracts that are more advantageous to type-k customers, $1 - F_k(u)$, and the smaller is the retention level $\Phi_k(u)$. The objective function (7) is simply a weighted average of the profits associated with each type of customer, with weights given by the retention levels.

Remark 1 *We will show that, in equilibrium, for each k, the retention function $\Phi_k(\cdot)$ is strictly positive and strictly increasing over an interval $[\underline{u}_k, \bar{u}_k]$, with $\Phi_k(\bar{u}_k) = 1$. Thus, $\Phi_k(\cdot)$ corresponds to a distribution function on $[\underline{u}_k, \bar{u}_k]$, with a mass point at $\underline{u}_k$. The problem (7) is therefore equivalent to that of a monopolistic firm that offers second-degree price-discrininatory schedules (a la Mussa and Rosen) in a setting where type-k customers have a (random) outside option u distributed according to $\Phi_k(u)$ (such a model is analysed in Rochet and Stole (2002)). Note, however, that in our setting the retention functions $\Phi_k(\cdot)$ are equilibrium objects that depend on the behavior of all firms in the economy.*

The result from Lemma 2 holds independently of which incentive constraints bind in equilibrium. In order to advance the analysis, we will proceed by assuming that IC_L is always slack in equilibrium, in which case IC_H is the only potentially binding constraint. As will become clear, this is indeed true in any CSE of this economy. If IC_L is always slack, we can use Lemma 1 to simplify the objective function (7) identified in Lemma 2. Accordingly, firms choose menus $\mathcal{M} = (u_L, u_H)$ to solve

$$\max_{(u_L,u_H)} \quad \pi(u_L, u_H) \qquad s.t. \qquad u_H \geq u_L \geq 0, \tag{8}$$

where

$$\pi(u_L, u_H) \equiv p_H \cdot \Phi_H(u_H) \cdot (S_H^* - u_H) + p_L \cdot \Phi_L(u_L) \cdot (S_L(u_L, u_H) - u_L)\,. \tag{9}$$

Note that the constraint $u_H \geq u_L \geq 0$ accounts for the IC_H and IR constraints that ensure implementability. To help understand the problem of a firm, let us assume for the moment that the retention functions $\Phi_k(\cdot)$ are differentiable in equilibrium (the equilibrium characterization from Proposition 1 below confirms this conjecture). At any point where $u_H > u_L > 0$, the first-order condition for the high types is

$$\underbrace{p_H \cdot \Phi'_H(u_H) \cdot (S_H^* - u_H)}_{\text{retention gains}} - \underbrace{p_H \cdot \Phi_H(u_H)}_{\text{profit losses}} + \underbrace{p_L \cdot \Phi_L(u_L) \cdot \frac{\partial S_L}{\partial u_H}(u_L, u_H)}_{\text{efficiency gains}} = 0, \tag{10}$$

and for the low types is

$$\underbrace{p_L \cdot \Phi'_L(u_L) \cdot (S_L(u_L, u_H) - u_L)}_{\text{retention gains}} - \underbrace{p_L \cdot \Phi_L(u_L)}_{\text{profit losses}} + \underbrace{p_L \cdot \Phi_L(u_L) \cdot \frac{\partial S_L}{\partial u_L}(u_L, u_H)}_{\text{efficiency losses}} = 0. \tag{11}$$

Intuitively, the firms' choice of menus balances retention, profit, and efficiency considerations. Let us start with the first-order condition for high types, given by equation (10). By increasing the flow indirect utility u_H, the firm increases retention (the first term), but decreases profits (the second term). More interestingly, increasing u_H also relaxes the incentive constraint IC_H, which enables the firm to reduce the quality distortion present in the low types' contract. This efficiency gain is the third term in equation (10).

Let us now consider the first-order condition for low types, given by equation (11). The first two terms capture the retention gains and the profit losses from increasing u_L. In contrast to (10), however, increasing u_L has the effect of tightening the incentive constraint IC_H, which implies that the quality distortion present in the low types' contract has to increase. This efficiency loss is the third term in equation (11).

We can directly compute from equations (5) and (6) that, whenever IC_H is binding,

$$\frac{\partial S_L}{\partial u_H}(u_L, u_H) = -\frac{\partial S_L}{\partial u_L}(u_L, u_H) = \big(\theta_L - \varphi'(q_L(u_L, u_H))\big) \cdot \frac{1}{\triangle\theta} > 0,$$

that is, the impact on surplus from changing u_H or u_L have the same magnitudes and opposite signs.

3.1 Equilibrium Characterization

We will now characterize the distribution over menus that prevail in equilibrium. As a first step, the next lemma uses standard arguments to establish that, for each $k \in \{L, H\}$, the distribution over flow indirect utilities, F_k, is continuous, and has support on an interval that starts at the indirect utility associated with the monopolistic (Mussa-Rosen) menu. We denote by $\Upsilon_k \subset \mathbb{R}_+$ the support of each distribution F_k, and by Υ the set of pairs (u_L, u_H) induced by the menus in the support of the distribution over mechanisms $\tilde{F}$. In light of Lemma 1, we will abuse notation and identify the support of $\tilde{F}$ with the set Υ.

Lemma 3 *[**Support**] In any CSE of this economy, the marginal cdf of flow indirect utilities, F_k, has no mass points, and has support*

$$\Upsilon_k = [u_k^M, \bar{u}_k],$$

where $\bar{u}_k < S_k^*$, for $k \in \{L, H\}$.

In the proof of this lemma, we construct deviations to any candidate equilibrium where F_k is not continuous, or where Υ_k is not a closed interval starting at u_k^M. To obtain some intuition, assume that Υ contains an implementable menu whose indirect utilities (u_L, u_H) are such that $u_k < u_k^M$ for some k (in which case $\inf \Upsilon_k < u_k^M$). By deviating to the menu $\left(\max\{u_k, u_k^M\}\right)_{k=L,H}$, the firm weakly raises its flow profit and retention for both customer types, and makes strict gains for at least one type of customer. Therefore, any menu in Υ has to satisfy $u_k \geq u_k^M$ for all k.

Alternatively, suppose there exists $\epsilon > 0$ such that $u_L > u_L^M + \epsilon$ for all $u_L \in \Upsilon_L$ (the argument for type $k = H$ is analogous). Pick the menu with the lowest flow indirect utility for low types, denoted $(u_L^\#, u_H^\#)$, and consider a deviation to $(u_L^M, u_H^\#)$. This deviation affects retention neither for low nor high types, but it strictly increases the flow profit obtained from the low types. Therefore, $\inf \Upsilon_k = u_k^M$ for all k. The proof contained in the appendix formalizes the heuristics above, and applies similar ideas to establish that each Υ_k is connected, and each F_k is continuous.

Before studying CSE of this economy, it is useful to introduce an important class of CSE equilibria, as described below.

Definition 2 *[**Ordered CSE**] A CSE is said to be* ordered *if, for any two menus $\mathcal{M} = (u_L, u_H)$ and $\mathcal{M}' = (u'_L, u'_H)$ offered in equilibrium, $u_L < u'_L$ if and only if $u_H < u'_H$. In this case, the menu (u'_L, u'_H) is said to be* more generous *than the menu (u_L, u_H).*

Intuitively, in an ordered CSE, if firm A offers a better deal to low types than firm B, then firm A also offers a better deal to high types than firm B. For some parameters of the model (in particular, whenever the cost of advertising c is sufficiently large), it turns out there is only one CSE, the ordered CSE. For other parameters (sufficiently small c), other CSE exist. We delay the details until after Proposition 1. Ordered CSE have the following important property.

Remark 2 ***[Support Function]*** *In every ordered CSE, the support of flow indirect utilities offered by firms can be described by a strictly increasing and bijective* support function $\hat{u}_L : \Upsilon_H \to \Upsilon_L$ *such that, for every menu* $\mathcal{M} = (u_L, u_H)$ *in* Υ, $u_L = \hat{u}_L(u_H)$. *By definition, a support function* $\hat{u}_L$ *is such that* $F_L(\hat{u}_L(u_H)) = F_H(u_H)$, *or equivalently,* $\Phi_L(\hat{u}_L(u_H)) = \Phi_H(u_H)$ *for all* $u_H \in \Upsilon_H$.

The next proposition characterizes the unique ordered CSE of the economy.

Proposition 1 ***[Equilibrium Characterization]*** *There exists a unique ordered CSE. In this equilibrium, the support of flow indirect utilities offered by firms,* Υ, *is described by the support function* $\hat{u}_L : [u_H^M, \bar{u}_H] \to [0, \bar{u}_L]$ *that solves the differential equation*

$$\hat{u}_L'(u_H) = \frac{S_L(\hat{u}_L(u_H), u_H) - \hat{u}_L(u_H)}{S_H^* - u_H} \cdot \frac{1 - \frac{p_L}{p_H} \cdot \frac{\partial S_L}{\partial u_H}(\hat{u}_L(u_H), u_H)}{1 - \frac{\partial S_L}{\partial u_L}(\hat{u}_L(u_H), u_H)} \tag{12}$$

with boundary condition $\hat{u}_L(u_H^M) = 0$. *In particular,* $\hat{u}_L'(u_H) \in (0,1)$ *for all* $u_H \in [u_H^M, \bar{u}_H]$.

The equilibrium market coverage, and the upper limit of the support Υ_H, $\bar{u}_H$, *are determined by the indifference conditions*

$$\pi(u_L^M, u_H^M) = \frac{c \cdot v}{\lambda(v)} \cdot (\gamma + r) = \pi(\hat{u}_L(\bar{u}_H), \bar{u}_H). \tag{13}$$

Finally, the equilibrium retention functions, Φ_k, *are the solutions to the differential equations defined by the first-order conditions (10) and (11) evaluated along the support function* $u_L = \hat{u}_L(u_H)$, *with boundary conditions* $\Phi_k(\bar{u}_k) = 1$, *for* $k \in \{L, H\}$.

Equilibrium Construction. Before discussing intuition, it is worth clarifying how we construct the equilibrium in Proposition 1. We start with the support function $\hat{u}_L(\cdot)$. By Lemma 3 and Remark 2, we know that $\Upsilon = \{(\hat{u}_L(u_H), u_H) : u_H^M \le u_H \le \bar{u}_H\}$. Therefore, the support function $\hat{u}_L(\cdot)$, and the supremum of Υ_H, $\bar{u}_H$, determines the support Υ. In particular, the Mussa-Rosen menu $\mathcal{M}^M$ is associated with the highest expected discounted profits but the lowest retention level among all menus in Υ. In turn, the "supremum" menu $\bar{\mathcal{M}} = (\hat{u}_L(\bar{u}_H), \bar{u}_H)$ is associated with the lowest expected discounted profits, but the highest retention level among all menus in Υ.

The differential equation (12), with boundary condition $\hat{u}_L(u_H^M) = 0$, delivers the support function $\hat{u}_L(\cdot)$. Remarkably, the function $\hat{u}_L(\cdot)$ does not depend on any search parameter (such as the market coverage v, the destruction rate γ, the discount rate r, or the advertising cost c). Rather, it depends only on the primitives of the price discrimination problem (that is, the types θ_k, the probabilities p_k, and the cost function $\varphi(\cdot)$).

Now consider the indifference condition (13). This condition requires that the expected discounted profits from the menus $\mathcal{M}^M$ and $\bar{\mathcal{M}}$ equal the expected advertising cost of a unit of capacity. The menus $\mathcal{M}^M$ and $\bar{\mathcal{M}}$ lie at the extremes of the support Υ. Together with first-order conditions

(10) and (11), the indifference condition (13) guarantees that firms are indifferent between offering any two menus in Υ. This implies that the expected profit associated with *any* menu offered in equilibrium equals the expected expenditures with advertising, in which case $\Pi = 0$.

Importantly, the first equality in (13), which pertains to the expected discounted profits from the Mussa-Rosen menu, determines the equilibrium mass of advertising firms v (which, together with the market coverage condition (2), determines the equilibrium market coverage μ). In turn, the second equality in (13), which pertains to the expected discounted profits from the "upper" menu $\bar{\mathcal{M}}$, determines $\bar{u}_H$, the highest flow indirect utility offered to high types in equilibrium.

Finally, evaluated along the support function $u_L = \hat{u}_L(u_H)$, the first-order conditions (10) and (11) define the retention functions $\Phi_k(\cdot)$ (and, therefore, the distributions over flow indirect utilities $F_k(\cdot)$) that render firms indifferent between choosing any menu in the support Υ.

Competition and Distortions. The characterization from Proposition 1 helps understanding the interplay between competition and asymmetric information about customers' types. In this regard, it is convenient to consider the *utility wedge* $\omega(u_H) \equiv u_H - \hat{u}_L(u_H)$, which measures the difference between the flow indirect utility of high and low types along the support Υ_H. As established in Lemma 1, this wedge determines whether (and which) incentive compatibility constraints bind, what, in turn, determines the quality provision to both customers types.

Because $\hat{u}'_L(u_H) \in (0, 1)$ along the support Υ_H, it follows that the wedge $\omega(u_H)$ is strictly increasing in u_H, reaching its maximum at the upper limit of Υ_H, $\bar{u}_H$. The fact that $\omega(u_H)$ is strictly increasing in u_H reflects the fact that competition for high types is fiercer than competition for low types in equilibrium, as the type-H customers "have more surplus to share" with firms.

As established in Lemma 3, $\bar{u}_k < S_k^*$, what then implies that $\omega(\bar{u}_H) < \omega(S_H^*) = S_H^* - S_L^* < \triangle\theta \cdot q_H^*$. Therefore, by virtue of Lemma 1, it follows that the incentive compatibility constraint for low types (IC_L) never binds in equilibrium, and the quality supplied to the high type is always efficient.

In turn, the quality supplied to low types may be downward distorted. Because $\triangle\theta \cdot q_L^* < S_H^* - S_L^* = \omega(S_H^*)$, it follows that there exists a threshold level $\tilde{u}_H \in (u_H^M, S_H^*)$ satisfying $\omega(\tilde{u}_H) = q_L^* \cdot \triangle\theta$. Therefore, for u_H sufficiently small, i.e. $u_H \in [u_H^M, \tilde{u}_H)$, the incentive constraint IC_H binds, and the quality provision to low-type customers is downward distorted:

$$q_L\left(\hat{u}_L\left(u_H\right), u_H\right) = \frac{\omega\left(u_H\right)}{\Delta\theta} < q_L^*. \tag{14}$$

In turn, for $u_H > \tilde{u}_H$, IC_H no longer binds, and the quality provision to low-type customers is efficient.[3]

[3]Interestingly, the support function (and hence the wedge $\omega(\cdot)$) is linear whenever $u_H \in [\tilde{u}_H, \bar{u}_H]$, i.e., in the region of the support where quality provision is efficient.

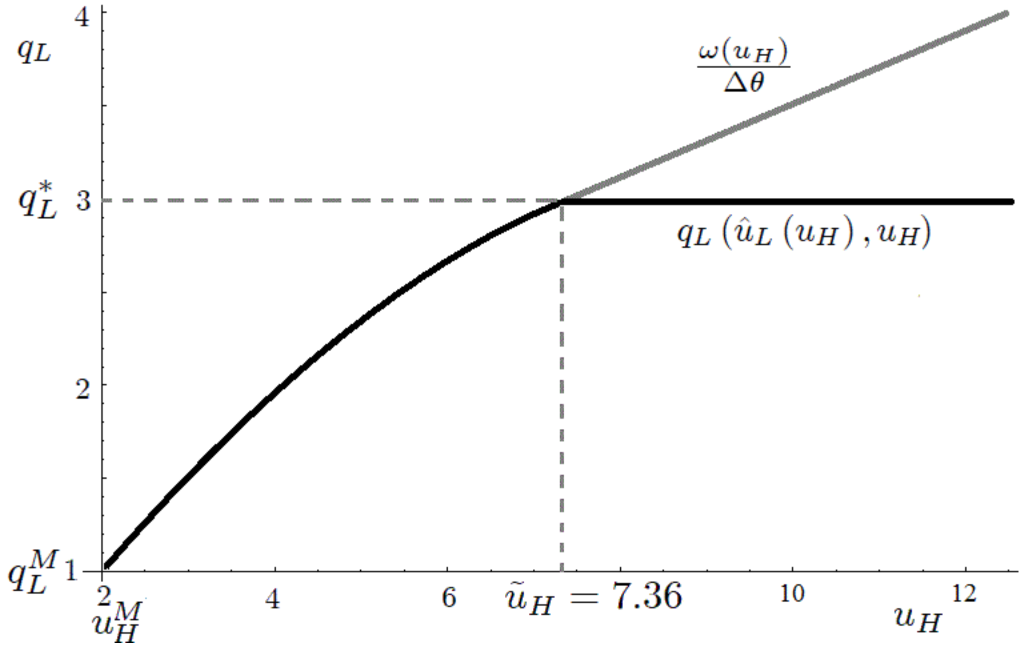

Figure 1: The schedule of quality provided to low-type customers as a function of the indirect utility enjoyed by high types. The parametrization is $p_H = p_L = \frac{1}{2}, \theta_L = 3, \theta_H = 5$, and $\varphi(q) = \frac{1}{2} \cdot q^2$.

An important implication of equation (14) is therefore that the distortion of low-type quality contracts depends on *how generous* the menu is in terms of customer payoffs. We summarize this as follows.

Corollary 1 *[**Efficiency and generosity**] Menus for which customers earn higher payoffs are more efficient. In particular, $S_L(\hat{u}_L(u_H), u_H)$ is strictly increasing whenever $u_H < \tilde{u}_H$ and equal to the efficient level S_L^* whenever $u_H \geq \tilde{u}_H$.*

The figure above illustrates Corollary 1 in the case where types are uniformly distributed and production costs are quadratic.

Equilibrium Uniqueness. We will now clarify under what conditions there is a unique CSE. As it turns out, the uniqueness of equilibria is intimately related to order properties of the objective function (8). The next lemma demonstrates that firms' profits have locally strict increasing (constant) differences if and only if the incentive constraint IC_H is binding (slack).

Remark 3 *[**Increasing Differences**] If the downward incentive constraint (IC_H) is binding, the discounted expected profit function $\pi(u_L, u_H)$ satisfies strict increasing differences in (u_L, u_H). In turn, when IC_H is slack, the discounted expected profit function $\pi(u_L, u_H)$ is modular in (u_L, u_H); that is, $\pi(u_L, u_H) = \sum_{k \in \{L,H\}} \pi_k(u_k)$, for some choice of $\pi_k(\cdot)$.*

The reason for this property is very simple. When IC_H binds, an increase in u_H, *ceteris paribus*, increases the quality that can be supplied to low-type customers in an incentive-compatible menu. Therefore, the profit that firms can earn in the relationship with low types is increased. This in turn increases the firms' benefit of low type retention and makes them more willing to cede a higher indirect utility for this purpose.

In constrast, when IC_H is slack, the firms' decisions of which indirect utility to leave to each type of customer are completely separable. In particular, the indirect utility that a firm leaves to a low type can be chosen independently of the payoff left to a high type.

The property in Remark 3 leads us to the important finding that, depending on parameters, the ordered CSE described in Proposition 1 can be the *only* CSE.

Corollary 2 *[**Incentive Constraints**] There exists a threshold $\tilde{c} \in \left(0, \frac{\lambda_0}{\gamma+r} \cdot \pi^M\right)$ such that*

1. *if $c \geq \tilde{c}$, the downward incentive constraint (IC_H) is binding for all menus offered in equilibrium. In this case, the only CSE is the ordered CSE.*

2. *if $c < \tilde{c}$, the downward incentive constraint (IC_H) is binding for all menus such that $u_H \leq \tilde{u}_H$. For such menus, the indirect utility, and the quality provision, of low-type customers is the same in any CSE. Yet, there exist multiple CSE equilibria that differ on the menus (u_L, u_H) with $u_H > \tilde{u}_H$. However, all such equilibria (including the non-ordered ones) lead to the same distributions over flow indirect utilities $F_k(\cdot)$.*

According to this result, the order requirement from Definition 2 "selects" equilibria only when $c < \tilde{c}$. In such a case, for all $u_H \in [\tilde{u}_H, \bar{u}_H]$, the discounted expected profit function $\pi(u_L, u_H)$ is modular in (u_L, u_H). As a consequence, for each given $u_H \in [\tilde{u}_H, \bar{u}_H]$, the L-type flow indirect utility that maximizes $\pi(u_L, u_H)$ is not unique. It is then possible to construct different equilibria in which firms randomize over efficient menus whose flow indirect utilities are not ordered. It is worth emphasizing, however, that all CSE equilibria (including the non-ordered ones) lead to the same distributions over flow indirect utilities $F_k(\cdot)$ (as implied by the first-order conditions (10) and (11)). In this sense, all CSE are "payoff-equivalent" to the unique ordered CSE identified in Proposition 1.

3.2 Comparative Statics

The characterization from Proposition 1 reveals that if ex-ante competition is intense (as captured by low advertising costs), then a positive mass of firms provide quality efficiently for both customer types. The next proposition investigates the effect of a higher degree of competition on the equilibrium distributions of flow indirect utilities. As one should expect, when advertising costs **decrease**, firms face more competitive pressure, and more often offer menus that lead to high flow indirect utilities for both customer types. This implies the mass of firms that offer inefficient qualities in equilibrium decreases as competition gets fiercer.

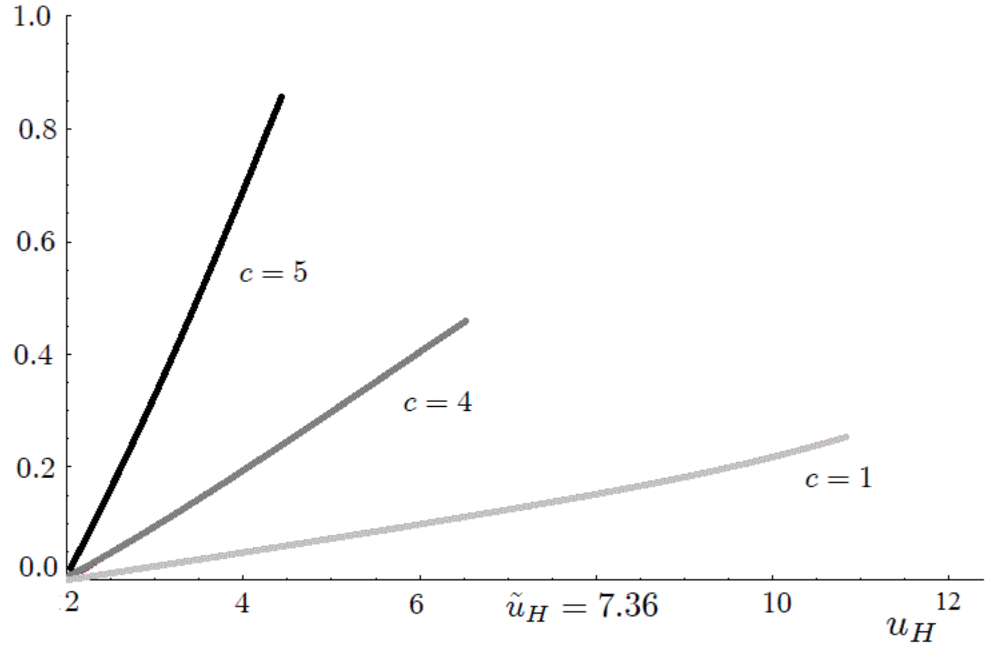

Figure 2: The probability density functions of the indirect utilities enjoyed by high types, f_H, under different costs of advertising. The search parameters are $\lambda(v) = v, \gamma = 1$, and $r = 0.2$. The Mussa-Rosen parameters are $p_H = p_L = \frac{1}{2}, \theta_L = 3, \theta_H = 5$, and $\varphi(q) = \frac{1}{2} \cdot q^2$.

Proposition 2 ***[Competition and Distortions]*** *Denote by F_k and $\check{F}_k$ the equilibrium distributions over flow indirect utilities when the advertising cost is c and $\check{c}$, respectively. If $c < \check{c}$, then F_k first-order stochastically dominates $\check{F}_k$. In particular, the mass of firms that offer inefficient qualities decreases as c decreases: $F_k(\tilde{u}_H) < \check{F}_k(\tilde{u}_H)$.*

Figure 2 illustrates the result from Proposition 2 by plotting the densities for the same constellation of parameters as Figure 1. The distribution of high-type payoffs are ordered in terms of first-order stochastic dominance, as stated by the proposition. For high values of c, the support of high-type payoffs lies below $\tilde{u}_H$, and so all menus involve distorted low-type qualities. For lower values of c, high-type payoffs exceed $\tilde{u}_H$ with positive probability, so that the probability of efficient contracting is also positive.

Next, we provide a result describing the distributions of offers made in equilibrium, as well as the distribution of contracts in existing matches. To describe these distributions it is enough to consider the distribution of high-type payoffs $F_H(u_H)$ and $G_H(u_H)$ respectively. As discussed above, the marginal distributions for low-type offers then satisfy $F_L(\hat{u}_L(u_H)) = F_H(u_H)$ and $G_L(\hat{u}_L(u_H)) = G_H(u_H)$, as discussed above.

Proposition 3 ***[Competitive and Monopolistic Benchmarks]***

1. *As the cost of advertising c approaches* $\frac{\lambda_0}{\gamma+r}\cdot\pi^M$*,* $F_H(u_H)$ *converges to a degenerate distribution centered at* u_H^M*. Thus, the distribution of offers converges to a degenerate distribution at the monopolistic (Mussa-Rosen) menu* $\mathcal{M}^M$*. The same is true for the distribution of contracts in matched pairs.*

2. *As the cost of advertising c approaches zero,* $F_H(u_H)$ *converges to a non-degenerate continuous distribution* $F_H^0(u_H)$ *with support* $\left[u_H^M, S_H^*\right]$*. The distribution over high-type payoffs in matched pairs,* $G_H(u_H)$*, converges to a degenerate distribution at* S_H^**. Thus, the distribution of contracts in matched pairs converges to a degenerate distribution at the competitive (Bertrand) menu* $\mathcal{M}^*$*.*

As the advertising cost gets closer to the the profit of a monopolist subject to matching frictions (which is the Mussa-Rosen profit discounted by both time and the match destruction rate γ), the mass of advertising firms vanishes, $v \to 0$, and the distribution over menus converges to a degenerate distribution centered at the Mussa-Rosen menu $\mathcal{M}^M$. On the other hand, as the advertising cost approaches zero, the mass of advertising firms explodes, i.e. $v \to \infty$. The distribution of high-type payoffs then converges to a limiting distribution $F_H^0(u_H)$ which stochastically dominates the distributions for all positive costs below $\frac{\lambda_0}{\gamma+r}\cdot\pi^M$. Because $\lambda(v)\to\infty$ as $v\to\infty$, (3) then implies that $G_H(u_H)$ converges to a degenerate distribution centered at S_H^*.

Importantly, Proposition 3 captures the entire spectrum of industry competitiveness. When c is large, ex-post competition is weak, and we obtain the sensible prediction that firms' behavior is close to that of a firm with complete market power. When c is small, ex-post competition is strong. While the distribution of offers does not converge to competitive (Bertrand) menus, those offers which are far from this competitive benchmark yield lower payoffs to customers and therefore result in shorter matches. This explains why the distribution of contracts in steady-state matches converges to point mass at the competitive menu.

3.3 The Lerner-Wilson Formula under Competition

In order to derive the firms' best responses, we wrote the firms' mechanism design problem in terms of the flow indirect utilities enjoyed by each type of customer. While this approach is convenient for characterizing equilibrium, in reality firms compete by posting menus of quality and price pairs.

In this subsection, we will translate the equilibrium characterization from Proposition 1 in terms of observable variables (which may render our model amenable to structural empirical analysis). To this end, let us define the retention semi-elasticity for type-k customers of a contract (q_k, x_k) as

$$\mathcal{E}_k(q_k,x_k)\equiv\frac{\Phi_k'(\theta_k\cdot q_k-x_k)}{\Phi_k(\theta_k\cdot q_k-x_k)}.$$

The semi-elasticity $\mathcal{E}_k(q_k, x_k)$ measures the percentage increase in retention among type-k customers as the price associated to quality q_k is decreased by one dollar. Moreover, let us denote the retention among type-k customers of a contract (q_k, x_k) as $\tilde{\Phi}_k(q_k, x_k) \equiv \Phi_k(\theta_k \cdot q_k - x_k)$. Then the following is true.

Proposition 4 ***[Lerner-Wilson Formula under Competition]*** *For every menu $\mathcal{M}$ offered in the unique CSE equilibrium, for $k \in \{L, H\}$,*

$$\underbrace{\mathcal{E}_k(q_k, x_k)}_{\textit{retention semi-elasticity}} \cdot \underbrace{(x_k - \varphi(q_k))}_{\textit{mark-up}} + sign_k \cdot \frac{p_L}{p_k} \cdot \underbrace{\frac{\theta_L - \varphi'(q_L)}{\triangle\theta}}_{\textit{distortion}} = 1, \tag{15}$$

where $sign_k$ equals 1 (resp., -1) when $k = H$ (resp., $k = L$). Moreover, the retentions of the contracts offered to low and high-type customers coincide:

$$\tilde{\Phi}_H(q_H, x_H) = \tilde{\Phi}_L(q_L, x_L). \tag{16}$$

Equation (15) expresses the first-order conditions (10) and (11) in terms of retention semi-elasticities, mark-up's, and distortions relative to efficient quality provision. To obtain intuition, let us consider first some menu $\mathcal{M}$ for which quality provision is efficient. In this case, equation (15) takes a familiar form: it states that, at the optimum, mark-ups equal inverse semi-elasticities. It is worth noting, however, that retention semi-elasticities are endogenous in our model, as they depend on the equilibrium distribution over mechanisms followed by firms.

Let us consider next some menu $\mathcal{M}$ for which low-type customers obtain distorted qualities. In this case, incentive constraints bind, and firms have to internalize the effect that contracts designed to each type of customer have on the profits obtained from the other type. As discussed in the previous section, offering more favorable deals to high types alleviates distortions (and increases profits), while offering more favorable deals to low types magnifies distortions (and decreases profits). The second term in Equation (15) expresses the distortion effect in terms of the low-type quality q_L, customer valuations, and the probability of each customer type.

Equation (16) is an immediate result of considering ordered CSE: for each given menu, retentions are identical across types. As discussed in subsection 3.1, the only CSE is the ordered CSE when advertising costs are high. If advertising costs are low, Equation (16) holds for all menus offering buyers sufficiently low payoffs in *any* CSE (i.e., for all menus sufficiently close to the Mussa-Rosen monopoly menu).

4 Discussion

to be added

5 Conclusion

to be added

6 Appendix

This Appendix collects proofs of all results.

Proof. [Proof of Lemma 1] If the low type is offered the quality q_L, then payoffs must satisfy IC_H, i.e.,

$$u_H \geq u_L + \Delta\theta q_L. \tag{17}$$

On the other hand, IC_L requires that

$$u_L \geq u_H - \Delta\theta q_H. \tag{18}$$

The firm would like to make its offer as efficient as possible subject to the payoffs it delivers to the customer.

If $u_H - u_L < \bar{\Delta}_L \equiv \Delta\theta q_L^*$, then offering the efficient quality q_L^* for the low type is inconsistent with (17), and the firm does best to choose the highest possible value. That is, the firm chooses quality $q_L(u_L, u_H)$ which satisfies (17) with equality, or

$$q_L(u_L, u_H) \equiv \frac{u_H - u_L}{\Delta\theta}.$$

If $u_H - u_L \geq \bar{\Delta}_L$, then the constraint (17) does not bind, and the firm chooses low-type quality efficiently: $q_L(u_L, u_H) \equiv q_L^*$. Similarly, let $\bar{\Delta}_H \equiv \Delta\theta q_H^*$. If $u_H - u_L > \bar{\Delta}_H$, then asking the quality q_H^* for the high type violates (18), and so the best the firm can do is to choose $q_H(u_L, u_H)$ defined by

$$q_H(u_L, u_H) \equiv \frac{u_H - u_L}{\Delta\theta}.$$

If $u_H - u_L < \bar{\Delta}_H$, the firm offers the high-type an efficient quality: $q_H(u_L, u_H) \equiv q_H^*$. ■

Proof. [Proof of Lemma 2] From 2, $\mu = \frac{\lambda(v)}{\gamma+\lambda(v)}$. Plugging this value for μ, together with the formula for G_k in (3) and $\Pi = 0$ into (4) yields

$$\frac{cv}{\lambda(v)} = \max_{\mathcal{M}\in\mathbb{I}} \left\{ \sum_{k=H,L} p_k \cdot \frac{\gamma}{\gamma + \lambda(v) \cdot (1 - F_k(u(q_k, x_k, \theta_k)))} \cdot \frac{\theta_k q_k - \varphi(q_k) - u(q_k, x_k, \theta_k)}{\gamma + r + \lambda(v) \cdot (1 - F_k(u(q_k, x_k, \theta_k)))} \right\}.$$

Multiplying by $(\gamma + r)$ and using the mapping between contracts and payoffs given in Lemma 1 yields (7). ■

Before proceeding to the proof of Lemma 3, we consider properties of the firm's problem (7). Rather than assuming that IC_L does not bind, as in (9), in the Appendix, we define expected profits conditional on matching with a buyer more generally by

$$\begin{aligned} \pi\left(u_L, u_H\right) &= p_H \Phi_H\left(u_H\right)\left(S_H(u_L, u_H) - u_H\right) \\ &\quad + p_L \Phi_L\left(u_L\right)\left(S_L(u_L, u_H)\right) - u_L\right). \end{aligned} \tag{19}$$

In particular, we permit $S_H(u_L, u_H) < S_H^*$. The next result simply states that $\pi(\cdot, \cdot)$ exhibits strictly increasing differences, provided that at least one of the constraints IC_L and IC_H binds.

Lemma 4 *Consider any two contracts (u_L^1, u_H^1) and (u_L^2, u_H^2), where $u_L^2 > u_L^1 \geq 0$ and $u_H^2 > u_H^1 \geq 0$. Then*

$$\pi\left(u_L^2, u_H^2\right) - \pi\left(u_L^2, u_H^1\right) \geq \pi\left(u_L^1, u_H^2\right) - \pi\left(u_L^1, u_H^1\right). \tag{20}$$

If, in addition, $u_H^1 - u_L^1 \notin [q_L^* \Delta\theta, q_H^* \Delta\theta]$ or $u_H^2 - u_L^2 \notin [q_L^* \Delta\theta, q_H^* \Delta\theta]$, then the inequality in (20) is strict.

Proof. Note that

$$\begin{aligned} \pi\left(u_L, u_H^2\right) - \pi\left(u_L, u_H^1\right) &= p_L \Phi_L\left(u_L\right)\left(S_L\left(u_L, u_H^2\right) - S_L\left(u_L, u_H^1\right)\right) \\ &\quad + p_H\left(\Phi_H\left(u_H^2\right) - \Phi_H\left(u_H^1\right)\right)\left(S_H\left(u_L, u_H^2\right) - u_H^2\right) \\ &\quad + p_H \Phi_H\left(u_H^1\right)\begin{pmatrix} S_H\left(u_L, u_H^2\right) - u_H^2 \\ -\left(S_H\left(u_L, u_H^1\right) - u_H^1\right) \end{pmatrix}. \end{aligned}$$

The first term is strictly increasing in case $u_H^1 - u_L < q_L^* \Delta\theta$, by strict concavity of $q_L \theta_L - \varphi(q_L)$, and non-decreasing otherwise. The second term is nondecreasing. This follows from noting that either $S_H\left(u_L, u_H^2\right) = S_H^*$ or $u_H^2 - u_L > q_H^* \Delta\theta$. In the latter case, $S_H\left(u_L, u_H^2\right)$ is increasing in u_L, since $q_H\left(u_L, u_H^2\right)$ is then decreasing in u_L, while $q_H\left(u_L, u_H^2\right) > q_H^*$. The third term is strictly increasing in u_L in case $u_H^2 - u_L > q_H^* \Delta\theta$ by strict concavity of $q_H \theta_H - \varphi(q_H)$, and is constant in u_L otherwise. ■

Lemma 5 *Consider two optimal contracts (u_L^1, u_H^1) and (u_L^2, u_H^2). i) If $u_H^1 - u_L^1 \geq \Delta\theta q_H^*$ then $u_H^2 > u_H^1$ implies $u_L^2 \geq u_L^1$. ii) If $u_H^1 - u_L^1 \leq \Delta\theta q_L^*$ then $u_H^2 > u_H^1$ implies $u_L^2 \geq u_L^1$.*

Proof. This follows immediately from Lemma 4. For Case (i), suppose that $u_H^1 - u_L^1 \geq \Delta\theta q_H^*$ and $u_H^2 > u_H^1$ but that $u_L^2 < u_L^1$. Then $u_H^2 - u_L^1 > \Delta\theta q_H^*$, and, by (20)

$$\pi\left(u_L^1, u_H^2\right) - \pi\left(u_L^1, u_H^1\right) > \pi\left(u_L^2, u_H^2\right) - \pi\left(u_L^2, u_H^1\right)$$

or

$$\pi\left(u_L^1, u_H^1\right) + \pi\left(u_L^2, u_H^2\right) < \pi\left(u_L^1, u_H^2\right) + \pi\left(u_L^2, u_H^1\right).$$

That (u_L^1, u_H^1) and (u_L^2, u_H^2) are optimal contracts implies $\pi\left(u_L^1, u_H^1\right) = \pi\left(u_L^2, u_H^2\right)$. Thus we must have either $\pi\left(u_L^1, u_H^1\right) < \pi\left(u_L^1, u_H^2\right)$ or $\pi\left(u_L^1, u_H^1\right) < \pi\left(u_L^2, u_H^1\right)$, which contradicts optimality of $\left(u_L^1, u_H^1\right)$. ■

Proof. [Proof of Lemma 3] **Step 1. Positive mass of firms making offers.** We first show that a positive mass of firms make offers in equilibrium, i.e. $v > 0$. Suppose to the contrary that $v = 0$. Then the expected payoff for a firm conditional on matching with a buyer and offering menu $\left(u_L^M, u_H^M\right) = \left(0, u_H^M\right)$ is $\frac{\pi^M}{\gamma+r}$, where $\pi^M = p_L S_L\left(0, u_H^M\right) + p_H\left(S_H\left(0, u_H^M\right) - u_H^M\right)$. Thus, the expected payoff of a firm posting a vacancy (as given by (7)) is

$$\Pi = -\frac{c}{r} + \lim_{v \searrow 0} \frac{\lambda(v)}{vr}\frac{\pi^M}{\gamma + r}.$$

That $\lim_{v\searrow 0}\frac{vc}{\lambda(v)} < \frac{\pi^M}{\gamma+r}$ implies $\Pi > 0$. Hence, each firm strictly prefers to deviate to making an offer; i.e., we cannot have $v = 0$ in any CSE.

Step 2. No mass points in the distribution of high-type offers.

Next, we show that F_H has no mass points. Assume towards a contradiction there is an atom of firms offering $\tilde{u}_H$.

We first show that, if a firm makes an equilibrium offer $(\tilde{u}_L, \tilde{u}_H)$, for some value $\tilde{u}_L$, then $S_H(\tilde{u}_L, \tilde{u}_H) - \tilde{u}_H > 0$. Suppose not. Then it must be that $S_L(\tilde{u}_L, \tilde{u}_H) - \tilde{u}_L \le 0$ (in case both $S_H(\tilde{u}_L, \tilde{u}_H) - \tilde{u}_H \le 0$ and $S_L(\tilde{u}_L, \tilde{u}_H) - \tilde{u}_L > 0$, offering only the option designed for the low type improve the seller's expected profit since the flow profits are then $S_L(\tilde{u}_L, \tilde{u}_H) - \tilde{u}_L > 0$ also for the high type, while the probability of the high type accepting this offer is positive). Hence, $\pi(\tilde{u}_L, \tilde{u}_H) \le 0$. This contradicts seller optimization, since $\pi\left(0, u_H^M\right) > 0$: this follows because $S_H(0, u_H^M) - u_H^M > 0$ and $S_L(0, u_H^M) \ge 0$, and because the menu $\left(0, u_H^M\right)$ is accepted with positive probability by high types — it is accepted, at the least, by unmatched high types.

Next, notice that $S_L(\tilde{u}_L, \tilde{u}_H) - \tilde{u}_L \ge 0$. If not, the seller can profit by offering the menu $(q_L, x_L) = (0, 0)$ and $(q_H, x_H) = (q_H^*, \theta_H q_H^* - \tilde{u}_H)$. Irrespective of whether the low type finds it incentive compatible to choose the option $(0, 0)$, the seller is guaranteed an expected profit at least as high as under the original menu.

These two observations imply that $\pi(\tilde{u}_L + \varepsilon, \tilde{u}_H + \varepsilon) > \pi(\tilde{u}_L, \tilde{u}_H)$ for $\varepsilon > 0$ sufficiently small, contradicting the optimality of $(\tilde{u}_L, \tilde{u}_H)$. To see this, note that $\pi(\tilde{u}_L + \varepsilon, \tilde{u}_H + \varepsilon)$ must be bounded below by

$$\pi(\tilde{u}_L, \tilde{u}_H) - \varepsilon\left[p_H \Phi_H(\tilde{u}_H + \varepsilon) + p_L \Phi_L(\tilde{u}_L + \varepsilon)\right]$$
$$+ p_H\left(S_H(\tilde{u}_L, \tilde{u}_H) - \tilde{u}_H - \varepsilon\right)\left[\Phi_H(\tilde{u}_H + \varepsilon) - \Phi_H(\tilde{u}_H)\right],$$

with π given by (19). Since $\Phi_H(\tilde{u}_H + \varepsilon) - \Phi_H(\tilde{u}_H)$ is bounded above zero as $\varepsilon \searrow 0$, and since $S_H(\tilde{u}_L, \tilde{u}_H) - \tilde{u}_H > 0$, this must be greater than $\pi(\tilde{u}_L, \tilde{u}_H)$ whenever ε is sufficiently small.

Step 3. No mass points in the distribution of low-type offers for any $u_L > 0$. Next, we show that there can be no mass point in F_L above zero. Suppose towards a contradiction that F_L has a mass point at some $\tilde{u}_L > 0$. Take a firm that offers $(\tilde{u}_L, \tilde{u}_H)$.

Since, as reasoned above, $S_L(\tilde{u}_L, \tilde{u}_H) - \tilde{u}_L \geq 0$, we can consider two cases.

Case 1: $S_L(\tilde{u}_L, \tilde{u}_H) - \tilde{u}_L > 0$.

As noted in Step 1, the expected profit conditional on matching with a high type must also be positive. Notice that in this case $\pi(\tilde{u}_L + \varepsilon, \tilde{u}_H + \varepsilon)$ is bounded below by

$$\begin{aligned} &\pi(\tilde{u}_L, \tilde{u}_H) - \varepsilon\left[p_H \Phi_H(\tilde{u}_H + \varepsilon) + p_L \Phi_L(\tilde{u}_L + \varepsilon)\right] \\ &+ p_L\left(S_L(\tilde{u}_L, \tilde{u}_H) - \tilde{u}_L - \varepsilon\right)\left(\Phi_L(\tilde{u}_L + \varepsilon) - \Phi_L(\tilde{u}_L)\right). \end{aligned}$$

Since Φ_L has a mass point at $\tilde{u}_L$, and since $S_L(\tilde{u}_L, \tilde{u}_H) - \tilde{u}_L > 0$, this must be strictly greater than $\pi(\tilde{u}_L, \tilde{u}_H)$ for $\varepsilon > 0$ but sufficiently small. Therefore, $\pi(\tilde{u}_L + \varepsilon, \tilde{u}_H + \varepsilon) > \pi(\tilde{u}_L, \tilde{u}_H)$, contradicting the optimality of $(\tilde{u}_L, \tilde{u}_H)$.

Case 2: $S_L(\tilde{u}_L, \tilde{u}_H) - \tilde{u}_L = 0$.

Let $\{(q_L, x_L), (q_H, x_H)\} = \{(q_L(\tilde{u}_L, \tilde{u}_H), x_L(\tilde{u}_L, \tilde{u}_H)), (q_H(\tilde{u}_L, \tilde{u}_H), x_H(\tilde{u}_L, \tilde{u}_H))\}$ be the menu offered by the firm. Consider a deviation to the menu $\{(q_L, x_L + \varepsilon), (q_H, x_H)\}$ for some small $\varepsilon > 0$. Since (by assumption) $\tilde{u}_L > 0$, provided that $\varepsilon < \tilde{u}_L$, the low-type buyer accepts the offer at least when he is unmatched. Whether the low-type buyer chooses the option $(q_L, x_L + \varepsilon)$ or (q_H, x_H), the seller's expected profit increases strictly. For the case where the buyer selects (q_H, x_H), this follows because $S_H(\tilde{u}_L, \tilde{u}_H) - \tilde{u}_H > 0$, as in the argument of Step 1.

Step 4: The supports Υ_k are bounded below by u_k^M, for each k. Note that the support Υ_L must be nowhere below $u_L^M = 0$, since buyers reject any offer which delivers a negative flow payoff. Now consider why the support Υ_H must be bounded below by u_H^M. If a seller offers $(\tilde{u}_L, \tilde{u}_H)$ with $\tilde{u}_L \geq 0$ and $\tilde{u}_H < u_H^M$, then $\pi\left(\tilde{u}_L, u_H^M\right) > \pi(\tilde{u}_L, \tilde{u}_H)$, since $S_H\left(\tilde{u}_L, u_H^M\right) - u_H^M > S_H(\tilde{u}_L, \tilde{u}_H) - \tilde{u}_H$, and since $\Phi_H(\cdot)$ is weakly increasing with $\Phi_H\left(u_H^M\right) > 0$.

Step 5. The supports Υ_k are intervals. Suppose for a contradiction that one or both of the supports are disconnected sets. We treat only the possibility where Υ_L is disconnected, as the proof for Υ_H is similar. We suppose for simplicity that payoffs on the boundary of the support are offered in equilibrium — the argument can easily be extended to the other case. Hence there exist payoffs u_L', u_L'' with $u_L' < u_L''$ which are on the boundary of the support, and $(u_L', u_L'') \cap \Upsilon_L = \emptyset$. We are considering the case where there exists some optimal menu (u_L'', u_H''). For any $\varepsilon \in (0, u_L'' - u_L')$, optimality requires $\pi(u_L'' - \varepsilon, u_H'') \leq \pi(u_L'', u_H'')$. This implies that $q_H(u_L'', u_H'') > q_H^*$, i.e. IC_L binds, so that, by Lemma 4, $u_H' \leq u_H''$. If $u_H' < u_H''$, then, for $\eta \in (0, \min\{u_H'' - u_H', u_L'' - u_L'\})$,

$\pi(u''_L - \eta, u''_H - \eta) > \pi(u''_L, u''_H)$ contradicting the optimality of (u''_L, u''_H). If $u'_H = u''_H$, then, because $\theta_H q - \varphi(q)$ is strictly concave in q, that $\pi(u''_L - \varepsilon, u''_H) \leq \pi(u''_L, u''_H)$ for all ε implies that $\pi(u'_L + \eta, u'_H) > \pi(u'_L, u'_H)$ for some $\eta > 0$ sufficiently small. This contradicts the optimality of (u'_L, u'_H).

Step 6. The minimum of the supports Υ_L and Υ_H are, respectively, $u_L^M = 0$ and $u_H^M > 0$. Suppose not, and consider the three possible cases. We again consider for simplicity the case where the minimum of the support corresponds to the payoff for a menu which is offered in equilibrium. We let $\underline{u}_L = \min \Upsilon_L$ and $\underline{u}_H = \min \Upsilon_H$.

Case 1. Suppose $\underline{u}_L > 0$ and $\underline{u}_H > u_H^M$. Let $\varepsilon = \min\{\underline{u}_L, \underline{u}_H - u_H^M\}$. Then, for $\alpha > 0$ but sufficiently small, $\pi(\underline{u}_L, \underline{u}_H) < \pi(\underline{u}_L - \varepsilon, \underline{u}_H - \varepsilon)$. This contradicts the optimality of the menu $(\underline{u}_L, \underline{u}_H)$.

Case 2. Suppose $\underline{u}_L > 0$ and $\underline{u}_H = u_H^M$. Let $(\underline{u}_L, \check{u}_H)$ be an optimal offer. There are two possibilities. If IC_H binds, so that $q_L(\underline{u}_L, \check{u}_H) < q_L^*$, then, by Lemma 4 (and our assumption that $\underline{u}_H$ is a payoff offered in equilibrium), $(\underline{u}_L, \underline{u}_H)$ is an optimal offer. However, $\pi(0, \underline{u}_H) > \pi(\underline{u}_L, \underline{u}_H)$, a contradiction. Otherwise, there exists $\varepsilon > 0$ sufficiently small that $\pi(\underline{u}_L - \varepsilon, \check{u}_H) > \pi(\underline{u}_L, \check{u}_H)$ (which follows because $\Phi_L(\underline{u}_L - \varepsilon) = \Phi_L(\underline{u}_L)$, while $S_L(\underline{u}_L - \varepsilon, \check{u}_H) - (\underline{u}_L - \varepsilon) > S_L(\underline{u}_L, \check{u}_H) - \underline{u}_L$), contradicting the optimality of $(\underline{u}_L, \check{u}_H)$.

Case 3. Suppose $\underline{u}_L = 0$ and $\underline{u}_H > u_H^M$. Let $(\check{u}_L, \underline{u}_H)$ be an optimal offer. If IC_H or IC_L binds, then $(0, \underline{u}_H)$ is an optimal offer. However, $\pi(0, \underline{u}_H) < \pi(0, u_H^M)$, a contradiction. Otherwise, neither constraint binds while $\pi(\check{u}_L, \underline{u}_H - \varepsilon) > \pi(\check{u}_L, \underline{u}_H)$ for some $\varepsilon > 0$ sufficiently small. This contradicts the optimality of $(\check{u}_L, \underline{u}_H)$.

Step 7. No mass points in the distribution of low-type offers at $u_L = 0$. Suppose to the contrary that F_L has a mass point at zero. Then there exists $\varepsilon > 0$ such that $\pi(0, u_H^M) < \pi(\varepsilon, u_H^M)$. This follows because (i) the increase in Φ_L from $\Phi_L(0)$ to $\Phi_L(\varepsilon)$ is bounded above zero, (ii) $q_L^M > 0$, so that $S_L(0, u_H^M) > 0$, and (iii) $S_L(\varepsilon, u_H^M) - \varepsilon \to S_L(0, u_H^M)$ and $S_H(\varepsilon, u_H^M) - u_H^M = S_H^* - u_H^M = S_H(0, u_H^M) - u_H^M$. ■

Proof. [Proof of Proposition 1] **Step 1. Construction of an equilibrium.**

Step 1.1. The differential equation. We begin with necessary conditions for an equilibrium, assuming that $\Phi_H(\cdot)$ and $\Phi_L(\cdot)$ are absolutely continuous with a support function $\hat{u}_L(\cdot)$ continuous and differentiable a.e.. For each $u_H \in \Upsilon_H$,

$$\begin{aligned} & \pi(\hat{u}_L(u_H), u_H) \\ = \; & \Phi_H(u_H)(p_L(S_L(\hat{u}_L(u_H), u_H) - \hat{u}_L(u_H)) + p_H(S_H^* - u_H)) = k \end{aligned}$$

for some $k > 0$. Differentiating w.r.t. u_H yields

$$\begin{aligned} &\Phi_H'(u_H)\left(\left(S_L\left(\hat{u}_L\left(u_H\right),u_H\right)-\hat{u}_L\left(u_H\right)\right)+p_H\left(S_H^*-u_H\right)\right) \\ &+\Phi_H\left(u_H\right)\begin{pmatrix} \hat{u}_L'\left(u_H\right)p_L\left(-1+\frac{\partial S_L(\hat{u}_L(u_H),u_H)}{\partial u_L}\right) \\ +p_L\frac{\partial S_L(\hat{u}_L(u_H),u_H)}{\partial u_H}-p_H \end{pmatrix} = 0. \end{aligned}$$

Equivalently,

$$\frac{\Phi_H'(u_H)}{\Phi_H\left(u_H\right)}=\frac{-\left(\hat{u}_L'\left(u_H\right)p_L\left(-1+\frac{\partial S_L(\hat{u}_L(u_H),u_H)}{\partial u_L}\right)+p_L\frac{\partial S_L(\hat{u}_L(u_H),u_H)}{\partial u_H}-p_H\right)}{p_L\left(S_L\left(\hat{u}_L\left(u_H\right),u_H\right)-\hat{u}_L\left(u_H\right)\right)+p_H\left(S_H^*-u_H\right)}.$$

The first-order condition (10) is

$$\frac{\Phi_H'(u_H)}{\Phi_H(u_H)}=\frac{p_H-p_L\frac{\partial S_L}{\partial u_H}\left(\hat{u}_L\left(u_H\right),u_H\right)}{p_H\cdot\left(S_H^*-u_H\right)}.$$

Setting the right-hand sides equal and rearranging yields

$$\hat{u}_L'\left(u_H\right)=h\left(\hat{u}_L\left(u_H\right),u_H\right)$$

where

$$h\left(u_L,u_H\right)=\frac{S_L\left(u_L,u_H\right)-u_L}{S_H^*-u_H}\cdot\frac{1-\frac{p_L}{p_H}\frac{\partial S_L}{\partial u_H}\left(u_L,u_H\right)}{1-\frac{\partial S_L}{\partial u_L}\left(u_L,u_H\right)}, \tag{21}$$

i.e. (12).

Step 1.2. Existence and properties of solution to ODE. For any $\varepsilon\in(0,S_H^*)$, the function $h(\cdot,\cdot)$ is Lipschitz continuous on $\Lambda(\varepsilon)\equiv\left\{(u_L,u_H)\in[0,S_L^*]\times[u_H^M,S_H^*-\varepsilon):u_L<u_H\right\}$. Hence, by the Picard-Lindelöf theorem, for any $\varepsilon\in(0,S_H^*)$, and for any (u_L,u_H) in the interior of $\Lambda(\varepsilon)$, there is a unique local solution to $\hat{u}_L'(u_H)=h(\hat{u}_L(u_H),u_H)$.

Now consider $\hat{u}_L'(u_H)=h(\hat{u}_L(u_H),u_H)$ with initial condition $\hat{u}_L\left(u_H^M\right)=0$ and note the existence of $\eta>0$ such that a unique solution exists on $\left[u_H^M,u_H^M+\eta\right]$ where $(\hat{u}_L(u_H),u_H)$ remains in $\Lambda(0)$.

We show that $h(\hat{u}_L(u_H),u_H)$ remains bounded and that $(\hat{u}_L(u_H),u_H)$ remains in $\Lambda(0)$ as u_H increases to S_H^*, implying the existence of a global solution to $\hat{u}_L'(u_H)=h(\hat{u}_L(u_H),u_H)$ on $[u_H^M,S_H^*)$. To do this, we first establish the existence of a value $\tilde{u}_H\in\left(u_H^M,S_H^*\right)$ such that $\tilde{u}_H-\hat{u}_L(\tilde{u}_H)=q_L^*\Delta\theta$ and such that $\hat{u}_L(u_H)$ remains in $(u_H,S_L^*]$ on $[u_H^M,\tilde{u}_H]$.

To see this, note that, provided that $u_H-\hat{u}_L(u_H)$ remains below $q_L^*\Delta\theta$, then $h(\hat{u}_L(u_H),u_H)$ remains in $[0,1)$. That $h(\hat{u}_L(u_H),u_H)$ remains below 1 follows because

$$\begin{aligned} & S_H^*-u_H-\left(S_L\left(\hat{u}_L\left(u_H\right),u_H\right)-\hat{u}_L\left(u_H\right)\right) \\ = \ & \theta_H q_H^*-\varphi\left(q_H^*\right)-\left(\theta_L q_L\left(\hat{u}_L\left(u_H\right),u_H\right)-\varphi\left(q_L\left(\hat{u}_L\left(u_H\right),u_H\right)\right)\right)-q_L\left(\hat{u}_L\left(u_H\right),u_H\right)\Delta\theta \\ = \ & \theta_H q_H^*-\varphi\left(q_H^*\right)-\left(\theta_H q_L\left(\hat{u}_L\left(u_H\right),u_H\right)-\varphi\left(q_L\left(\hat{u}_L\left(u_H\right),u_H\right)\right)\right) \\ > \ & 0 \end{aligned} \tag{22}$$

whenever $u_H - \hat{u}_L(u_H) < q_L^* \Delta\theta$. That it remains non-negative follows from two observations. First, if $S_L(\hat{u}_L(u_H), u_H) = \hat{u}_L(u_H)$ for some $u_H \in [u_H^M, S_H^*)$ such that $u_H - \hat{u}_L(u_H) < q_L^* \Delta\theta$, then $\frac{d}{du_H}[S_L(\hat{u}_L(u_H), u_H) - \hat{u}_L(u_H)] > 0$. Second, $\frac{p_L}{p_H}\frac{\partial S_L}{\partial u_H}(\hat{u}_L(u_H), u_H) < 1$ provided $u_H - \hat{u}_L(u_H) \geq u_H^M$, which is guaranteed by the initial condition and that $h(\hat{u}_L(u_H), u_H)$ remains less than 1.

Now, suppose with a view to contradiction that there is no value $\tilde{u}_H \in (u_H^M, S_H^*)$. Because $q_L(\hat{u}_L(u_H), u_H) \leq q_L^* < q_H^*$, (22) implies that $\hat{u}_L(u_H)$ must remain bounded below $S_L(\hat{u}_L(u_H), u_H)$ as $u_H \nearrow S_H^*$. This contradicts that $h(\hat{u}_L(u_H), u_H)$ remains below 1.

Next, consider extending the solution to $u_H \in (\tilde{u}_H, S_H^*)$. It is easily checked that $\hat{u}_L(u_H) = S_L^* - \alpha(S_H^* - u_H)$ with $\alpha = \frac{S_L^* - \hat{u}_L(\tilde{u}_H)}{S_H^* - \tilde{u}_H}$satisfies $\hat{u}_L'(u_H) = h(\hat{u}_L(u_H), u_H)$ and remains in $\Lambda(0)$. That $\hat{u}_L(u_H)$ remains below u_H follows because $\alpha < 1$, since

$$
\begin{aligned}
S_L^* - \hat{u}_L(\tilde{u}_H) &= S_H^* - q_H^* \cdot \triangle\theta - \hat{u}_L(\tilde{u}_H) \\
&= q_H^* \cdot \theta_L - \varphi(q_H^*) - \hat{u}_L(\tilde{u}_H) \\
&< q_L^* \cdot \theta_L - \varphi(q_L^*) - \hat{u}_L(\tilde{u}_H) \\
&= S_L^* - \hat{u}_L(\tilde{u}_H).
\end{aligned}
$$

Step 1.3. Equilibrium mass of advertisements and distribution of offers. Consider the first equality in (13), i.e.

$$
\begin{aligned}
&\frac{\gamma}{\gamma+\lambda(v)}\frac{\gamma+r}{\gamma+r+\lambda(v)}\left\{p_H\left(S_H(0,u_H^M)-u_H^M\right)+p_L S_L(0,u_H^M)\right\} \\
=\ &\frac{cv}{\lambda(v)}(\gamma+r).
\end{aligned} \tag{23}
$$

By assumption on c, the limit of the right-hand side as $v \searrow 0$ is less than the limit of the right-hand side. The right-hand side is weakly increasing in v (since $\lambda(\cdot)$ is weakly concave) while the left-hand side is strictly decreasing and converges to zero as $v \to +\infty$ by our assumption that $\lambda(\cdot)$ is strictly increasing and onto all of $\mathbb{R}_+$. Hence, there exists a unique value for the mass of advertising that satisfies the indifference condition in Proposition 1. The distribution function F_H (and hence F_L) can be determined from the differential equations (10) via Φ_H after using that $\Phi_L(\hat{u}_L(u_H)) = \Phi_H(u_H)$. Equivalently, we can solve

$$
\begin{aligned}
&\frac{\gamma}{\gamma+\lambda(v)}\frac{\gamma+r}{\gamma+r+\lambda(v)}\left\{p_H\left(S_H(0,u_H^M)-u_H^M\right)+p_L S_L(0,u_H^M)\right\} \\
=\ &\frac{\gamma}{\gamma+\lambda(v)\cdot(1-F_H(u_H))}\cdot\frac{\gamma+r}{\gamma+r+\lambda(v)\cdot(1-F_H(u_H))} \\
&\cdot\left\{p_H\left(S_H(\hat{u}_L(u_H),u_H)-u_H\right)+p_L\left(S_L(\hat{u}_L(u_H),u_H)-\hat{u}_L(u_H)\right)\right\}
\end{aligned}
$$

for $F_H(u_H)$ which is strictly increasing in u_H from $F_H(u_H^M) = 0$. We define $\bar{u}_H = \max \Upsilon_H$ to be the top of the support Υ_H, with $\bar{u}_L = \hat{u}_L(\bar{u}_H) = \max \Upsilon_L$. Since $F_H(\cdot)$ is continuous and increasing with range $[0,1]$, it is a c.d.f.. We define $F_L(\cdot)$ by $F_L(\hat{u}_L(u_H))$ for all $u_H \in \Upsilon_H$, which is similarly a c.d.f.. From hereon, we denote these distributions by F_L^e and F_H^e, with the support function $\hat{u}_L^e$ (this defines the distribution over payoffs $\tilde{F}^e$, which has support on $\{(u_L, u_H) : u_L = \hat{u}_L(u_H), u_H \in \Upsilon_H\}$).

Step 1.4. Verification of equilibrium. Now consider the equilibrium defined by F_L^e, F_H^e (with supports Υ_L^e and Υ_H^e) and $\hat{u}_L^e$. For any menu $(\hat{u}_L^e(u'_H), u'_H)$ with $u'_H \in \Upsilon_H^e$, $\pi(\hat{u}_L^e(u'_H), u'_H) \geq \pi(u_L, u_H)$ for all $(u_L, u_H) \in \Upsilon_L \times \Upsilon_H$. This follows because $\pi(\hat{u}_L(\cdot), \cdot)$ is constant on Υ_H and by Lemma 4. In addition, profits must be less for any menu (u_L, u_H) with $u_L < u_L^M = 0$ or $u_H < u_H^M$ as argued in Step 4 of the proof of Lemma 3. Finally, note that a higher profit cannot be obtained by choosing $u_L > \max \Upsilon_L^e$ or $u_H > \max \Upsilon_H^e$. This is clear from comparison to the menu $(\max \Upsilon_L^e, \max \Upsilon_H^e)$. Indeed, $\Phi_H(u_H) = \Phi_H(\max \Upsilon_H^e)$ and $\Phi_L(u_L) = \Phi_L(\max \Upsilon_L^e)$ while $S_L(u_L, u_H) - u_L \leq S_L(\max \Upsilon_L^e, \max \Upsilon_H^e) - \max \Upsilon_L^e$, and $S_H(u_L, u_H) - u_H \leq S_H(\max \Upsilon_L^e, \max \Upsilon_H^e) - \max \Upsilon_H^e$.

Step 2. Equilibrium uniqueness.

Step 2.1. Equilibrium measure of advertisements. In any equilibrium, the measure of vacancies advertised is equal to the unique value v identified as a solution to the equality (23). Indeed, this must be the case for the expected payoff from advertising to be zero; i.e., $\Pi = 0$ and for the monopoly menu $\left(0, u_H^M\right)$ to lie on the support of the equilibrium menus.

Step 2.2. Agreement of any distribution of offers. Now consider any equilibrium with the mass of advertisements given in the previous step, and let $\tilde{F}^{\#}$ be the distribution over the menus offered in this equilibrium. We show that this distribution must agree with that in the equilibrium $\tilde{F}^e$ constructed above on the complement of $\{(u_L, u_H) : u_L > \hat{u}_L^e(\tilde{u}_H), u_H > \tilde{u}_H\}$, where $\tilde{u}_H$ is the threshold identified in Step 1.2 such that $u_H - \hat{u}_L^e(u_H) \geq \Delta\theta q_L^*$ for $u_H \geq \tilde{u}_H$. In particular, we snow that if $A \subset \Re^2$ is (Borel) measurable then

$$\begin{aligned}
\tilde{F}^{\#}\left(A \cap [0, \hat{u}_L^e(\tilde{u}_H)] \times [u_H^M, \infty)\right) &= \tilde{F}^e\left(A \cap [0, \hat{u}_L^e(u_H)] \times [\underline{u}_H, \infty)\right); \text{ and} \\
\tilde{F}^{\#}\left(A \cap [0, \infty) \times [u_H^M, \tilde{u}_H]\right) &= \tilde{F}^e\left(A \cap [0, \infty) \times [\underline{u}_H, \tilde{u}_H]\right).
\end{aligned}$$

We show this by the following steps.

Step 2.2.1. $\tilde{F}^{\#}$ and $\tilde{F}^e$ agree on a subset.

Take $\varepsilon > 0$ such that $q_L(0, \underline{u}_H + \varepsilon) < q_L^*$. Define $u_L^\alpha(\underline{u}_H + \varepsilon)$ such that a menu $(u_L^\alpha(\underline{u}_H + \varepsilon), \underline{u}_H + \varepsilon)$ is optimal in the equilibrium with offer distribution $\tilde{F}^{\#}$ (that such a value $u_L^\alpha(\underline{u}_H + \varepsilon)$ must exist follows from Lemma 3).

Step 2.2.1.a. The marginal $F_H^{\#}$ is Lipschitz continuous on $[\underline{u}_H, \underline{u}_H + \varepsilon]$.

The marginal distribution $F_H^{\#}$ over high-type payoffs offered in the equilibrium with distribution $\tilde{F}^{\#}$ is Lipschitz continuous. Take any $\nu > 0$ and u_H such that $u_H, u_H + \nu \in (\underline{u}_H, \underline{u}_H + \varepsilon)$ and take an optimal menu offering (u_L, u_H). Notice that since (u_L, u_H) is optimal we have $\pi^{\#}(u_L, u_H + \nu) \leq \pi^{\#}(u_L, u_H)$ or

$$\begin{aligned}
&\Phi_L^{\#}(u_L)\left[S_L(u_L, u_H + \nu) - u_L\right] + \Phi_H^{\#}(u_H + \nu)\left[S_H^* - u_H - \nu\right] \\
&\leq \Phi_L^{\#}(u_L)\left[S_L(u_L, u_H) - u_L\right] + \Phi_H^{\#}(u_H)\left[S_H^* - u_H\right],
\end{aligned}$$

and hence, since $S_L(u_L, u_H + \nu) > S_L(u_L, u_H)$, we have

$$\Phi_H^{\#}(u_H + \nu)\left[S_H^* - u_H - \nu\right] \leq \Phi_H^{\#}(u_H)\left[S_H^* - u_H\right],$$

which implies

$$\left(\frac{\Phi_H^{\#}(u_H + \nu) - \Phi_H^{\#}(u_H)}{v}\right) \leq \left(\frac{\Phi_H^{\#}(\underline{u}_H + \varepsilon)}{S_H^* - \underline{u}_H - \varepsilon}\right),$$

proving the claim for $\Phi_H^{\#}$. Straightforward algebra, and the definition of $\Phi_H^{\#}$ shows that F_H is Lipschitz continuous.

Step 2.2.1.b. The marginal $F_L^{\#}$ is Lipschitz continuous on $[0, u_L^{\alpha}(\underline{u}_H + \varepsilon)]$.

The proof is analogous to the proof of Step 2.2.1.a. and is omitted.

Step 2.2.1.c. $F_L^{\#}(u_L) = F_H^{\#}(u_H)$ for almost all (u_L, u_H) offered contracts such that $u_H \leq \underline{u}_H + \varepsilon$.

This argument follows from the fact that the profit function has strictly increasing differences in (u_L, u_H) when the constraint (**??**) binds (see Lemma 4).

Step 2.2.1.d. Agreement of $\tilde{F}^{\#} = \tilde{F}^e$ on the stated set:

$$\begin{aligned}
\tilde{F}^{\#}\left(A \cap [0, u_L^{\alpha}(\underline{u}_H + \varepsilon)] \times [\underline{u}_H, \infty)\right) &= \tilde{F}^{e}\left(A \cap [0, u_L^{\alpha}(\underline{u}_H + \varepsilon)] \times [\underline{u}_H, \infty)\right) \\
\text{and } \tilde{F}^{\#}\left(A \cap [0, \infty) \times [\underline{u}_H, \underline{u}_H + \varepsilon]\right) &= \tilde{F}^{e}\left(A \cap [0, \infty) \times [\underline{u}_H, \underline{u}_H + \varepsilon]\right)
\end{aligned}$$

From Steps 2.2.1.a. and 2.2.1.b. the marginals $F_L^{\#}$ and $F_H^{\#}$ must be absolutely continuous. Hence, so too are the functions $\Phi_L^{\#}(\cdot)$ and $\Phi_H^{\#}(\cdot)$. It follows that any equilibrium distribution $\tilde{F}^{\#}$ must be such that all menus satisfy the differential equations (10) and (11) almost everywhere. Hence, the distribution must be defined by the unique solution to the ordinary differential equation described above (see Step 1.1 above).

Step 2.2.2 Extending Step 2.2.1 to the complement of $\{(u_L, u_H) : u_L > \hat{u}_L(\tilde{u}_H), u_H > \tilde{u}_H\}$.

In step 2.2.1 we showed that there is $\varepsilon > 0$ such that we have uniqueness over the complement of

$$\{(u_L, u_H) : u_L > u_L^{\alpha}(\underline{u}_H + \varepsilon), u_H > \underline{u}_H + \varepsilon\}.$$

We claim that we can make the same argument for all $\varepsilon < \tilde{u}_H - \underline{u}_H$. Suppose that the claim is false and let $\varepsilon^{\sup} \in (0, \tilde{u}_H - \underline{u}_H)$ denote the supremum of values for which the claim is true. Then $q_L(\hat{u}_L^e(\underline{u}_H + \varepsilon^{\sup}), \underline{u}_H + \varepsilon^{\sup}) < q_L^*$. Thus, from the same argument as in Step 2.2.1 we can find $\eta > 0$ such that

$$\begin{aligned}
\tilde{F}^{\#}(A \cap [0, u_L^{\alpha}(\underline{u}_H + \varepsilon^{\sup} + \eta)] \times [\underline{u}_H, \infty)) &= \tilde{F}^e(A \cap [0, u_L^{\alpha}(\underline{u}_H + \varepsilon^{\sup} + \eta)] \times [\underline{u}_H, \infty)) \\
\text{and } \tilde{F}^{\#}(A \cap [0, \infty) \times [\underline{u}_H, \underline{u}_H + \varepsilon^{\sup} + \eta]) &= \tilde{F}^e(A \cap [0, \infty) \times [\underline{u}_H, \underline{u}_H + \varepsilon^{\sup} + \eta]),
\end{aligned}$$

contradicting the definition of $\varepsilon^{\sup}$.

Step 2.3. Distributions over the set $\{(u_L, u_H) : u_L > \hat{u}_L(\tilde{u}_H), u_H > \tilde{u}_H\}$. Consider the equilibrium that we constructed. There are two possibilities. Either $\max \Upsilon_H^e \leq \tilde{u}_H$, in which case the equilibrium is unique (i.e., there is only one CSE), or $\max \Upsilon_H^e > \tilde{u}_H$ in which case CSE are not unique. However, we can show that if $\max \Upsilon_H^e > \tilde{u}_H$, then $q_L(u_L, u_H) = q_L^*$ and $q_H(u_L, u_H) = q_H^*$ for all offers with $u_H > \tilde{u}_H$. Once we restrict attention to ordered CSE, this follows from arguments closely related to those above. In particular, we can confirm that the unique ordered CSE coincides with the solution described in the proposition. ■

Proof. [Proof of Corollary 1] Consider the differential equation $\hat{u}_L'(u_H) = h(\hat{u}_L(u_H), u_H)$ with h given by (21). Notice that $u_H - \hat{u}_L(u_H)$ is strictly increasing at $u_H = u_H^M$, since $\frac{p_L}{p_H}\frac{\partial S_L}{\partial u_H}(0, u_H^M) = 1$. This implies that, for $\varepsilon > 0$ sufficiently small, $h(\hat{u}_L(u_H), u_H) > 0$ for all $u_H \in (u_H^M, u_H^M + \varepsilon)$. Therefore, if the claim is not true, there must be some first value $u_H^{\#} \in (u_H^M, \tilde{u}_H)$ such that $h\left(\hat{u}_L\left(u_H^{\#}\right), u_H^{\#}\right) = 0$. However, we must then have both $h(\hat{u}_L(u_H), u_H)$ and $\hat{u}_L'(u_H)$ converge to zero (while remaining positive) as $u_H \nearrow u_H^{\#}$. Thus, while both $1 - \frac{p_L}{p_H}\frac{\partial S_L}{\partial u_H}(u_L, u_H)$ and $S_L(u_L, \hat{u}_L(u_H)) - \hat{u}_L(u_H)$ are positive for $u_H \in \left(u_H^M, u_H^{\#}\right)$, they are also strictly increasing for all sufficiently large $u_H < u_H^{\#}$. This contradicts $h\left(\hat{u}_L\left(u_H^{\#}\right), u_H^{\#}\right) = 0$. ■

Proof. [Proof of Remark 3] See Lemma 4. ■

Proof. [Proof of Corollary 2] See Step 1.2 in the proof of Proposition 1. ■

Proof. [Proof of Proposition 2] To come. ■

Proof. [Proof of Proposition 3] Part 1 is obvious. For Part 2, note that we must have, for all $u_H \in [u_H^M, \bar{u}_H]$,

$$\begin{aligned}
& \Phi_H(u_H^M)\left(p_L\left(S_L\left(0, u_H^M\right)\right) + p_H\left(S_H^* - u_H^M\right)\right) \\
= \; & \Phi_H(u_H)\left(p_L\left(S_L(\hat{u}_L(u_H), u_H) - \hat{u}_L(u_H)\right) + p_H\left(S_H^* - u_H\right)\right).
\end{aligned}$$

Differentiating with respect to u_H yields

$$\begin{aligned} & \Phi_H'(u_H)\left(p_L\left(S_L\left(\hat{u}_L(u_H),u_H\right)-\hat{u}_L(u_H)\right)+p_H\left(S_H^*-u_H\right)\right) \\ +\Phi(u_H)\left(p_L\left(\frac{\partial S_L\left(\hat{u}_L(u_H),u_H\right)}{u_L}\hat{u}_L'(u_H)+\frac{\partial S_L\left(\hat{u}_L(u_H),u_H\right)}{u_H}-\hat{u}_L'(u_H)\right)-p_H\right) &= 0, \end{aligned}$$

or

$$\frac{\Phi_H'(u_H)}{\Phi(u_H)}=\frac{p_L\left(\frac{\partial S_L(\hat{u}_L(u_H),u_H)}{\partial u_H}\left(\hat{u}_L'(u_H)-1\right)-\hat{u}_L'(u_H)\right)+p_H}{p_L\left(S_L\left(\hat{u}_L(u_H),u_H\right)-\hat{u}_L(u_H)\right)+p_H\left(S_H^*-u_H\right)}. \tag{24}$$

Next note that, by definition of Φ_H,

$$\frac{\Phi_H'(u_H)}{\Phi(u_H)}=f_H(u_H)\frac{\lambda\left(\gamma+r+\lambda\left(1-F_H(u_H)\right)\right)+\lambda\left(\gamma+\lambda\left(1-F_H(u_H)\right)\right)}{\left(\gamma+\lambda\left(1-F_H(u_H)\right)\right)\left(\gamma+r+\lambda\left(1-F_H(u_H)\right)\right)}. \tag{25}$$

Take a sequence of equilibria with entry costs $\{c_n\}$ such that $c_n\to 0$. Clearly, the respective measure of firms $v_n\to\infty$ and hence $\lambda_n\equiv\lambda(v_n)$ increases to infinite. From (24) and (25) we have:

$$\begin{aligned} & f_{Hn}(u_H) \\ =& \left[\begin{array}{c}\left(\frac{\lambda_n(\gamma+r+\lambda_n(1-F_{Hn}(u_H)))+\lambda_n(\gamma+\lambda(1-F_{Hn}(u_H)))}{(\gamma+\lambda_n(1-F_{Hn}(u_H)))(\gamma+r+\lambda_n(1-F_{Hn}(u_H)))}\right)^{-1} \\ \times\frac{p_L\left(\frac{\partial S_L(\hat{u}_L(u_H),u_H)}{\partial u_H}\left(\hat{u}_L'(u_H)-1\right)-\hat{u}_L'(u_H)\right)+p_H}{p_L(S_L(\hat{u}_L(u_H),u_H)-\hat{u}_L(u_H))+p_H\left(S_H^*-u_H\right)}\end{array}\right]. \end{aligned} \tag{26}$$

As the RHS of (26) is uniformly bounded we conclude that $\{F_{Hn}\}$ is equicontinuous and hence by Arzela Ascoli Theorem F_{Hn} has a subsequence converging to F_0 (with density f_0). Using (26) and straightforward algebra we conclude that:

$$\frac{2f_{H0}(u_H)}{1-F_{H0}(u_H)}=\frac{p_L\left(\frac{\partial S_L(\hat{u}_L(u_H),u_H)}{\partial u_H}\left(\hat{u}_L'(u_H)-1\right)-\hat{u}_L'(u_H)\right)+p_H}{p_L\left(S_L\left(\hat{u}_L(u_H),u_H\right)-\hat{u}_L(u_H)\right)+p_H\left(S_H^*-u_H\right)}.$$

■

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