Communication in Repeated Network Games with Private Monitoring*

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May 16, 2012

Abstract

I consider repeated games with private monitoring played on a social network. Each player has a set of neighbors with whom he interacts: a player’s payoff depends on his own and his neighbors’ actions only. Monitoring is private and imperfect: each player observes his stage payoff but not his neighbors’ actions. I introduce costless communication among players at each stage: communication can be public, private or a mixture of both. I prove that a folk theorem holds for a large class of payoff functions if and only if any two players have a non-common neighbor.

Keywords: communication, folk theorem, imperfect private monitoring, networks, repeated games.

JEL codes: C72, C73

*I deeply thank my advisor Tristan Tomala for his invaluable guidance and constant encouragement. I also thank Olivier Compte, Olivier Gossner, Johannes Hörner, Frédéric Koessler, Tomasz Michalski, Guillaume Vigeral, Yannick Viossat and seminar audiences at Ecole Polytechnique, Paris Game Theory Seminar in Institut Poincaré, Paris School of Economics and the 22nd Stony Brook Game Theory Festival.
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1 Introduction

Consider a competition with local interaction: only a few other firms—its nearest competitors—have a direct impact on a firm’s profit. However, this does not rule out that, step by step, a firm’s conduct affects the pattern of behavior on the entire industry. In line with the assumption that interactions are local, I assume that monitoring also is local: a firm observes its own profit only. In particular, firms might lack incentives or the expertise to monitor even direct competitors. In this setup, firms may not be able to identify possible deviators, even among neighbors. This issue stands in the way of collusion since it may be impossible for a firm to punish several competitors (see Example 3.1). I introduce costless communication among firms. The nature of communication is free: firms can send private emails to each other, or communicate publicly with a subset of players (Carbon Copy). The Carbon Copy list of players is certifiable. (Players could also use Blind Carbon Copy, that is a mixture of private and public communication.) On one hand, the introduction of communication might help reaching collusion by spreading information. On the other hand, communication also enlarges the set of possible deviations. In particular, with private emails (i.e. firms can send different messages to different competitors at each stage) as assumed here, a firm may send spurious messages to a subset of competitors or may send different messages to different competitors. Both situations might stand in the way of coordination. For instance, during a punishment phase, a punisher might coordinate secretly with the punished player. The possibility of collusion crucially depends on the geometry of the network, since both interaction and monitoring revolve around it. This paper answers the following question: for which networks is collusion sustainable? The main result is that in this setup, cooperation is achievable if and only if any two firms have a non-common neighbor (Condition C).

Such a market is modeled as a repeated game played on a network: nodes represent firms, and edges link direct competitors. Each player has a set of neighbors with whom he interacts: his payoff depends on his own and his neighbors’ actions only. Monitoring is private and local in that players observe their stage payoff only. Hence, both interaction and monitoring revolve around the network. In addition, players can send costless messages at each stage and communication can be public, private or a mixture of both. This paper circumscribes the networks for which a full folk theorem holds in this setup, i.e. under which conditions all feasible, strictly individually rational payoffs are equilibrium payoffs in the repeated game with low discounting. For a wide class of payoff functions, I construct a sequential equilibrium strategy for the family of networks that satisfy Condition C. (In fact, I consider a refinement of sequential equilibrium, namely belief-free equilibrium.) Condition C also is necessary: if Condition C is not satisfied, then the folk theorem does not hold, even
with private or public communication.

The key to my characterization lies in understanding when communication makes it possible to (i) transmit precise information about players’ deviations (detection and identification) and (ii) coordinate players’ behavior. Detection of neighbors’ deviations is solved by considering a large class of payoff functions. I use a refinement of sequential equilibrium, namely belief-free equilibrium ([7]), which requires sequential rationality for all private beliefs. To be clear, the proofs would be the same whether I would consider sequential equilibria ([18]) since the specification of the beliefs is useless here.

I assume that payoff functions are such that any unilateral change of a player’s action affects each neighbor’s payoff. Hence, neighbors’ deviations are detectable, although deviators may not be identifiable. Such payoffs arise naturally in Bertrand or Cournot oligopolies in which a competitor’s decisions have an impact on a given firm’s profit. For this wide class of payoff functions, the condition on the networks’ topology for a folk theorem to hold is that any two players must have at least one non-common neighbor (Condition C). It has a simple interpretation. Assume that player $i$ detects a neighbor’s deviation and wants to punish the deviator. According to player $i$, the deviator could either be his neighbor $j$ of $k$. The condition states that there must be another player $\ell$ who (without loss of generality) is a neighbor of $j$ but not of $k$. If indeed player $j$ is the deviator, then player $\ell$ can confirm this to player $i$, since he also detected a unilateral deviation; whereas if player $k$ is the deviator, then $\ell$ can also inform player $i$ of this because he did not detect a deviation. Yet, player $\ell$ could deviate by not reporting his information. I handle this issue by requiring deviating players to confess their deviation afterwards.\(^1\) Under unilateral deviations, if player $\ell$ does not report his information truthfully, then the initial deviating player, either $j$ or $k$, should confess to player $i$.

Condition C does not rule out networks for which a player, say $k$, has a single neighbor, say $\ell$. In that case, player $\ell$ might have an incentive not to report player $k$’s deviations for which player $\ell$ is the unique monitor. However, with the same argument as before which relies on unilateral deviations, if player $\ell$ does not report player $k$’s deviation, then player $k$ confesses. Nevertheless, it might be impossible for players other than $k$ and $\ell$ to distinguish between the two following histories: “player $k$ deviates in action at stage $t$ and his unique monitor $\ell$ truthfully reports the deviation” and “player $k$ does not deviate at stage $t$ whereas player $\ell$ lies when reporting player $k$’s deviation”. However, it is then possible to punish both players $k$ and $\ell$ since no player is a neighbor of both of them: while being minmaxed, I require player $k$ (respectively $\ell$) to minmax player $\ell$ (respectively $k$). Regarding point (ii), public

\(^1\)Confessing an old deviation is a weakly dominated strategy. However, it is possible to construct rewards in order to give players incentive to confess after a deviation (see Section 7).
communication serves the purpose of coordinating players' actions. In particular, minmax strategies might be mixed, and pure actions are not monitored even by neighbors. This stands in the way of detecting deviations during punishment phases as well as providing incentives (rewards) for the minmaxing players to randomize according to the right distribution of their minmax strategies. To tackle this problem, players announce the pure actions they actually play during the punishment phase. Therefore, players can detect their neighbors' deviations during the punishment phase.

Condition C is also necessary for a folk theorem to hold. Indeed, if two players, say $j$ and $k$, have the same neighbors, it is possible to construct a particular payoff function for which a common neighbor of both, say player $i$, cannot differentiate between players $j$ and $k$'s deviations. In addition, player $i$ is unable to punish both: intuitively, player $i$ rewards player $k$ (respectively $j$) when he punishes player $j$ (respectively player $k$). This is a failure of joint rationality in the terminology of Renault and Tomala ([26]).

Condition C is satisfied in many environments with product differentiation. For vertical differentiation, consider a market in which a firm sells a luxury product and may not be competing with a firm producing a low quality product. However, both might be competing with a firm selling a product of intermediate quality. For horizontal differentiation, a good example is Hotelling market model ([15]) in which firms compete with the closest rivals. The networks $G_1$ and $G_3$ displayed in Figure 5, which both satisfy Condition C, depict such environments.

**Application.** An application of interest is a game of partnership (see [24]). Consider a partnership in which production is conducted by a team whose goal is to maintain a certain level of individual effort among its members. Each member's effort is not observable and there is moral hazard issue (effort is costly). However, each member's wage depends on the effort of a subset of members only, called the direct colleagues. For instance, the head of a subteam's remuneration may depend on his own's, his direct subordinates' and on his own chief's levels of effort: this defines a subteam and the remuneration of each of its members depends on the results of the subteam. In addition, agents may communicate with each other via emails, either privately or publicly. If a member is denounced by his direct colleagues, the group can punish him by reducing his share in the total profit, which rises other members' shares. This paper shows that coordination is sustainable if and only if any two members have a non-common colleague. In particular, the following network structures ($G_1$ and $G_2$) prevent a folk theorem to hold.
Also, no complete network satisfies Condition C, which entails that remuneration must not depend on the results of the whole firm in order to enable coordination. However, if there is no hierarchy but the members form a circle (with at least five members, network $G_3$ in the figure below), then coordination is supportable. Moreover, the tree structure depicted by the network $G_4$ also enables a folk theorem to hold.

Related literature and contributions. This paper lies at the juncture of two independent literatures: repeated games and social networks. Regarding repeated games, the folk theorem was originally established for Nash equilibria ([1, 10, 27, 28]) and extended by Fudenberg and Maskin ([10, 11]) to subgame-perfect equilibria. A key assumption is perfect monitoring. Lots of papers on folk theorems with imperfect monitoring have focused on imperfect public information (see [9]). The model of collusion by Green and Porter ([14]) also considers imperfect public monitoring in that the market price serves as a commonly observable signal. In the undiscounted case, Lehrer (see for instance [20, 21]) provides a fairly comprehensive study of the equilibrium payoff set for two-player games with imperfect private monitoring. With more than two players, the difficulty relies on the necessity for players to coordinate their behavior to punish potential deviators. Under discounting, as assumed here, much less is known. There is a large recent literature on imperfect private monitoring and belief-free equilibria (see [22] for a general survey), but these papers consider different monitoring structures than here, for instance they assume full support of
the signal distributions. Fudenberg and Levine ([8]) establish a folk theorem with imperfect private monitoring without explicit communication. They consider private random signals induced by the action profile of all other players. With public communication, Compte ([5]) and Kandori and Matsushima ([17]) provide sufficient conditions for a folk theorem to hold. Both consider probabilistic signals, which is not the case here. Yet, their assumptions do not apply here: in particular, they assume that the distributions of private signals given the action profile have the same supports. Besides, both [5] and [17] provide sufficient conditions only for a folk theorem to hold, whereas Condition C is also necessary in my setup. Finally, cooperation can be obtained with equilibria that are simple than in [5] and [17] for the private monitoring setup I consider. Closer to my setting, Ben-Porath and Kahneman ([2]) establish a folk theorem for the case in which (i) each player observes his neighbors’ moves, and (ii) the assumption of public communication is maintained. Renault and Tomala ([25]) and Tomala ([30]) studied repeated games with the same signals as in [2] (i.e. each player observes his neighbors’ moves) but communication is constrained by the network structure and they do not impose sequential rationality. All these papers ([2, 25, 30]) assume that monitoring among neighbors is perfect. To the contrary, I assume here that it is imperfect: payoffs encapsulate all an agent’s feedback about his neighbors’ play (for instance, firms infer rivals’ likely behavior from their own profits). Besides, interaction is global in all these papers, i.e. each player interacts with all other players. Here, interactions revolve around a network. Moreover, because of Condition C, I prove that the folk theorem even fails if the network is complete (see Example 4.2). Recently, Nava and Piccionne ([23]) and Cho ([4]) study games in networks with local interaction. Yet, both assume that each player perfectly observes his neighbors’ moves.

This paper is also related to the literature on social and economic networks (for an overview of the networks literature, see Goyal, [13], and Jackson, [16]). Networks in which a player’s payoff depends on his own and his neighbor’s actions have been studied by Galeotti and al. ([12]) and Bramoullé and Kranton ([3]). However, this literature does not account for repeated games in general.

The paper is organized as follows. The model is introduced in Section 2. In Section 3, I discuss the assumption on payoff functions. The main result is displayed in Section 4. Section 5 is devoted to the construction of an equilibrium strategy, provided Condition C on networks is satisfied: this proves that this condition is sufficient for a folk theorem to hold. Section 6 proves that the folk theorem fails if Condition C is not satisfied. Finally, Section 7 develops some extensions and raises open questions.
2 The setup

Consider a repeated game played on a fixed network where players interact with their neighbors only. This is described by the following data.

- A finite set $N = \{1, \ldots, n\}$ of players ($n \geq 3$).

- For each player $i \in N$, a non-empty finite set $A^i$ of his actions (with $|A^i| \geq 2$).

- An undirected graph $G = (N, E)$ in which the vertices are the players and $E \subseteq N \times N$ is a set of links. Let $\mathcal{N}(i) = \{j \neq i : ij \in E\}$ be the set of player $i$’s neighbors. Since $G$ is undirected, the following holds: $i \in \mathcal{N}(j) \iff j \in \mathcal{N}(i)$.

- Finally, each player $i \in N$ has a payoff function of the form $g^i : \prod_{j \in \mathcal{N}(i) \cup \{i\}} A^j \to \mathbb{R}$, i.e. player $i$’s stage payoff depends on his own and his neighbors’ actions only.

I use the following notations: $A = \prod_{i \in N} A^i$, $N^{-i} = N \setminus \{i\}$, $A^{N(i) \cup \{i\}} = \prod_{j \in \mathcal{N}(i) \cup \{i\}} A^j$, $a^{N(i)} = (a^j)_{j \in \mathcal{N}(i)}$ and $g = (g^1, \ldots, g^n)$ denotes the payoff vector.

In addition, I introduce costless communication. Players are able to communicate both privately and publicly. First, each player can send different messages to distinct players. Second, players can make public announcements to all players or to a subset of players only. For instance, if a player $i$ makes a public announcement to a subset $S$ of players, then the list $S$ is certifiable, that is: each player $s$ in $S$ knows that all members in $S$ received the same message, although he does not know the messages received by players who are not in $S$. Let $M^i$ be a non-empty finite set of player $i$’s messages. Let $m^i_t(j)$ represent the private message sent by player $i$ to player $j \in N$ at stage $t$ and $m^i_t(S)$ the public message sent by $i$ at stage $t$ to players in $S \subseteq N$ (hence, $m^i_t(N)$ is a public announcement to all players). The specification of the set $M^i$ is part of the solution and is described in Section 5.

The repeated game unfolds as follows. At every stage $t \in \mathbb{N}^*$:

(i) simultaneously, players choose actions in their action sets and send messages to all players, either publicly or privately as described above.

(ii) Let $a_t = (a^i_t)$ be the action profile at stage $t$. At the end of stage $t$, each player $i \in N$ observes his stage payoff $g^i(a^i_t, a^{N(i)}_t)$. A player cannot observe the actions chosen by others, even by his neighbors.

Hence, both interaction and monitoring revolve around the network $G$. In addition, I assume perfect recall and that the whole description of the game is common knowledge. For each

\[2\text{The two-player case comes down to perfect monitoring, see Section 3.}\]
stage $t$, denote by $H^i_t$ the set of player $i$’s private histories up to stage $t$, that is $H^i_t = (A^i \times (M^i)^{N-i} \times (M^j)_{j \in N-i} \times \{g^i\})^t$, where $\{g^i\}$ is the range of $g^i$ ($H^i_0$ is a singleton). An element of $h^i_t$ is called an $i$-history of length $t$. An action strategy for player $i$ is denoted by $\sigma^i = (\sigma^i_t)_{t \geq 1}$ where for each stage $t$, $\sigma^i_t$ is a mapping from $H^i_{t-1}$ to $\Delta(A^i)$ (where $\Delta(A^i)$ denotes the set of probabilities distributions over $A^i$). A communication strategy for player $i$ is denoted by $\phi^i = (\phi^i_t)_{t \geq 1}$ where for each stage $t$, $\phi^i_t$ is a mapping from $H^i_{t-1}$ to $\Delta((M^i)^{N-i})$. Each player can deviate from $\sigma^i$ or from $\phi^i$, henceforth I shall distinguish between action and communication deviations accordingly. I call a behavior strategy of a player $i$ the pair $(\sigma^i, \phi^i)$. Let $\Sigma^i$ be the set of player $i$’s action strategies and $\Phi^i$ his set of communication strategies. I denote by $\sigma = (\sigma^i)_{i \in N} \in \prod_{i \in N} \Sigma^i$ the players’ joint action strategy and by $\phi = (\phi^i)_{i \in N} \in \prod_{i \in N} \Phi^i$ their joint communication strategy. Let $H_t$ be the set of histories of length $t$ that consists of the sequences of actions, payoffs and messages for $t$ stages. Let $H_{\infty}$ be the set of all possible infinite histories. A profile $(\sigma, \phi)$ defines a probability distribution, $P_{\sigma, \phi}$, over the set of plays $H_{\infty}$, and I denote $E_{\sigma, \phi}$ the corresponding expectation. I consider the infinitely discounted repeated game, in which the overall payoff function of each player $i$ in $N$ is the expected normalized sum of discounted payoffs. That is, for each player $i$ in $N$:

$$
\gamma^i_B(\sigma, \phi) = E_{\sigma, \phi} \left[ (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} g^i_t(a^i_t, d^N_t) \right],
$$

where $\delta \in [0, 1)$ is a common discount factor.

I consider a refinement of sequential equilibrium, namely belief-free equilibrium. For that, I denote by $(\sigma^i, \phi^i)_{|h^i_t}$ player $i$’s continuation strategy after private history $h^i_t$. Let also $(\sigma^{-i}, \phi^{-i})_{|h^{-i}_t}$ be the profile of continuation strategies of players $j \neq i$ after private histories $h^{-i}_t$. The solution concept I use is the following.

**Definition 2.1.** Belief-free equilibrium (see [7])

A joint behavior strategy $(\sigma, \phi)$ is a belief-free equilibrium if $\forall h_t, \forall i \in N$,

$$(\sigma^i, \phi^i)_{|h^i_t} \in BR((\sigma^{-i}, \phi^{-i})_{|h^{-i}_t}).$$

A belief-free equilibrium payoff is an element $(\gamma^1(\sigma, \phi), \ldots, \gamma^n(\sigma, \phi))$ in $\mathbb{R}^n$, where $(\sigma, \phi)$ is a belief-free equilibrium.

**Remark 2.2.** Belief-free equilibrium refines the concept of sequential equilibrium introduced by Kreps and Wilson ([18]). Belief-free equilibria allow for sequential rationality for all private beliefs. In other words, a sequential equilibrium is a belief-free strategy profile if it has the additional property that a player’s continuation strategy is still the player’s best response
when he secretly learns about his opponents’ private histories. The main advantages of belief-free equilibria are: (i) robustness with respect to private beliefs, (ii) easiness to define, and (iii) tractability.

Remark 2.3. With complete information as assumed here, there always exists a belief-free equilibrium, since any sequence of static equilibrium profiles that is independent of the history of play is a belief-free equilibrium.

Let $\Gamma_\delta(G, g)$ be the $\delta$-discounted game and $E_\delta(G, g)$ its associated set of belief-free equilibrium payoffs. For each $a \in A$, let $g(a) = (g^1(a^1, a^{N(1)}), \ldots, g^n(a^n, a^{N(n)}))$ and $g(A) = \{g(a) : a \in A\}$. Let $co\ g(A)$ be the convex hull of $g(A)$, which is the set of feasible payoffs. Player $i$’s (independent) minmax level is defined by:

$$p^i = \min_{x^{N(i)} \in \prod_{j \in N(i)} \Delta(A_j)} \max_{x^i \in \Delta(A_i)} g^i(x^i, x^{N(i)}).$$

Henceforth, I shall normalize the payoffs of the game such that $(p^1, \ldots, p^n) = (0, \ldots, 0)$. I denote by $IR^*(G, g) = \{g = (g^1, \ldots, g^n) \in \mathbb{R}^n : \forall i \in N, g^i > 0\}$ the set of strictly individually rational payoffs. Finally, let $V^* = co\ g(A) \cap IR^*(G, g)$ be the set of feasible and strictly individually rational payoffs.

The aim of this paper is to characterize the networks $G$ for which a folk theorem holds, that is: each feasible and strictly individually rational payoff is a belief-free equilibrium payoff of the repeated game for a large enough discount factor. In the next section, I display and motivate the class of payoff functions I consider.

Remark 2.4. Henceforth, only connected networks are taken into account. Indeed, interaction is local, hence players in distinct connected components do not interact with each other. This makes communication useless between players in different connected components. Therefore, I model different connected components as different games.

3 A class of payoff functions

We first show that a necessary condition for a folk theorem is that the payoff functions are sufficiently rich to enable players to detect deviations.

Example 3.1. Consider the 4-player game played on the following network:

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3It is possible here to drive equilibrium payoffs below this bound, see Section 7.
The action sets of the players are $A^1 = A^2 = A^4 = \{0, 1\}$ and $A^3 = \{-1, 0, 1\}$. Payoff functions are the following:

\[
\begin{align*}
g^1(a^1, a^2) &= 3a^2 - a^1 \\
g^2(a^1, a^2, a^3) &= 2(a^1 + a^2 + a^3) \\
g^3(a^2, a^3, a^4) &= 3 - a^3 - a^2 - 3(1 - a^4)a^2 \\
g^4(a^3, a^4) &= \begin{cases} 
4 & \text{if } a^3 \geq 0 \\
0 & \text{otherwise.}
\end{cases}
\end{align*}
\]

Each player’s minmax level is $p^i = 0$, $i \in \{1, 4\}$. Therefore, the payoff vector $(2, 6, 1, 4)$, only obtained when all players choose action 1, is feasible and strictly individually rational. Notice also that $(2, 6, 1, 4)$ is an extreme point of $V^*$ and that $\text{int} V^* \neq \emptyset$. However, $(2, 6, 1, 4)$ is not a Nash equilibrium payoff of the repeated game: for any discount factor $\delta \in [0, 1)$, either player 1 has an incentive to deviate by playing $a^1 = 0$ and gets a payoff of 3, or player 3 has an incentive to deviate by playing $a^3 = 0$ to get a payoff of 2. The key argument is that both deviations induce the same signals for player 2. Indeed, both deviations first induce a decrease of 2 in player 2’s payoffs. In addition, player 4 has no useful information: first, he does not detect any of these deviations since he is not a neighbor of player 1, and in addition, player 4’s payoff remains the same whether player 3 chooses $a^3 = 1$ or $a^3 = 0$. Therefore, although player 2 detects when player 1 or player 3 plays 0 instead of 1, he is unable to learn who the deviator is. Moreover, it is not possible for player 2 to punish both players 1 and 3. On the one hand, player 2 has to play $a^2 = 0$ to minmax player 1, in which case player 3 gets a payoff of 3. On the other hand, player 2 has to choose $a^2 = 1$ to minmax player 3, in which case player 1 gets a payoff of 3. Hence, player 2 rewards player 3 (respectively player 1) when he punishes player 1 (respectively player 3). So, either player 1’s or player 3’s deviation is profitable, which contradicts the fact that $(2, 6, 1, 4)$ is a Nash equilibrium payoff of the repeated game.\footnote{For a formal proof in a general setup, see Section 6.} One can also check that payoffs are bounded away from $(2, 6, 1, 4)$.

On the other hand, let $g^1$, $g^2$ and $g^3$ remain unchanged and assume that player 4’s payoff

\[\text{Figure 3:} \quad \begin{array}{cccc}
1 & 2 & 3 & 4
\end{array}\]
is the following:

\[ g^4(a^3, a^4) = \begin{cases} 
4 & \text{if } a^3 = 1 \\
4 + \epsilon & \text{if } a^3 = 0 \\
0 & \text{if } a^3 = -1 
\end{cases} \]

with \( \epsilon > 0 \). Player 4 is now able to detect player 3’s deviations. Therefore, if player 2 detects either player 1’s or player 3’s deviation, he can obtain from player 4 the name of the deviator (player 4 clears either player 3 if he does not detect any deviation, or player 1 otherwise). For a large enough discount factor, it is possible to construct a belief-free equilibrium with payoff’s vector \((2, 6, 1, 4)\) (see Section 5 for the construction).

The previous example shows that if a deviation of a player \( i \) does not alter all his neighbors’ payoffs, then it may possible for some feasible and strictly individually rational payoffs not to be equilibrium payoffs of the repeated game\(^5\). Hence, it is not possible to get a folk theorem for all payoff functions \( g \). I introduce the following assumption.\(^6\)

**Assumption 3.2.** For each player \( i \in N \), each neighbor \( j \in \mathcal{N}(i) \), every actions \( b^j, c^j \) in \( A^j \), \( a^i \) in \( A^i \), \( a^{\mathcal{N}(i)\setminus\{j\}} \) in \( A^{\mathcal{N}(i)\setminus\{j\}} \):

\[ g^i(a^i, a^{\mathcal{N}(i)\setminus\{j\}}, b^j) \neq g^i(a^i, a^{\mathcal{N}(i)\setminus\{j\}}, c^j). \]

**Example 3.3.** The following payoff functions satisfy Assumption 3.2:

- for each player \( i \) in \( N \), let \( A^i \subset \mathbb{N} \) and \( g^i(a^i, a^{\mathcal{N}(i)}) = f\left(\sum_{j \in \mathcal{N}(i)\cup\{i\}} a^j\right) \) with \( f \) strictly monotone;

- for each player \( i \) in \( N \), let \( A^i \subset \mathbb{R} \) and \( g^i \) strictly monotone with respect to each argument;

- for each player \( i \) in \( N \), let \( A^i \subset \mathbb{R} \) and \( g^i(a^i, a^{\mathcal{N}(i)}) = \sum_{j \in \mathcal{N}(i)} a^j - a^i \) (this game can be seen as a generalized prisoner’s dilemma for \( n \) players);

- firms’ profits in Cournot and Bertrand games.

In the next section, I introduce the necessary and sufficient condition on networks for a folk theorem to hold.

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\(^5\)For a similar phenomenon, see [20].

\(^6\)This assumption entails that the two-player case comes down to perfect monitoring.
4 The main result

Theorem 4.1. The following statements are equivalent:

1. For each payoff function $g$ that satisfies Assumption 3.2, if the interior of $V^*$ is nonempty relatively to $\mathbb{R}^n$, then for any payoff $v$ in $V^*$, there exists $\delta \in (0,1)$ such that for all $\delta \in (\bar{\delta},1)$, $v$ is a belief-free equilibrium vector payoff of the $\delta$-discounted game.

2. The graph $G$ is such that, for every pair of players $i,j \in N$, $i$ and $j$ have at least one non-common neighbor, that is:

$$\forall i,j \in N, \ N(i) \setminus \{j\} \triangle N(j) \setminus \{i\} \neq \emptyset$$

(C)

where $\triangle$ stands for the symmetric difference.$^7$

If Condition C is satisfied, $g$ fulfills Assumption 3.2 and $V^*$ is of full dimension, then, for any $v \in V^*$, it is possible to construct a belief-free equilibrium with payoff $v$ for large enough discount factors. I construct this strategy in Section 5. Intuitively, Condition C makes it possible to construct a communication strategy that enables players to identify the deviator if a deviation occurs.

In addition, Condition C prevents the network from having too many links, because it may lead to less information. The idea is that too many links increase the probability for two players to have the same neighbors, which violates Condition C. If two players have the same neighbors, their deviations may not be distinguishable and I exhibit particular payoff function for which the folk theorem fails (see Section 6 for the general proof). In particular, the next example shows that the folk theorem fails for complete networks.

Example 4.2. Consider the 5-player game played on the complete network, i.e. for every player $i$ in $N$, $\mathcal{N}(i) = N \setminus \{i\}$:

Figure 4:

![Figure 4](image)

$^7$The set $\mathcal{N}(i) \setminus \{j\} \triangle \mathcal{N}(j) \setminus \{i\}$ contains the players $\ell \neq i,j$ such that $\ell$ is a neighbor of $i$ or $j$, but not of both.
Assume $A^i = \{-1, 0, 1\}$ for each player $i$ in $N$. The payoff functions are the following:

$g^i(a^i, a^{N(i)}) = g^i(a^1, \ldots, a^5) = g^i(a) = \sum_{j \in \mathcal{N}(i)} a^j - a^i$ for each player $i$ in $\{1, 2, 3\}$, and

$g^m(a) = -\sum_{j \in \mathcal{N}(m)} a^j - a^m$ for each player $m$ in $\{4, 5\}$. The minmax level is $v^i = -3$ for each player $i \in N$. Consider the case in which players coordinate on playing action 0 at each stage, which yields a payoff of 0 for each player. It is then not possible to identify the deviator if an action deviation occurs.\footnote{Notice that the players can deviate in action or in communication since they have both action and communication strategies. Henceforth, we make the distinction between action deviations and communication deviations if needed.}

For instance, it is not possible to differentiate between player 1's and player 5's deviations. Indeed, if player 1 deviates in action at some stage $t$, everybody detects the deviation because of Assumption 3.2. However, the graph being complete, each player suspects all other players and nobody can differentiate between player 1’s and player 5’s deviations in particular. In this example, an action deviation of any player is detected, but the deviator is not identified. Therefore, either player 1 or player 5 has an incentive to deviate by playing $-1$ instead of 0, and the folk theorem fails.

I now show some properties of the networks satisfying Condition C. First, I display some examples of graphs that satisfy Condition C (Figure 5) and some that do not (Figure 6). Notice that Condition C is not monotonic with respect to the number of links, contrary to connectivity. For instance, both $G_9$ and $G_{10}$ are 2-connected, whereas only $G_9$ satisfies Condition C. Also, neither $G_3$ nor $G_4$ are 2-connected, whereas only $G_3$ satisfies Condition C.
I now exhibit some properties of the networks that satisfy Condition C.

**Proposition 4.3.** Assume that the network $G$ is connected and satisfies Condition C. The following statements hold.

(i) There are at least $n = 4$ players.

(ii) If $n = 4$, then $G$ is a line (graph $G_3$ in Figure 5).

The proof of Proposition 4.3 is given in the appendix. The next proposition shows that, for the family of networks that satisfy Condition C, a player cannot be the unique neighbor of more than one player.

**Proposition 4.4.** If a connected network satisfies Condition C, then for every players $j$ and $j^1$ such that $\mathcal{N}(j^1) = \{j\}$ and every $\ell \in \mathcal{N}(j)$ so that $\ell \neq j^1$, $\sharp \mathcal{N}(\ell) \geq 2$.

**Proof.** Take any network $G$, a pair of players $j$ and $j^1$, and assume $\mathcal{N}(j^1) = \{j\}$. Consider any $\ell \in \mathcal{N}(j)$ so that $\ell \neq j^1$. If $\sharp \mathcal{N}(\ell) = 1$, then $\mathcal{N}(\ell) = \{j\}$. Therefore, $\mathcal{N}(\ell) = \mathcal{N}(j^1)$ and Condition C is violated. \hfill $\Box$
I now study the trees that satisfy Condition C. Recall that a tree is a connected graph without cycles and that the distance in a graph between two nodes $i$ and $j$ is the length of the shortest path from $i$ to $j$ (the reader is referred to [6] for usual definitions of graph theory).

**Corollary 4.5.** The following statements hold.

(i) A tree satisfies Condition C if and only if the distance between any two terminal nodes is greater than three.

(ii) No star satisfies Condition C.

Going back to Figures 5 and 6, notice that the distance between two terminal nodes in $G_7$ is at least three, whereas the distance between nodes 4 and 5 in $G_8$ (or between 6 and 7) is equal to two.

**Proof.** The proof of point (ii) is straightforward given Proposition 4.4. Regarding point (i), consider first a tree $G$ that satisfies Condition C. Proposition 4.4 directly implies that there are no terminal nodes $i$ and $j$ in $G$ such that the distance between $i$ and $j$ is 2.

For the converse, take any tree $G$ for which the distance between two terminal nodes is greater than three. Obviously, any two terminal nodes have a non-common neighbor. Take now any two nodes $i$ and $j$ in $G$ such that at least one of them is not terminal, say $i$. Any node that is not terminal has at least two neighbors, let $\mathcal{N}(i) = \{i_1, i_2\}$ without loss of generality. By definition of a tree, there is a unique path $p$ between $i$ and $j$. Therefore, if $i_1$ is in $p$ (respectively $i_2$), then $i_2$ (respectively $i_1$) is in $\mathcal{N}(i) \setminus \{j\} \triangle \mathcal{N}(j) \setminus \{i\}$. Hence, $G$ satisfies Condition C.

\[ \square \]

5 Construction of the equilibrium strategy

In this section, I assume that the network $G$ satisfies Condition C of Theorem 4.1, namely: for any pair of players $i$, $j$ in $N$, $i$ and $j$ have at least one non-common neighbor. Take a payoff function that satisfies Assumption 3.2 (this is assumed throughout this section). Henceforth, assume that $\text{int} V^*$ is non-empty. I take a point $v = (v^1, \ldots, v^n)$ in $V^*$ and I construct a belief-free equilibrium of the repeated game $(\sigma^*, \phi^*)$ with payoff $v$ for a large enough discount factor. The strategy is made of four parts: a stream of pure actions leading to the payoff $v$ and messages to be played in case of no deviation, periods of communication allowing the players to identify the deviator when there is a deviation, a punishment phase and finally a reward phase. Before constructing the equilibrium strategy, I set out two remarks.
(i) Deviations from the equilibrium strategy are twofold: action and communication deviations. Since communication is costless, communication deviations need to be punished only if they affect continuation payoffs. In the strategy I construct, communication deviations that do not affect continuation payoffs are not punished. More precisely, if a player starts sending spurious messages although no player has deviated in action, and if in addition all players learn that there was no action deviation, then the deviator is not punished. On the other hand, for some communication deviations without any action deviation, it may be impossible for some players to be aware that there was no action deviation: in this case, punishments of several players are needed. In any case, a player can deviate both in communication and in action, and is then punished (action deviations always lead to the deviator’s punishment).

(ii) Each player $i$ detects an action deviation from a pure action profile if and only if he observes a stage payoff’s change (because of Assumption 3.2). Therefore, there is an action deviation from a pure action profile of some player $k$ at stage $t$ if and only if all player $k$’s neighbors detect it stage $t$. Nevertheless, player $k$’s neighbors may not be able to identify who the deviator is.

I now construct the four phases of the strategy and then derive the equilibrium property. First of all, if a player observes a private history incompatible with unilateral deviations, he should play his coordinate of an arbitrary Nash equilibrium of the one-shot game.

5.1 Phase I: equilibrium path

For each player $i$ in $N$ and each stage $t > 0$, choose $\bar{a}_i^t \in A_i$ such that

$$(1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} g^i_t(\bar{a}_i^t, \bar{a}_{N(i)}^t) = v^i.$$ 

This is possible when $\delta \geq 1 - \frac{1}{n}$ (existence is proved by Sorin, Proposition 4 p.151 in [28]). Moreover, Fudenberg and Maskin (1991) prove that for every $\epsilon > 0$, there exists $\delta_\epsilon < 1$ such that for all $\delta \geq \delta_\epsilon$ and every $v \in V^*$ such that $v^i \geq v$ for all $i$, the deterministic sequence of pure actions $\bar{a}_t$ can be constructed so that the continuation payoffs at each stage are within $\epsilon$ of $v$ (Lemma 2 p. 432 in [11]).

During this phase, player $i$ should play action $\bar{a}_i^t$ at stage $t$. Moreover, at every period, player $i$ should announce $\bar{m}_i^t(N) = N(i) \cup \{i\}$ publicly to all players. This message means that

\[9\text{If it was not the case, some player would prefer to deviate from } \bar{a}, \text{ even if doing so caused his opponents to minmax him thereafter.}\]
player \(i\) did not deviate and did not detect any action deviation at stage \(t - 1\).\(^{10}\) According to player \(i\), his neighbors and himself are innocent regarding any possible action deviation at stage \(t - 1\). Player \(i\) then announces his set of innocents publicly to all players at stage \(t\), denoted \(I_i^t\), made of himself and his neighbors: \(I_i^t = \mathcal{N}(i) \cup \{i\}\).

### 5.2 Phase II: communication phase

This phase aims at identifying the deviator when a deviation occurs. Like in Example 3.1, it is not always possible to punish several players and the identification of the deviator is therefore needed. Nevertheless, for the family of networks that satisfy Condition C, it is not always possible to identify the deviator. Take for instance a player \(k\) who has a single neighbor \(\ell\). It may be impossible for players \(i \neq k, \ell\) to differentiate between the two following deviations:

- player \(k\) deviates in action at stage \(t\) and claims that he is innocent regarding stages \(t\) and \(t + 1\);
- player \(\ell\) deviates in communication at stage \(t + 1\) and claims that player \(k\) deviated at stage \(t\).

These deviations still need to be punished, otherwise either player \(\ell\) or player \(k\) might have an incentive to deviate. Yet, this is not an obstacle for a folk theorem to hold. Indeed, no player is neighbor of both players \(k\) and \(\ell\), hence it is feasible to minmax both players \(k\) and \(\ell\): while being minmaxed, player \(\ell\) is prescribed to minmax player \(k\). Therefore, if one of these deviations occurs, players only need to know that the deviator is either \(k\) or \(\ell\). More precisely, if \(G\) satisfies Condition C and \(g\) fulfills Assumption 3.2, the communication strategy in phase II is such that the following properties hold.

**Claim 5.1.** For every player \(j\) in \(N\) such that \(\#\mathcal{N}(j) \geq 2\), if \(j\) deviates in action at stage \(t\), then each player identifies \(j\) as deviator at stage \(t + 1\) and clears all players \(i \neq j\) at stage \(t + 1\).

**Claim 5.2.** For every pair of players \(k\) and \(\ell\) such that \(\mathcal{N}(k) = \{\ell\}\), if:\(^{11}\)

(a) either \(k\) deviates in action at stage \(t\);

\(^{10}\)Recall that, at each stage, messages are sent before observing stage payoffs. This assumption is not needed: with a slight modification, the strategy construction is still valid for the case in which messages are sent after the observation of stage payoffs.

\(^{11}\)Recall that Condition C entails that \(k\) is unique given \(\ell\) (see Proposition 4.4). Moreover, \(n \geq 3\) and \(G\) is connected, hence \(\#\mathcal{N}(\ell) \geq 2\).
(b) or \(k\) deviates in communication at stage \(t+1\) by claiming he deviated in action at stage \(t\), although there is no action deviation at stage \(t\);

(c) or \(\ell\) deviates in communication at stage \(t+1\) by making player \(k\) identified as a deviator, although there is no action deviation at stage \(t\);

then both \(k\) and \(\ell\) are identified as suspects by all players at stage \(t+1\) and all other players are cleared.

**Claim 5.3.** For any other communication deviation from (b) and (c) at stage \(t+1\) and if there is no action deviation at stage \(t\), then every player is cleared at stage \(t+1\).

From these claims, it turns out that if a player \(j\) deviates in action at some stage \(t\), then he is identified as deviator and can be punished (either he is the unique suspect, or he has a single neighbor who also is suspected, in which case both can be punished). Moreover, for any pair of players \(k\) and \(\ell\) such that \(N(k) = \{\ell\}\) and if there is no action deviation at stage \(t\), both \(k\) and \(\ell\) are identified under communication deviations (b) and (c); for all other communication deviations, everybody is cleared. For the latter case, all players know that there was no action deviation at stage \(t\), thus only a communication deviation at stage \(t+1\). That is why no player need to be punished since there is no effect on continuation payoffs. Only players \(k\)'s and \(\ell\)'s communication deviations which are indistinguishable with player \(k\)'s action deviation need to be punished.

Formally, the strategy during phase II is the following. Each player \(i \in N\) enters phase II every time he detects any kind of deviation from \((\sigma^*, \phi^*)\). For instance, when in phase I, player \(i\) enters phase II at the end of stage \(t\) either if he detects an action deviation at stage \(t\), or if he receives a public message different from \(\bar{m}_j(N) = N(j) \cup \{j\}\) from some player \(j \in N^{-i}\) at stage \(t\) (only public announcements are taken into account here). Players may enter phase II at different stages. Indeed, consider the situation in which there is an action deviation of some player \(j\) at stage \(t\). Player \(j\)'s neighbors start phase II at the end of stage \(t\), whereas other players may not start phase II before the end of stage \(t+1\) when they receive messages of player \(j\)'s neighbors.

During phase II, players should stick to the action strategy they were playing in the previous phase. For instance, if players are following the equilibrium path at stage \(t\) when they detect some deviation, they should enter the communication phase and keep playing \(\bar{a}_{t+1}\) at stage \(t+1\). This part of the strategy is thereby purely communicative.

In what follows, I construct the communication strategy used by player \(i\) in phase II and denoted \(\tilde{\phi}^i\). I then set out how identification works and how transition to another phase is made.
Communication strategy. The strategy $\tilde{\varphi}_i$ of each player $i \in N$ consists in announcing sets of innocents publicly to all players as follows:

- if player $i$ detects an action deviation at stage $t$, then he announces $\tilde{m}^{i}_{t+1}(N) \supseteq N \setminus N(i)$ at stage $t + 1$.
  This message means that player $i$ claims that all his neighbors are suspects regarding a deviation at stage $t$; or, in other words, that all other players, including himself, are innocent.

- If player $i$ deviates in action at stage $t$, then he should announce $\tilde{m}^{i}_{t+1}(N) \supseteq N^{-i}$ at stage $t + 1$. This message means player $i$ confesses at stage $t + 1$ that he deviated at stage $t$.

- Otherwise, player $i$ directly goes to the next phase using the transition rule given below.

Identification. Denote by $X^{i}_{t+1} \subset N$ player $i$’s set of suspected players at stage $t + 1$ regarding a possible deviation at stage $t$. For each player $i$ in $N$, the set $X^{i}_{t+1}$ is computed as follows.

(i) Only announcements of the form $m^{j}_{t+1}(N) = N(j) \cup \{j\}$, $m^{j}_{t+1}(N) = N \setminus N(j)$ or $m^{j}_{t+1}(N) = N^{-j}$ for each player $j \in N$ are taken into account. Other announcements are disregarded: in particular, private messages are ignored. Notice also that player $i$ takes into account his own announcement $m^{i}_{t+1}(N)$.

(ii) For every player $j$ such that $\#N(j) \geq 2$ for each $j^1 \in N(j)$:

• if there exist at least two players $j^1$ and $j^2$ such that $j^1 \neq j^2$, $j \in m^{j^1}_{t+1}(N)$ and $j \in m^{j^2}_{t+1}(N)$, then $j \notin X^{i}_{t+1}$ (i.e. $j$ is cleared);

• otherwise, $j \in X^{i}_{t+1}$ (i.e. $j$ is identified as suspect).

(iii) For every pair of players $k$ and $\ell$ such that $N(k) = \{\ell\}$:

• if there exist at least two players $k^1$ and $k^2$ such that $k^1 \neq k^2$, $k \in m^{k^1}_{t+1}(N)$ and $k \in m^{k^2}_{t+1}(N)$:
  
  – if there exist at least two players $\ell^1$ and $\ell^2$ such that $\ell^1 \neq \ell^2$, $\ell \in m^{\ell^1}_{t+1}(N)$ and $\ell \in m^{\ell^2}_{t+1}(N)$, then $\ell \notin X^{i}_{t+1}$ (i.e. $\ell$ is suspected);
  
  – otherwise, $\ell \in X^{i}_{t+1}$ (i.e. $\ell$ is identified as suspect);

• otherwise, $\{k, \ell\} \in X^{i}_{t+1}$ (i.e. both $k$ and $\ell$ are identified as suspects).

\[\text{If player } i \text{ is in phase I at stage } t, \text{ then } \tilde{m}^{i}_{t+1}(N) = N \setminus N(i). \text{ Yet, it is not always the case. In particular, when player } i \text{ is in phase III, he may send in addition other announcements (see Section 5.2).}\]
I prove in Section 5.5 that this strategy guarantees Claims 5.1, 5.2 and 5.3 to be satisfied. These properties are true whenever players are in phase I, III or IV. Formal proofs are given in Section 5.5. I now set out how transition is made from phase II to another state.

**Transition.** Denote by plan \( \{j\} \) (respectively \( \{k, \ell\} \)) the punishment phase (phase III) in which player \( j \) (respectively players \( k \) and \( \ell \)) is minmaxed. The transition rule to another phase is the following:

- if \( X_{t+1}^i = \emptyset \), then keep playing according to the current action phase and use the corresponding communication strategy;
- if \( X_{t+1}^i = \{j\} \) for some player \( j \in N \) such that \( \#N(j) \geq 2 \) and \( \#N(j^1) \geq 2 \) for each \( j^1 \in N(j) \), then start plan \( \{j\} \);
- if either \( X_{t+1}^i = \{\ell\} \) or \( X_{t+1}^i = \{k, \ell\} \), with \( N(k) = \{\ell\} \), then start plan \( \{k, \ell\} \);
- otherwise, play an arbitrary Nash equilibrium of the one-shot game (history incompatible with unilateral deviations).

**Remark 5.4.** Assume \( N(k) = \{\ell\} \). Players \( \ell \) may prefer plan \( \{\ell\} \) to plan \( \{k, \ell\} \), or the opposite. That is the reason why the transition rule prevents plan \( \{\ell\} \) to happen: if player \( \ell \) is identified as a suspect, then players start plan \( \{k, \ell\} \) in order to minmax both, no matter if player \( k \) is also suspected. Moreover, the identification is such that player \( \ell \) is always suspected when player \( k \) is, so that plan \( \{k\} \) never occurs either.

The next example shows how the strategy constructed so far works.

**Example 5.5.** Consider the 4-player game played on the following network:

```
  1 --- 2 --- 3 --- 4
```

for which Condition C is satisfied. In addition, consider \( g \) that fulfills Assumption 3.2.

Assume first that player 2 deviates in action at some stage \( t \) and, for brevity, does not deviate in communication at stage \( t \) (but possibly at stage \( t + 1 \)). Furthermore, assume this deviation to be the first, so that all players are in phase I at stage \( t \).\(^{13}\) Hence, each player \( i \), except possibly \( k \), announces publicly \( \bar{m}_t^i(N) = N(i) \cup \{i\} \) at stage \( t \). At stage \( t + 1 \), players 1 and 3 should enter phase II and stick to actions \( \bar{a}_t^1 \) and \( \bar{a}_t^3 \) respectively. In addition, strategies \( \bar{\phi}_{t+1}^1 \) and \( \bar{\phi}_{t+1}^3 \) prescribe them to announce publicly \( m_{t+1}^1(N) = N \setminus N(1) = \{1, 3, 4\} \) and \( m_{t+1}^3(N) = N \setminus N(3) = \{1, 3\} \) respectively. Player 4 starts phase II at the end of stage \( t + 1 \). Similar arguments apply if it is not the case.
should announce $m^4_{t+1}(N) = \bar{m}^4_{t+1} = \{3, 4\}$. Finally, player 2 should announce \{1, 3, 4\} publicly to all players at stage $t+1$. Under unilateral deviations, players 1, 3 and 4 appear in the public announcements at stage $t+1$ of at least two different players, hence each player $i$ clears players 1, 3 and 4. Moreover, player 2 appears in at most one public announcement, therefore each player $i$ identifies player 2 as the deviator. The transition rule prescribes players to enter phase III to minmax both players 1 and 2 (plan \{1, 2\}).

Assume now that there is no action deviation at stage $t$ and that player 2 deviates in communication at stage $t+1$ by announcing $m^2_{t+1}(N) = N \setminus N'(2) = \{2, 4\}$ publicly to all players (recall that private messages are not taken into account in the strategy constructed). Under $\phi^*$, Players 1, 3 and 4 announce respectively \{1, 2\}, \{2, 3, 4\} and \{3, 4\} publicly to all players at stage $t+1$ since there is no action deviation at stage $t$ by assumption. Hence, $X^i_t = \{1, 2\}$ for each player $i$ in $N$. Players then start plan \{1, 2\}. The reason of this joint punishment is that players 3 and 4 do not differentiate between the histories “player 1 deviates in action at stage $t$ and in communication at stage $t+1$ since he does not confess” and “player 1 deviates neither in action at stage $t$ nor in communication at stage $t+1$ and player 2 deviates in communication at stage $t+1$”. The two following cases are then possible.

- If player 2 also deviates in action at stage $t+1$. At stage $t+2$, players 1, 2, 3 and 4 should announce respectively \{1, 3, 4\}, \{1, 3, 4\}, \{1, 3\} and \{3, 4\} publicly to all players. Even if one player deviates at stage $t+2$, each player clears players 1, 3 and 4 and identifies player 2, hence $X^i_{t+2} = \{2\}$ for each player $i$. Nevertheless, players then start again plan \{1, 2\}.

- If player 2 does not deviate in action at stage $t+1$, then there is no action deviation at stage $t+1$ under unilateral deviations. Players keep playing plan \{1, 2\} until a new possible deviation.

Assume now that player 1 deviates in action and, for brevity, does not deviate in communication at stage $t$. In addition, Furthermore this deviation to be the first.\footnote{Again, similar arguments apply if it is not the case.} At stage $t+1$, players 1, 2, 3 and 4 should announce respectively \{2, 3, 4\}, \{2, 4\}, \{2, 3, 4\} and \{3, 4\} publicly to all players. Therefore, the name of player 1 appears in at most one public announcement, whereas the names of players 2, 3 and 4 are in at least two distinct players’ public announcements. Hence, $X^i_{t+1} = \{1, 2\}$ for each player $i$. Again, players 3 and 4 cannot differentiate between this deviation of player 1 and the deviation of player 2 described in the previous case. Therefore, all players start plan \{1, 2\}. Assume now that there is no action deviation at stage $t$ and that player 1 deviates in communication at stage $t+1$ by announcing $m^1_{t+1}(N) = N^{-1} = \{2, 3, 4\}$ publicly to all
players: player 1 lies when he confesses his action deviation at stage $t$. At stage $t+1$, the public announcements of players 2, 3 and 4 are respectively $\{1,2\}$, $\{2,3,4\}$ and $\{3,4\}$, hence $X^i_t = \{1,2\}$ for each player $i$ in $N$. Again, all players start plan $\{1,2\}$.

Finally, assume that there is no action deviation at stage $t$ and that player 1 deviates in communication at stage $t+1$ by announcing publicly $m^{i+1}_t(N) = N \setminus N(1) = \{1,3,4\}$. The public announcements of players 2, 3 and 4 at stage $t+1$ are respectively $\{1,2\}$, $\{2,3,4\}$ and $\{3,4\}$. All players appears in at least two messages, hence $X^i_t = \emptyset$ for each player $i$. Intuitively, each player deduces that there was no action deviation at stage $t$ and that player 1 deviates in communication at stage $t+1$. However, communication being costless, there is no need to punish player 1. The transition rule prescribes all players to keep playing according to the phase in which the game is and to use the corresponding communication strategy. ⊳

This concludes the description of the communication phase.

5.3 Phase III: punishment phase

First case: plan $\{j\}$.

Consider the situation in which each player enters a punishment phase in order to minmax player $j$. Notice first that only player $j$’s neighbors are able to punish him. Besides, since minmax strategies might be mixed, minmaxing players’ deviations might be undetectable: players may not know the sequences of pure actions their neighbors should play. For that reason, announcements are needed. Players’ strategies during phase III, denoted $\hat{\sigma}$ and $\hat{\phi}$, are as follows.

During this phase, each player’s communication strategy is twofold:

(i) first, at each stage $s$, each player $i$ in $N$ announces his set of innocents $I^i_s$ publicly to all players (as in phase I). For instance, $I^i_s = N(i) \cup \{i\}$ belongs to player $i$’s public announcement at stage $s$ if player $i$ does not detect any action deviation at stage $s-1$.

(ii) In addition, each player $i$ reveals his pure action $a^i_s$ publicly to all players at each stage $s$. With these announcements, all players know the pure actions each player should have played at each stage $s$. This enables the players to detect deviations and start phase II if needed.

Recall that players choose actions and messages simultaneously at each stage, hence player $j$ has not received his opponents’ announcements when he chooses his action and sends his messages. Therefore, player $j$ is indeed punished when minmax strategies are mixed.
A message of any player $i$ during phase III has then the following form: for each stage $s$ in phase III, $m_i^s(N) = (I_i^s, a_i^s)$.

Action strategies of the players are as follows. At each stage $s \geq t + 2$ during phase III (the length of phase III, denoted $\mu(\delta)$, is adapted in Section 5.4):

- each player $i \in {\cal N}(j)$ plays according to his minmax strategy against $j$, denoted ($P_j^i$) (recall that $P_j^i$ can be a mixed strategy). Denote by $P(j) = (P_j^i)_{i \in {\cal N}(j)}$ the profile of minmax strategies of player $j$’s neighbors against him. For any strategy $(\sigma^j, \psi^j)$ of player $j$:

$$
\gamma^j_\delta(\sigma^j, P(j), \psi^j, (\phi^i)_{i \in {\cal N}(j)}) \leq \sum_{t=1}^{\infty} (1 - \delta)\delta^{t-1}p^j \leq 0
$$

where $\phi^i$ stands for any communication strategy of each minmaxing player $i \in {\cal N}(j)$.

- Player $j$ commits himself to an arbitrary pure action $P^j$ during his punishment, hence his deviations are detectable. In particular, player $j$ is supposed to play $P^j$ even if a player $i \in {\cal N}(j)$ deviates by privately reporting to player $j$ the sequence of pure actions he will play during plan $\{j\}$. Hence, player $j$ cannot use this information to get a higher payoff. If he deviates from $P^j$, then player $j$ only lengthens his punishment (see Section 5.5).

- The actions of other players $m \neq i, j$ are arbitrary. Recall that each player $m$ should announce publicly the pure action chosen at each stage, hence his deviations are detectable.

During phase III, each player $i$ in $N$ starts the communication phase at some stage $s$ if:

- either player $i$ detects an action deviation at stage $s - 1$, i.e. his payoff at stage $s - 1$ is not the one he would get if all his neighbors would have played the pure actions they announce at stage $s - 1$;

- or, there exists $\tilde{i} \in {\cal N}(j) \cap {\cal N}(i)$ such that $a_{\tilde{i}}^{s-1}$ is not in the support of $P_{\tilde{i}}^j(k)$ (this deviation is regarded as an action deviation);\(^{15}\)

- or, there exists a player $m \in N$ such that $I_m^s \neq {\cal N}(s) \cup \{s\}$.

If player $i$ never starts phase II (or if all players are cleared), then player $i$ goes to phase IV at the end of the punishment phase, hence at stage $t + 2 + \mu(\delta)$.

\(^{15}\)Notice that one can have $i = j$.  

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Second case: plan \{k, \ell\}.

Consider now the situation in which two players \(k\) and \(\ell\), with \(\mathcal{N}(k) = \ell\), are minmaxed from stage \(t + 2\) on. It means that either player \(k\) or player \(\ell\) have been identified as suspect at the end of stage \(t + 1\).

Remark 5.6. The strategy constructed in phase II is such that, whenever the state is \(\{k, \ell\}\), it must be the case that \(\mathcal{N}(k) = \{\ell\}\) without loss of generality (see Lemmas 5.10, 5.11 and 5.12 in Section 5.5). In this case, phase III is such that both \(k\) and \(\ell\) are punished. It is possible since each player has to punish only one suspect among \(k\) or \(\ell\).

Each player’s communication strategy during phase III is the same as for the first case. However, action strategies differ and are as follows. At each stage \(s \geq t + 2\) during phase III:

- each player \(i \in \mathcal{N}(\ell)\), including player \(k\), plays according to his minmax strategy against \(\ell\), denoted \((P^i(\ell))\) (recall that \(P^i(\ell)\) can be a mixed strategy).

- Player \(\ell\) plays according to his minmax strategy against \(k\), \(P^\ell(k)\).

- The actions of other players \(m \neq i, k, \ell\) are arbitrary. As explained above, each player \(m\) should still announce publicly the pure action chosen at each stage.

Each player \(i\) in \(\mathcal{N}\) starts phase II at some stage \(s\) if:

- either player \(i\) detects an action deviation at stage \(s - 1\), i.e. his payoff at stage \(s - 1\) is not the one he would get if all his neighbors would have played the pure action they announced at stage \(s - 1\);

- or there exists \(\tilde{i} \in \mathcal{N}(\ell) \cap \mathcal{N}(i)\) such that \(a^\tilde{i}_{s-1}\) is not in the support of \(P^\tilde{i}(\ell)\);\(^{16}\)

- or \(i \in \mathcal{N}(\ell)\) and \(a^\ell_{s-1}\) is not in the support of \(P^\ell(k)\);

- or there exists a player \(m\) such that \(I^{ma}_s \neq \mathcal{N}(s) \cup \{s\}\).

If player \(i\) never starts phase II (or if all players are cleared), then player \(i\) goes to phase IV at the end of the punishment phase, hence at stage \(t + 2 + \mu(\delta)\). This concludes the description of the punishment phase.

Remark 5.7. The reward phase is such that players \(k\)’s and \(\ell\)’s continuation payoffs after the punishment phase vary according to their realized payoff during phase III (see Section 5.4). Therefore, they are indifferent between the pure actions in the support of their minmax strategies. In particular, they cannot benefit from a minmaxing player reporting privately his sequence of pure actions.

\(^{16}\)Notice that one can have \(i = \ell\) and \(\tilde{i} = k\), or the contrary.
5.4 Phase IV: reward phase

The aim of this phase is twofold.

(i) In order to provide each minmaxing player, who is not minmaxed himself, an incentive to play his minmax strategy in phase III, one need to add a reward in the form of an additional bonus $\rho > 0$ in his average payoff until a new possible deviation. If the discount factor is large enough, the loss in punishing is compensated by future bonus.

(ii) Moreover, to induce each minmaxing player to draw his pure actions according to the right distribution of his minmax strategy, I add a phase so that his payoff in the continuation game vary with his realized payoff in a way that makes him indifferent between the pure actions in the support of his minmax strategy. As in Fudenberg and Maskin ([10, 11]), any feasible and strictly individually rational continuation payoff must be exactly attained. Otherwise, a minmaxing player might not be exactly indifferent between his pure actions in the support of his minmax strategy.

The possibility of providing such rewards relies on the full dimensionality of the payoff set (recall that $\text{int} V^* \neq \emptyset$). I use the insights of Fudenberg and Maskin (1991). The formal construction of these rewards is described in what follows.

Choose $(v^1_r, \ldots, v^n_r) \in \text{int} V^*$ such that for each player $i \in N$, $v^i > v^i_r$. Since $V^*$ has full dimension, there exists $\rho > 0$ such that for each player $j$ such that $\#N(j) \geq 2$ and $\#N(j^1) \geq 2$ for every $j^1 \in N(j)$, the following holds:

$$v_r(j) = (v^1_r + \rho, \ldots, v^{j-1}_r + \rho, v^j_r, v^{j+1}_r + \rho, \ldots, v^n_r + \rho) \in V^*,$$

and for every players $k$ and $\ell$ such that $N(k) = \{\ell\}$:

$$v_r(k, \ell) = (v^1_r + \rho, \ldots, v^{k-1}_r + \rho, v^k_r, v^{k+1}_r + \rho, v^{\ell-1}_r + \rho, v^\ell_r + \frac{\rho}{2}, v^{\ell+1}_r + \rho, \ldots, v^n_r + \rho) \in V^*.$$

For each player $i$ in $N$, let $w^i(d)$ be player $i$’s realized discounted average payoff during phase III when punishing player $d$ in $N$. Denote by $\bar{g}^i$ player $i$’s greatest one-shot payoff, that is:

$$\bar{g}^i = \max_{a^i, (a^j)_{j \in N(i)}} g^i(a^i, a^j).$$

**Lemma 5.8.** For every $\epsilon > 0$ small enough, there exists $\delta^* > \delta_\epsilon^{17}$ such that for all $\delta > \delta^*$,

\[17\] Recall that $\delta_\epsilon$ is defined in Section 5.1 such that for every $\epsilon > 0$, there exists $\delta_\epsilon < 1$ such that for all $\delta \geq \delta_\epsilon$, and every $v \in V^*$ such that $v^i \geq v$ for all $i$, the deterministic sequence of pure actions $\bar{a}_t$ can be constructed so that the continuation payoffs at each stage are within $\epsilon$ of $v$. 

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there exists an integer $\mu(\delta)$\textsuperscript{18} (if several take the smallest) such that for every pair of players $i$ and $d$:

\[
(1 - \delta^2)\bar{v}_r^i + \delta^{\mu(\delta) + 3} v_r^i < v_r^i - \epsilon, \quad (1) \\
(1 - \delta^2)\bar{v}_r^i + \delta^{\mu(\delta) + 3} v_r^i < (1 - \delta^{\mu(\delta) + 2}) w^i(d) + \delta^{\mu(\delta) + 2}(v_r^i + \rho), \quad (2) \\
(1 - \delta^2)\bar{v}_r^i + \delta^{\mu(\delta) + 3} v_r^i < (1 - \delta^2) w^i(d) + \delta^2(v_r^i + \rho). \quad (3)
\]

Proof. Choose $\epsilon > 0$ such that, for every players $i$ and $d$, $\epsilon < \min_i v_r^i$ and

\[
-w^i(d) < \frac{v_r^i - \epsilon}{v_r^i} (\rho - w^i(d)). \quad (4)
\]

This is possible since $\rho > 0$. By Equation (4), $\delta^*$ exists. Indeed, I can rewrite Equation (4) as follows:

\[
0 < \left( \frac{v_r^i - \epsilon}{v_r^i} \right) \rho - \left( 1 - \frac{v_r^i - \epsilon}{v_r^i} \right) w^i(d).
\]

Hence, it is possible to find $x \in (0, 1)$ such that $x v_r^i < v_r^i - \epsilon$ and $x v_r^i < (1 - x) w^i(d) + x (v_r^i + \rho)$ (take $x$ close but lower than $\frac{v_r^i - \epsilon}{v_r^i}$). I then choose $\delta^*$ close to one and $\delta^{\mu(\delta)}$ close to $x$ as $\epsilon$ tends to zero (notice that $x$ is close to $1$ as $\epsilon$ tends to zero). The left hand side of Equations (1), (2) and (3) tends to $x v_r^i$ as $\epsilon$ tends to zero and Equations (1) and (2) directly follow. The right hand side of Equation (3) tends to $v_r^i + \frac{\rho}{2}$ as $\epsilon$ tends to zero and Equation (3) follows directly since $\rho > 0$.

Define now, for every pair of players $i$ and $d$:

\[
z^i(d) = \begin{cases} 
 w^i(d) \frac{1 - \delta^{\mu(\delta)}}{\delta^{\mu(\delta)}}, & \text{if } i \in \mathcal{N}(d), \\
 0, & \text{otherwise}.
\end{cases}
\]

Finally, let $a_t(j, \delta, (z^i(j))_{i \in \mathcal{N}(j)})$ and $a_t(k, \ell, \delta, (z^i(k)), (z^i(\ell))_{i \in \mathcal{N}(\ell)})$ be deterministic sequences of pure actions that result in the following payoffs:

\[
(v_r^j, (v_r^i + \rho - z^i(j))_{i \neq j}) \in V^*, \quad (5) \\
(v_r^k - z_k^\ell, v_r^k - z^k(\ell), (v_r^i + \rho - z^i(\ell))_{i \neq k, \ell}) \in V^*, \quad (6)
\]

and whose continuation payoffs are within $\epsilon$ of (5) and (6) respectively.

**Lemma 5.9.** The sequences $a_t(j, \delta, (z^i(j))_{i \in \mathcal{N}(j)})$ and $a_t(k, \ell, \delta, (z^i(k)), (z^i(\ell))_{i \in \mathcal{N}(\ell)})$ exist for $\epsilon$ close to zero and $\delta$ close to one.

\textsuperscript{18}Recall that $\mu(\delta)$ is the length of phase III.
Proof. Consider a sequence \((\epsilon_n, \delta_n)\) such that \(\epsilon_n\) tends to zero and \(\delta_n\) tends to one as \(n\) tends to infinity. By construction, \(\mu(\delta)\) is the smallest integer satisfying equations (3), (4) and (5) and \(\delta^\mu(\delta)\) is close to \(\frac{\gamma^f - \epsilon}{\gamma^f}\). Hence, \(\delta^\mu_n(\delta_n)\) close to one for \(n\) sufficiently large, which implies that \(z^i(j) = w^i(j)\frac{1}{\delta^\mu_n(\delta_n)}\) tends to zero as \(n\) tends to infinity. As a consequence, for \(n\) sufficiently large, \(\rho - z^i(j) > 0\) and the payoffs in (8) are in \(V^*\) and bounded away from the axes by at least \(\epsilon_n\). By Lemma 2 in Fudenberg and Maskin (page 432 in [11]), this implies that for \(n\) sufficiently large, there exists \(\delta\) close to one (and greater than \(\delta^* > \delta\)) such that, for every \(\delta > \delta\), there exists a sequence of pure actions \(a_t(j, \delta_n, (z^i(j))_{i \in N})\) with payoffs (8) and whose continuation payoffs are within \(\epsilon_n\) of (8). Similar arguments apply to prove that \(a_t(k, \ell, \delta, (z^\ell(k)), (z^i(\ell))_{i \in N(\ell)})\) exists. \(\square\)

The strategy \(\sigma^i*\) for any player \(i\) in phase IV is then the following. On one hand, if only player \(j\) is minmaxed in phase III, player \(i\) starts playing \(a_t(j, \delta, (z^i(j))_{i \in N})\) at stage \(t + 3 + \mu(\delta)\) until a new possible deviation. On the other hand, if both players \(k\) and \(\ell\) are minmaxed in phase III, then player \(i\) starts playing \(a_t(k, \ell, \delta, (z^\ell(k))_{i \in N(k)}, (z^i(\ell))_{i \in N(\ell)})\) at stage \(t + 3 + \mu(\delta)\) until a new possible deviation.

Intuitively, these rewards are such that, if plan \(\{j\}\) or plan \(\{k, \ell\}\), with \(N(k) = \{\ell\}\), is played, each player \(i \neq k, \ell\) has an incentive to play his minmax strategy against \(j\) due to the additional bonus of \(\rho\) thereafter. In addition, players \(k\) and \(\ell\) have no incentive to deviate during plan \(\{k, \ell\}\), otherwise they would only lengthen their punishment, postponing positive payoffs (recall that by construction plans \(\{k\}\) and \(\{\ell\}\) are never played). Finally, when punishing any player \(d\) in \(N\), each minmaxing player \(i\) has no incentive to draw the sequence of pure actions according to a false distribution in the support of \(P^i(d)\): any expected advantage that player \(i\) gets from playing some pure action in phase III is subsequently removed in phase IV.

For each player \(i \in N\), \(\phi^*\) is the same as before: when a player \(i\) does not detect any deviation at stage \(t\), then he should send the message \(m^i_{t+1} = N(i) \cup \{i\}\). When a player \(i\) detects a deviation, he starts phase II. This last part concludes the construction of \((\sigma^*, \phi^*)\).

The next section shows that it is a belief-free equilibrium with payoff \(v^*\).

5.5 Equilibrium property

Before proving the sufficiency part of Theorem 4.1, I exhibit some properties of the strategy \((\sigma^*, \phi^*)\). The following lemmas show that, for the family of networks that satisfy Condition C, if Assumption 3.2 is satisfied, then the strategy \((\sigma^*, \phi^*)\) guarantees Claims 5.1, 5.2 and 5.3 in Section 5.2 to be true. All the proofs are given in the appendix. The next lemma proves Claim 5.1: if a player who has more than two neighbors deviates in action at some stage \(t\),
then he is identified as suspect and all other players are cleared at stage $t + 1$.

**Lemma 5.10.** Consider a connected network $G$ that satisfies Condition $C$ and suppose Assumption 3.2 holds. For each player $j$ such that $\sharp N(j) \geq 2$, if $j$ deviates in action at some stage $t$ and if all players, except possibly a deviator, follow $\phi^*$ at stages $t$ and $t + 1$, then $X_{t+1}^i = \{j\}$ for each player $i \in N$.\(^{19}\)

Notice that there may be a deviation from a player $d \neq j$ at stage $t + 1$. Yet, in that case, player $j$ does not deviate at stage $t + 1$ under unilateral deviations. The next lemma proves Claim 5.2: it shows how $(\sigma^*, \phi^*)$ works in case a player $k$ has a single neighbor $\ell$.

**Lemma 5.11.** Consider a connected network $G$ that satisfies Condition $C$ and suppose Assumption 3.2 holds. For each pair of players $k$ and $\ell$ such that $N(k) = \{\ell\}$ and if:

- either $k$ deviates in action at stage $t$;

- or there is no action deviation at stage $t$, $m_{t+1}^k = m_{t+1}^k(N)$ and $m_{t+1}^k(N) \supseteq N^{-k}$ (i.e. player $k$ confesses publicly to all players);

- or there is no action deviation at stage $t$, $m_{t+1}^\ell = m_{t+1}^\ell(N)$ and $m_{t+1}^\ell(N) \supseteq N \setminus N(\ell)$ (i.e. player $\ell$ claims publicly he detected a deviation at stage $t$);

and if all players, except possibly a deviator, follow $\phi^*$ at stages $t$ and $t + 1$, then $X_{t+1}^i = \{k, \ell\}$ for each player $i \in N$.

The next lemma shows that no communication deviation other than those of Lemma 5.11 can induce the identification of an innocent player as a deviator (Claim 5.3).

**Lemma 5.12.** Consider a connected network $G$ that satisfies Condition $C$ and suppose Assumption 3.2 holds. For each player $k$ in $N$, if there is no action deviation at stage $t$ and if player $k$ deviates in communication at stage $t + 1$ such that:

- either $N(k) = \{\ell\}$, $m_{t+1}^k \neq m_{t+1}^k(N)$ or $\{N^{-k}, N(k) \cup \{k\}\} \notin m_{t+1}^k(N)$;

- or there exists $k'$ such that $N(k') = k$, $m_{t+1}^k \neq m_{t+1}^k(N)$ or $\{N \setminus N(k), N(k) \cup \{k\}\} \notin m_{t+1}^k(N)$;

- or $m_{t+1}^k \neq m_{t+1}^k(N)$ or $\{N(k) \cup \{k\}\} \notin m_{t+1}^k(N)$;

and if all players, except possibly a deviator, follow $\phi^*$ at stage $t + 1$, then $X_{t+1}^i = \emptyset$ for each player $i \in N$.

\(^{19}\)Notice that, with the previous notations, player $j$ could be player $\ell$ in Lemma 5.10.
I now prove the following proposition, which implies the sufficiency part of Theorem 4.1.

**Proposition 5.13.** Assume that \( G \) satisfies Condition C of Theorem 4.1, that \( g \) satisfies Assumption 3.2 and that \( \text{int} \, V^* \neq \emptyset \). Then, there exists \( \delta \in (0, 1) \) such that for all \( \delta \in (\delta, 1) \), the strategy \((\sigma^*, \phi^*)\) is a belief-free equilibrium with payoff \( v^* \) in the \( \delta \)-discounted game.

**Proof.** Assume that \( G \) satisfies Condition C of Theorem 4.1, that \( g \) fulfills Assumption 3.2 and that \( \text{int} \, V^* \neq \emptyset \). Take \( \delta \) defined in Section 5.4 and consider \( \delta > \delta > \delta \). Recall that \( \delta > \delta \) so that the sequence of pure actions \( \bar{a}_t \) defined in Section 5.1 exists and the continuation payoffs are within \( \epsilon \) of \( v \). Consider the strategy \((\sigma^*, \phi^*)\) constructed in the previous sections.

Take first a player \( j \) such that \( \sharp N(j) \geq 2 \) and \( \sharp N(i) \geq 2 \) for each \( i \in N(j) \). By Lemma 5.12, player \( j \)'s communication deviations do not change continuation strategies, provided player \( j \) does not deviate in action (recall that the deviation which consists in falsely reporting his pure actions during phase III is regarded as an action deviation). Henceforth, I focus on player \( j \)'s action deviations. Assume that player \( j \) stops playing action \( \bar{a}_t \) at some stage \( t \) during phase I and then conforms; without loss of generality, let \( t = 1 \). Lemmas 5.10 and 5.11 imply that player \( j \) is identified as the deviator at stage \( t + 1 = 2 \) by all players and the state becomes \( \{j\} \). Player \( j \) is thus minmaxed at stage \( t + 2 = 3 \) during \( \mu(\delta) \) periods. Player \( j \)'s discounted expected payoff is then no more than:

\[
(1 - \delta)^2 \sum_{t=1}^{2} \delta^{t-1} g^j + (1 - \delta) \sum_{t=\mu(\delta)+3} \delta^{t-1} v^j = (1 - \delta^2) g^j + \delta^{\mu(\delta)+3} v^j.
\]

Since \( \delta > \delta > \delta^* \), Equation (1) ensures that this is less than \( v^j - \epsilon \) which is a lower bound for player \( j \)'s continuation payoff to conforming from date \( t \) on.

If player \( j \) deviates in action during phase III when he is being punished, he obtains at most zero the stage in which he deviates, and then only lengthens his punishment, postponing the positive payoff \( v^j \). The case where player \( j \) deviates in action in phase III when a player \( d \notin N(j) \) is being punished is also trivial, since by construction player \( j \)'s action is arbitrary and may be a best-response.

Assume now that player \( j \) deviates in action at stage \( t \) during phase III when player \( d \in N(j) \) is being punished and then conforms. Two cases are possible. Assume first that player \( j \) deviates at stage \( t \) by playing an action which is not in the support of his minmax strategy. By construction, player \( j \) is identified as deviator at stage \( t + 1 \). Player \( j \)'s discounted expected payoff from the beginning of stage \( t \) is thus no more than:

\[
(1 - \delta)^2 \sum_{t=1}^{2} \delta^{t-1} g^j + (1 - \delta) \sum_{t=\mu(\delta)+3} \delta^{t-1} v^j = (1 - \delta^2) g^j + \delta^{\mu(\delta)+3} v^j.
\]
On the contrary, if he conforms, he gets at least:

$$(1 - \delta^\mu(\delta) - t + 2)w^j(d) + \delta^\mu(\delta) - t + 2(v^j_r + \rho).$$

If $t = 1$, then Equation (2) implies that Equation (8) exceeds Equation (7). If $t = \mu(\delta)$, it follows from Equation (3) that Equation (8) exceeds Equation (7). Finally, the cases for which $1 < t < \mu(\delta)$ follow from Equations (2) and (3) combined.

Second, Phase IV is constructed so that player $j$ is indifferent among all actions in the support of his minmax strategy during phase III when player $i$ is punished (if player $j$ conforms during phase IV). Regardless of player $j$’s actions in this phase, his continuation payoff from the beginning of phase III is within $\epsilon$ of:

$$(1 - \delta)\sum_{t=1}^{\mu(\delta)} \delta^{t-1}g^j(a^j_t, a^N_t(j)) + \delta^\mu(\delta)(v^j_r + \rho - z^j(i)) = \delta^\mu(\delta)(v^j_r + \rho)$$

Hence, player $j$ has no incentive to deviate in phase III by randomizing according to a false distribution in the support of his minmax strategy.

Finally, if player $j$ deviates in action at stage $t$ during phase IV, his discounted expected payoff is no more than:

$$(1 - \delta)\sum_{t=1}^{2} \delta^{t-1}g^j + (1 - \delta)\sum_{t>\mu(\delta)+3} \delta^{t-1}v^j_r = (1 - \delta^2)\bar{g}^j + \delta^{\mu(\delta)+3}v^j_r$$

If player $j$ conforms, his continuation payoff is at least $v^j_r - \epsilon$, and so Equation (1) ensures that deviation is not profitable.

Take now a pair of players $k$ and $\ell$ such that $N(k) = \ell$. If player $k$ (respectively player $\ell$) deviates only in communication, then either continuation strategies do not change (in the case in which all players are cleared, see Lemma 5.12), or both players $k$ and $\ell$ are minmaxed (see Lemma 5.11) and similar arguments as for action deviations apply. Moreover, similar arguments as before (cases in which player $j$ deviates) show that neither player $k$ nor player $\ell$ has an incentive to deviate in action during any phase of the game. (Notice that even if neither player $k$ nor $\ell$ obtain a reward after plan $\{k, \ell\}$ although they are minmaxing players, they still have an incentive to play their minmax strategies, otherwise they would lengthen their punishment, postponing positive payoffs.)
6 Necessary condition

In this section, I prove that Condition C of Theorem 4.1 is necessary for the folk theorem to hold.

**Proposition 6.1.** Assume that \( G \) does not satisfy Condition C. Then, there exists a payoff function \( g \) satisfying Assumption 3.2 such that \( \text{int} \ V^* \neq \emptyset \) and there exists a feasible and strictly individually rational payoff \( v \in V^* \) such that \( v \) is not a Nash equilibrium payoff of the repeated game.

Intuitively, if Condition C is violated, then there exists a player \( i \) and two of his neighbors \( j \) and \( k \) such that, for any pure action profile, both \( j \)'s and \( k \)'s deviations induce the payoffs for player \( i \). It entails that for any mixed action profile, both \( j \)'s and \( k \)'s deviations induce the same payoff distribution for player \( i \). Moreover, I prove that \( j \)'s and \( k \)'s deviations induce the same distribution of messages for player \( i \). In the terminology of Fudenberg, Levine and Maskin ([9]), pairwise identifiability fails. Hence, there exists a feasible and individually rational payoff which is not jointly rational (in the sense of Renault and Tomala in [26]; see Example 3.1 therein for a similar phenomenon). I now prove Proposition 6.1.

**Proof.** Take a network \( G \) such that Condition C is not satisfied. It implies that there exists a pair of players \( j \) and \( k \) in \( N \) who have the same neighbors: \( \mathcal{N}(j) \setminus \{k\} \triangle \mathcal{N}(k) \setminus \{j\} = \emptyset \).

For brevity, I focus on the case in which players \( j \) and \( k \) are not neighbors. The proof can be extended easily to the case in which \( j \) and \( k \) are neighbors.

Take a player \( i \in \mathcal{N}(j) \), so that \( i \in \mathcal{N}(k) \). Notice that all other players, \( m \neq i, j, k \) are either neighbors of both \( j \) and \( k \), or of none. Consider the payoff function for players \( i, j \) and \( k \) represented by the following matrix (where player \( i \) chooses the row, player \( j \) the column and player \( k \) the matrix):\(^{20}\)

\[
\begin{array}{c|cc}
    & C & D \\
\hline
C & 1,0,2 & 0,0,2 \\
D & 1,2,0 & 0,6,0 \\
\end{array}
\]

I write \( u(a^i, a^j, a^k) \) for this payoff vector. Since player \( j \)'s payoff does not depend on \( k \)'s action, nor does \( k \)'s depend on \( j \)'s, one shall drop the relevant argument in what follows. To

\(^{20}\)In order to satisfy Assumption 3.2 for the case in which players \( j \) and \( k \) are neighbors, I assume that if player \( j \) plays \( D \) this adds \( \epsilon > 0 \) to player \( k \)'s payoff, and symmetrically if player \( k \) plays \( D \) this adds \( \epsilon > 0 \) to player \( j \)'s payoff.
complete the description of $g$, let also assume that each player $m \neq i, j, k$ has two actions $C$ and $D$ such that:

(i) player $m$’s payoff at stage $t$ is:

$$g^m(a^m, A_t^{N(m)}) = \ell_t \frac{\epsilon}{n}$$

for some $\epsilon > 0$ and where $\ell_t = \sharp\{ \ell : \ell \in N(m) \cup \{ m \} \text{ and } a^\ell_t = C \} \ (\ell_t \text{ is the number of } m \text{’s neighbors including himself who play } C \text{ at stage } t)$.

(ii) For players $i, j$ and $k$, the payoff is:

$$g^i(a^i, A_t^{N(i)}) = u^i(a^i, A_t^i, a^k_t) + \ell_t \frac{\epsilon}{n},$$
$$g^j(a^j, A_t^{N(j)}) = u^j(a^j, a^k_t) + \ell_t \frac{\epsilon}{n},$$
$$g^k(a^k, A_t^{N(k)}) = u^k(a^k, A_t^i) + \ell_t \frac{\epsilon}{n},$$

where $\ell_t^i = \sharp\{ \ell : \ell \in N(i) \setminus \{ j, k \} \text{ and } a^\ell_t = D \}$, $\ell_t^j = \sharp\{ \ell : \ell \in N(j) \setminus \{ i \} \text{ and } a^\ell_t = D \}$, $\ell_t^k = \sharp\{ \ell : \ell \in N(k) \setminus \{ i \} \text{ and } a^\ell_t = D \}$.

The payoff function $g$ so defined has the following properties:\(^{21}\)

(i) $g$ satisfies Assumption 3.2;

(ii) $\text{int } V^* \neq \emptyset$;

(iii) $v^i = 0$, $v^j = 0$ and $v^k = 0$;

(iv) $C$ is a dominant strategy for each player $\ell \neq i, j, k$;

(v) the payoff $(1, 1, 1)$ (for players $i$, $j$ and $k$) is in $V^*$;

\(^{21}\)If a player has more than two actions, one shall duplicate rows, columns, matrices...in the following manner. For each player $p \in N$, number each action by $k_p > 1 \ (C = 1, D = 2$, etc). Payoff functions are such that:

- for each player $m \neq i, j, k$, his payoff at stage $t$ is $g^m(a^m_t, A_t^{N(m)}) = \ell_t \frac{\epsilon}{n} + \frac{1}{\sum_{p \in N(m) \cup \{ m \}} k_p}$.
- For each player $n \in \{i, j, k\}$, his payoff at stage $t$ is:

$$g^n(a^n_t, A_t^{N(n)}) = \begin{cases} 
  u^n(a^n_t, a^i_t, a^k_t) \text{ if } a^n_p \in \{1, 2\} \text{ for each } p \in N, \\
  u^n(a^n_t, a^i_t, a^k_t) + \frac{1}{\sum_{p \in N(n) \cup \{ n \}} k_p} \text{ if } a^n_t, a^i_t, a^k_t \in \{1, 2\}, a^n_t \geq 3 \forall m \neq i, j, k, \\
  6 + \frac{1}{\sum_{p \in N(n) \cup \{ n \}} k_p} \text{ otherwise.}
\end{cases}$$
(vi) by fixing \( \epsilon < \frac{1}{4} \), it follows from \( a^i \neq C \) for any \( \ell \in \mathcal{N}(i) \) that \( g^i(a^i, a^\mathcal{N}(i)) < 1 \), so the unique way to get the payoff of \((1,1,1)\) is that player \( i \) randomizes between \( C \) and \( D \) with probability \( \frac{1}{2} \) each, and all his neighbors (including players \( j \) and \( k \)) take action \( C \);

(vii) player \( i \) cannot punish both players \( j \) and \( k \): player \( i \) has to play \( C \) in order to minmax player \( j \), leading player \( k \) to guarantee a payoff of 6; and player \( i \) has to choose action \( D \) to minmax player \( k \), leading player \( j \) to guarantee 6.

Assume now that \((1,1,1)\) is a Nash equilibrium of the repeated game and let \( \bar{\sigma} = (\bar{\sigma}^i, \bar{\sigma}^j, \bar{\sigma}^k, (\bar{\sigma}^m)_{m \neq i,j,k}) \) and \( \bar{\phi} = (\bar{\phi}^i, \bar{\phi}^j, \bar{\phi}^k, (\bar{\phi}^m)_{m \neq i,j,k}) \) be a profile of equilibrium strategies which induces a payoff of \( \gamma_\delta = (1,1,1) \) for players \( i, j \) and \( k \). I construct deviations \( (\tau^j, \psi^j) \) and \( (\tau^k, \psi^k) \) for players \( j \) and \( k \) such that:

1. both deviations induce the same probability distributions over the sequences of announcements and payoffs received by player \( i \), i.e. the deviations are indistinguishable.

2. I will deduce from (1) that:

\[
\gamma^j(\tau^j, \bar{\sigma}^{-j}, \psi^j, \bar{\phi}^{-j}) + \gamma^k(\tau^k, \bar{\sigma}^{-k}, \psi^k, \bar{\phi}^{-k}) \geq 5.
\]

This latter equation contradicts that \((\bar{\sigma}, \bar{\phi})\) is an equilibrium of the repeated game. Indeed, \( \gamma^j(\bar{\sigma}, \bar{\phi}) + \gamma^k(\bar{\sigma}, \bar{\phi}) = 2 \) and \( \gamma^j(\tau^j, \bar{\sigma}^{-j}, \psi^j, \bar{\phi}^{-j}) + \gamma^k(\tau^k, \bar{\sigma}^{-k}, \psi^k, \bar{\phi}^{-k}) \geq 5 \), thus either \((\tau^j, \psi^j)\) is a profitable deviation for player \( j \), or \((\tau^k, \sigma^k)\) is a profitable deviation for player \( k \). I now construct these deviations. Define \((\tau^j, \psi^j)\) as follows (the construction of \((\tau^k, \psi^k)\) is completely symmetric):

(i) at each stage, player \( j \) plays \( \tau^j = D \) (instead of \( C \));

(ii) player \( j \) uses the communication strategy \( \psi^j = \bar{\phi}^j(h^j(\tau^k, \psi^k)) \): player \( j \) follows the equilibrium communication strategy but replaces the true history by the fictitious one in which player \( k \) is the deviator.

Under such a strategy, player \( i \) has no way to deduce whether the deviator is \( j \) or \( k \): in both cases, player \( i \)'s payoff decreases from 1 to 0 and no player \( m \neq i,j,k \) has useful information (even if \((\bar{\sigma}, \bar{\phi})\) is a mixed strategy). Indeed, according to \( \psi^j \), player \( j \) uses \( \bar{\phi}^j \) with a fictitious history. If \( \bar{\phi}^j \) is mixed, so is \( \psi^j \). Hence, for all private histories of player \( i \):

\[
P_{\tau^j, \bar{\sigma}^{-j}, \psi^j, \bar{\phi}^{-j}}(h^i) = P_{\tau^k, \bar{\sigma}^{-k}, \psi^k, \bar{\phi}^{-k}}(h^i).
\]
Now, I define the probabilities $b_t$ and $c_t$ as follows:

\[ b_t = \mathbb{P}_{\tau, \bar{\sigma}, \bar{\phi}, \psi, \phi}(a_t^i = C) = \mathbb{P}_{\tau, \bar{\sigma}, \bar{\phi}, \psi, \phi}(a_t^i = C), \]

\[ c_t = \mathbb{P}_{\tau, \bar{\sigma}, \bar{\phi}, \psi, \phi}(a_t^i = D) = \mathbb{P}_{\tau, \bar{\sigma}, \bar{\phi}, \psi, \phi}(a_t^i = D). \]

Under $(\tau^j, \psi^j, \bar{\sigma}^{-j}, \bar{\phi}^{-j})$, the expected payoff of player $j$ at stage $t$ is:

\[ g^j_t(\tau^j, \bar{\sigma}^N(j)) \geq 6b_t \]

\[ \geq 6(1 - b_t). \]

We then have: $\gamma^j(\tau^j, \psi^j, \bar{\sigma}^{-j}, \bar{\phi}^{-j}) \geq 6(1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1}(1 - b_t)$. Since $(\bar{\sigma}, \bar{\phi})$ is a Nash equilibrium of the repeated game, it is necessary that there exists $\tilde{\delta} \in (0, 1)$ such that for any $\delta \in (\tilde{\delta}, 1)$, $6(1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1}(1 - b_t) \leq 1$, so that $(1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1}b_t \geq \frac{5}{6}$. In the same way, the expected payoff of player $k$ at stage $t$ under $(\tau^k, \psi^k, \bar{\sigma}^{-k}, \bar{\phi}^{-k})$ is:

\[ g^k_t(\tau^k, \bar{\sigma}^N(k)) \geq 6b_t. \]

Hence, $\gamma^k(\tau^k, \psi^k, \bar{\sigma}^{-k}, \bar{\phi}^{-k}) \geq 6(1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1}b_t$. But there exists $\tilde{\delta} \in (0, 1)$ such that for any $\delta \in (\tilde{\delta}, 1)$, $(1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1}b_t \geq \frac{5}{6}$, so $\gamma^k(\tau^k, \psi^k, \bar{\sigma}^{-k}, \bar{\phi}^{-k}) \geq 5$. This contradicts the fact that $(\bar{\sigma}, \bar{\phi})$ is a Nash equilibrium of the repeated game.

\[ \Box \]

## 7 Concluding remarks

In this section, I discuss some extensions of the model and state some open problems.

**Weakly dominated strategies.** Part of the equilibrium strategy constructed in Section 5 is weakly dominated. Indeed, a player who deviates at some stage $t$ is required to confess at stage $t + 1$. Nevertheless, it is possible to make this part of the strategy dominant by adjusting accordingly the reward in phase IV: indeed, the reward phase can be such that a player who deviates and confesses publicly his deviation after that obtains a bonus of $\frac{\rho}{2} > 0$ during phase IV. Hence, a player who deviates has an incentive to confess, and doing so is not weakly dominated anymore. In addition, this bonus is strictly lower than $\rho$, otherwise a minmaxing player might have an incentive to deviate. Indeed, a minmaxing player’s realized payoff during phase III may be less than if he would get his minmax level instead. However, a minmaxing player has still no incentive to deviate, otherwise he would loose $\frac{\rho}{2}$ at each stage during phase IV.
**Correlated minmax.** In some repeated games with imperfect monitoring, it is possible to drive equilibrium payoffs below the independent minmax, see Renault and Tomala ([25]) for illuminating examples. It is the case here: Theorem 4.1 remains unchanged if I rather consider correlated minmax, defined as follows:

\[
p_{\text{corr}}^j = \min_{x^{N(i)} \in \Delta(A^{N(i)})} \max_{x^i \in \Delta(A^i)} g^j(x^i, x^{N(i)}).
\]

To prove Theorem 4.1 in that case, I need to modify the strategy constructed in Section 5 in the following way. The idea is that players can correlate their actions when punishing player \(k\) (respectively players \(k\) and \(\ell\)) in phase III, without revealing information to the minmaxed player(s). For that, define \(Q^{N(k)}(k) \in \Delta(A^{N(k)})\) (respectively \(Q^{N(\ell)}(\ell) \in \Delta(A^{N(\ell)})\)) a correlated strategy that realizes the minimum in \(p_{\text{corr}}^k\) (respectively \(p_{\text{corr}}^\ell\)). Choose a player \(j \neq k\) (respectively \(j \neq k, \ell\)). At the beginning of phase III, I add a stage in which player \(j\) draws i.i.d. sequences of pure actions according to \(Q^{N(k)}(k)\) (respectively \(Q^{N(\ell)}(\ell)\) when both players \(k\) and \(\ell\) are minmaxed)\(^22\) for the minmaxing players for \(\mu(\delta)\) periods. Player \(j\) announces the sequences publicly to all players except \(k\) (respectively player \(\ell\)). Deviations of player \(j\) are punished as before, and the reward phase makes player \(j\) indifferent between the pure actions actually played by him and his neighbors (recall that the reward phase is based on player \(j\)’s realized payoff).

**Private communication** An alternative model would be to consider private announcements, i.e. the list of receivers of a message is not certifiable. The possibility for the players to make public announcements is needed for two cases only.

(i) First, if there exists a player \(k\) such that \(\mathcal{N}(k) = \{\ell\}\), then players \(k\) and \(\ell\) need to make public announcements in phase II. Otherwise, a communication deviation of player \(k\) (respectively player \(\ell\)) could be to send spurious messages to a subset of players only. With the possibility of public communication, the strategy constructed in Section 5 ignores such deviations. If public communication is not allowed, this implies a lack of common knowledge of the deviation’s date, and there may be a coordination failure with some players starting phase III whereas other do not.

(ii) Second, public announcements are needed in the punishment phase (phase III). Otherwise, some communication deviations may entail a coordination failure, since players could have different informations on the pure actions chosen by their opponents (recall that pure actions are announced in phase III).

\(^{22}\)Notice that if \(\mathcal{N}(k) = \{\ell\}\), and both players \(k\) and \(\ell\) are minmaxed, there is no need to correlate in order to punish player \(k\) since only player \(\ell\) is a neighbor of \(k\).
Otherwise, only private communication is needed. Hence, the following corollary holds:

**Corollary 7.1.** Assume that players are only allowed to communicate privately with each other (no certifiability). If Condition C is satisfied and if each player has more than two neighbors, then the Folk theorem holds with minmax levels in pure strategies (and belief-free equilibrium as solution concept).

The proof is a straightforward application of the proof of Theorem 4.1.

**Folk theorem without discounting.** Condition C of Theorem 4.1 is also the necessary and sufficient condition for a folk theorem to hold if I consider uniform sequential equilibria of the undiscounted repeated game (see [8] and [29]). Namely, every feasible and individually rational payoff is a uniform sequential equilibrium payoff for any payoff function \( g \) that satisfies Assumption 3.2 if and only if Condition C is satisfied. Moreover, in that case, there is no need of public announcements, and players can be only allowed to send private messages (coordination in cases (i) and (ii) above is not needed). In addition, it is possible to restrict communication along the network: the necessary and sufficient conditions for a folk theorem to hold are known (but are different from Condition C) if (i) only private communication is allowed (no public announcements) and (ii) I consider Nash equilibrium of the repeated game or uniform sequential equilibrium for the undiscounted case ([19]). If communication is restricted along the network, conditions for a folk theorem to hold is an open problem for (i) public announcements and (ii) sequential equilibria of repeated games with discounting.

**Appendix**

**A  Proof of Proposition 4.3**

**Proof.** Take a connected network \( G \) that satisfies Condition C. We first prove point (i). Assume \( n = 3 \), with \( N = \{i, j, k\} \). Since \( G \) is connected, there exists a player, say \( i \), who has exactly two neighbors. Therefore, \( \mathcal{N}(k) = \{i, j\} \). But then, there is no player in \( \mathcal{N}(i) \setminus \{j\} \triangle \mathcal{N}(j) \setminus \{i\} \) and Condition C is violated.

We now prove point (ii). Assume \( n = 4 \), let \( N = \{i, j, k, \ell\} \). Since \( G \) is connected, at least one player, say \( i \), has at least two neighbors, say \( j \) and \( k \). Because of Condition C, it must be the case that \( \ell \in \mathcal{N}(j) \setminus \{k\} \triangle \mathcal{N}(k) \setminus \{j\} \). Assume without loss of generality that \( \ell \in \mathcal{N}(k) \), so \( \ell \not\in \mathcal{N}(j) \). I prove the following:

(1) \( j \) and \( k \) are not neighbors;
(2) $i$ and $\ell$ are not neighbors;
so that: $\mathcal{N}(i) = \{j, k\}$, $\mathcal{N}(j) = \{i\}$, $\mathcal{N}(k) = \{i, \ell\}$ and $\mathcal{N}(\ell) = \{k\}$, which is represented by the following network:

\[ j \rightarrow i \rightarrow k \rightarrow \ell \]

and proves point (ii) of Proposition 4.3.

For (1), assume that $j$ and $k$ are neighbors. Then $\mathcal{N}(j) = \{i, k\}$ (recall that $\ell \notin \mathcal{N}(j)$) and $\{j, k\} \subseteq \mathcal{N}(i)$. For Condition C to be satisfied, it must be that $\ell \in \mathcal{N}(i)$, so $\mathcal{N}(i) = \{j, k, \ell\}$. Yet, $\mathcal{N}(k) = \{i, j, \ell\}$, hence $\mathcal{N}(i) \setminus \{k\} \triangle \mathcal{N}(k) \setminus \{i\} = \emptyset$. This contradicts Condition C, so that $j$ and $k$ cannot be neighbors.

For (2), assume that $i$ and $\ell$ are neighbors. Then, $\{i, \ell\} \subseteq \mathcal{N}(k)$ and $\mathcal{N}(\ell) = \{i, k\}$ (again, recall that $j \notin \mathcal{N}(\ell)$). Therefore, it must be the case that $j \in \mathcal{N}(k)$ for Condition C to be satisfied, which is impossible by (i). As a consequence, $i$ and $\ell$ cannot be neighbors. \hfill $\square$

\section{B Proof of Lemma 5.10}

\textit{Proof.} Take a network $G$ that satisfies Condition C and a player $k \in N$ such that $\sharp \mathcal{N}(k) \geq 2$. Assume that player $k$ deviates in action at stage $t$. Take any player $j \neq k$. I first prove that $j$ is cleared at stage $t+1$ by every player $i \in N$. Two cases are possible.

(1) Assume first that for each player $j^1 \in \mathcal{N}(j)$, $\sharp \mathcal{N}(j^1) \geq 2$. Then, each player $i$ in $N$ clears player $j$ at stage $t+1$. Indeed, the following holds:

- first, if $j$ plays $\phi^{*j}$ at stage $t+1$, then $j \in m_{t+1}^j(N)$. Indeed, either $j \in \mathcal{N}(k)$ and $\tilde{\phi}^j$ prescribes player $j$ to announce publicly $N \setminus \mathcal{N}(j)$ to all players at stage $t+1$, so $j \in \tilde{m}_{t+1}^j(N)$. Or $j \notin \mathcal{N}(k)$ and player $j$ starts phase II at the end of stage $t+1$,\footnote{Notice that player $j$ could have started phase II at the end of stage $t$ if (a) one of his neighbor deviated at stage $t$, or (b) if a player who is not his neighbor deviated at stage $t-1$, or (c) if player $k$ deviated also in communication at stage $t$ by sending to $j$ a message different from $\tilde{m}_{t+1}^k(N)$. However, in any case, player $j$ ends this “previous” phase II at the end of stage $t$ and starts a “new” phase II at the end of stage $t+1$.} so $j$ is prescribed to announce $\mathcal{N}(j) \cup \{j\}$ publicly to all players at stage $t+1$, so $j \in \tilde{m}_{t+1}^j(N)$.

- Second, since Condition C is satisfied, there exists a player $m \neq j, k$ such that $m \in \mathcal{N}(k) \setminus \{j\} \triangle \mathcal{N}(j) \setminus \{k\}$. Moreover, if $m$ plays $\phi^m$ at stage $t+1$, then $j \in m_{t+1}^m(N)$. Indeed, either, $m \in \mathcal{N}(k)$ and $\tilde{\phi}^m$ prescribes player $m$ to announce $N \setminus \mathcal{N}(m)$ publicly to all players at stage $t+1$, so $j \in \tilde{m}_{t+1}^m(N)$. Or $m \notin \mathcal{N}(k)$, and $\tilde{\phi}^m$ prescribes player $m$ to announce $\mathcal{N}(m) \cup \{m\}$ publicly to all players at stage $t+1$, so $j \in \tilde{\phi}^m(N)$.

(2) Assume next that for each player $j^1 \in \mathcal{N}(j)$, $\sharp \mathcal{N}(j^1) = 1$. Then, each player $i$ in $N$ clears player $k$ at stage $t+1$. Indeed, the following holds:

\[ m_{t+1}^j(N) = m_{t+1}^m(N) = \{k\} \]

(3) Assume finally that for each player $j^1 \in \mathcal{N}(j)$, $\sharp \mathcal{N}(j^1) = 0$. Then, each player $i$ in $N$ clears player $k$ at stage $t+1$. Indeed, the following holds:

\[ m_{t+1}^j(N) = \emptyset \]

(4) Assume further that for each player $j^1 \in \mathcal{N}(j)$, $\sharp \mathcal{N}(j^1) = -1$. Then, each player $i$ in $N$ clears player $k$ at stage $t+1$. Indeed, the following holds:

\[ m_{t+1}^j(N) = \{k\} \]

(5) Assume finally that for each player $j^1 \in \mathcal{N}(j)$, $\sharp \mathcal{N}(j^1) = -2$. Then, each player $i$ in $N$ clears player $k$ at stage $t+1$. Indeed, the following holds:

\[ m_{t+1}^j(N) = \emptyset \]

\[ m_{t+1}^m(N) = \{k\} \]

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- Third, if \( k \) follows \( \phi^* \) at stage \( t + 1 \), then \( \tilde{\phi}^k \) prescribes player \( k \) to announce \( N \setminus \{ k \} \) publicly to all players at stage \( t + 1 \), so \( j \in m_{t+1}^k(N) \).

Since at most one player in \( \{ j, k, m \} \) deviates in communication at stage \( t + 1 \), then \( j \notin X_{t+1}^i \) for each player \( i \in N \).

(2) Assume now that there exists \( j^1 \in \mathcal{N}(j) \) such that \( \mathcal{N}(j^1) = \{ j \} \). First, proposition 4.4 implies that \( j^1 \) is unique. Second, \( G \) is connected and \( n \geq 3 \), so \( \sharp\mathcal{N}(j) \geq 2 \). Finally, \( j^1 \neq k \) since \( \sharp\mathcal{N}(k) \geq 2 \) by assumption. Hence, \( j^1 \) is cleared at stage \( t + 1 \) by every player \( i \) in \( N \) (see (1) above). In addition, with the same reasoning as before, at least two players in \( \{ j, k, m \} \) make public announcements including \( j \) to all players at stage \( t + 1 \). As a consequence, for each player \( i \in N \), \( j \notin X_{t+1}^i \).

Finally, I prove that no player \( i \in N \) clears player \( k \), i.e. \( k \in X_{t+1}^i \) for each player \( i \in N \). By construction, \( k \notin m_{t+1}^j(N) \) for any \( j \in N \) who follows \( \phi^*j \). Since at most one player deviates in communication at stage \( t + 1 \), \( k \in X_{t+1}^i \) for each player \( i \in N \).

Hence, \( X_{t+1}^i = \{ k \} \) for each player \( i \) in \( N \). \( \square \)

C Proof of Lemma 5.11

Proof. Take a connected network \( G \) that satisfies Condition C and suppose Assumption 3.2 holds. Take a pair of players \( k \) and \( \ell \) such that \( \mathcal{N}(k) = \{ \ell \} \).

Assume first that player \( k \) deviates in action at stage \( t \). At stage \( t + 1 \), players \( k \) and \( \ell \) should announce \( N^{-k} \) and \( N \setminus \mathcal{N}(k) \) respectively publicly to all players (and possibly make other announcements if the game is in phase III). All other players \( j \neq k, \ell \) should announce \( \mathcal{N}(j) \cup \{ j \} \) and \( k \notin m_{t+1}^j(N) \) publicly to all players since \( \mathcal{N}(k) = \{ \ell \} \). As a consequence, the name of \( k \) appears in at most one public announcement at stage \( t \) and \( X_{t+1}^i \supseteq \{ k, \ell \} \) for each player \( i \in N \). In addition, no player \( j \neq k \), including player \( \ell \), deviates in action at stage \( t \) under unilateral deviations. So, each player \( j \neq k \) such that \( \sharp\mathcal{N}(j) \geq 2 \) appears in the public announcements of at least two distinct players (among his two neighbors and himself), and \( j \notin X_{t+1}^i \) for each player \( i \in N \). On the other hand, each player \( j \neq k \) such that \( \sharp\mathcal{N}(j) = 1 \) also appears in the public announcements of at least two distinct players: either \( j \) and his single neighbor do not deviate and \( j \) appears in both of their public announcements, or one of them deviates at stage \( t + 1 \), which implies that \( k \) follows \( \phi_{t+1}^k \) and \( j \in m_{t+1}^k(N) \). As a consequence, each player \( j \neq k \) is cleared by all players at stage \( t + 1 \). Hence, \( X_{t+1}^i = \{ k, \ell \} \) for each player \( i \) in \( N \).

Assume now that there is no action deviation at stage \( t \) and that \( m_{t+1}^k = N^{-k} \). Player \( k \) thus deviates in communication at stage \( t + 1 \) and no other player does under unilateral
deviations. It follows that all other players \( j \neq k \) announce \( \mathcal{N}(j) \cup \{j\} \) publicly since there was no action deviation at stage \( t \). As a consequence, each player \( j \neq k \) is cleared by at least two players at stage \( t + 1 \), and player \( k \) appears in the public announcement of player \( \ell \) only. Henceforth, \( X^i_{t+1} = \{k, \ell\} \) for each player \( i \in N \).

Finally, assume that there is no action deviation at stage \( t \) and that \( m^t_{t+1}(N) = N \setminus \mathcal{N}(\ell) \). Player \( \ell \) thus deviates in communication at stage \( t + 1 \) and no other player does under unilateral deviations. It follows that all other players \( j \neq k \) announce \( \mathcal{N}(j) \cup \{j\} \) publicly to all players since there was no action deviation at stage \( t \). As a consequence, each player \( j \notin \mathcal{N}(\ell) \) is cleared by at least two players at stage \( t + 1 \). Moreover, for each player \( j \in \mathcal{N}(\ell) \setminus \{k\}, N(j) \geq 2 \). Indeed, if it was not the case, then \( \mathcal{N}(j) = \{\ell\} = \mathcal{N}(k) \) which contradicts Condition C. Therefore, each player \( j \in \mathcal{N}(\ell) \setminus \{k\} \) also appears in the public announcements of at least two distinct players at stage \( t + 1 \). Finally, player \( k \) appears in his own public announcement only. As a conclusion, \( X^i_{t+1} = \{k, \ell\} \) for every player \( i \in N \).

\[ \square \]

D Proof of Lemma 5.12

Proof. Take a connected network \( G \) that satisfies Condition C and suppose Assumption 3.2 holds. Take a player \( k \) in \( N \) and assume that there is no action deviation at stage \( t \).

Assume first that \( \mathcal{N}(k) = \{\ell\} \) and that \( k \) deviates in communication at stage \( t + 1 \). If \( m^t_{t+1}(N) \neq m^t_{t+1}(N) \) (\( k \) does not make a public announcement to all players), then the message of player \( k \) is ignored. Assume now that \( m^t_{t+1}(N) = m^t_{t+1}(N) \) and \( \{N^{-k}, \mathcal{N}(k) \cup \{k\}\} \notin m^t_{t+1}(N) \). Under unilateral deviations, all other players \( j \neq k \) announce \( \mathcal{N}(j) \cup \{j\} \) publicly. Two cases are then possible. Either \( N \setminus \mathcal{N}(k) \notin m^t_{t+1}(N) \) and \( m^t_{t+1}(N) \) is not taken into account under \( \phi^* \) since it implies that player \( k \)'s message is different from the kinds of messages regarded in phase II. Then, \( X^i_{t+1} = \emptyset \) for every player \( i \in N \). Or, \( N \setminus \mathcal{N}(k) \in m^t_{t+1}(N) \). Player \( k \) is then cleared by all players since his name is in the public announcements of \( k \) and \( \ell \). Player \( \ell \) is also cleared by all players because his name is in at least two public announcements among his own and his other neighbor than \( k \) (since \( n \geq 3 \) and \( G \) is connected, \( \sharp \mathcal{N}(k) \geq 2 \)). Each other player \( i \) is cleared by all players since his name appears in the public announcements of at least two players among him and his neighbors (each has at least one neighbor since \( G \) is connected). Hence, \( X^i_{t+1} = \emptyset \) for every player \( i \in N \).

Assume now that there exists \( k' \) such that \( \mathcal{N}(k') = k \), and that \( k \) deviates in communication at stage \( t + 1 \). If \( m^t_{t+1}(N) \neq m^t_{t+1}(N) \), then the message of player \( k \) is ignored. Assume now that \( m^t_{t+1}(N) = m^t_{t+1}(N) \) and that \( \{N \setminus \mathcal{N}(k), \mathcal{N}(k) \cup \{k\}\} \notin m^t_{t+1}(N) \). Under unilateral deviations, all other players \( j \neq k \) announce \( \mathcal{N}(j) \cup \{j\} \) publicly to all players. Two cases are possible. Either \( N^{-k} \notin m^t_{t+1}(N) \) and player \( k \)'s public announcement is not taken into
account as before. Or $N^{-k} \in m_{t+1}^k(N)$, in which case it is obvious that all players are cleared (recall that player $k$ has at least two neighbors). In any case, $X_{i}^{t+1} = \emptyset$ for each player $i$ in $N$.

Finally, assume $\sharp N(k) \geq 2$, $\sharp N(j) \geq 2$ for every $j \in N(k)$, and that $k$ deviates in communication at stage $t + 1$. If $m_{t+1}^k \neq m_{t+1}^k(N)$, then player $k$’s message is ignored. Assume now $m_{t+1}^k = m_{t+1}^k(N)$ and $\{N(k) \cup \{k\}\} \notin m_{t+1}^k(N)$. Under unilateral deviations, all other players $j \neq k$ announce $N(j) \cup \{j\}$ publicly to all players. Three cases are possible. Assume first that $N^{-k} \in m_{t+1}^k(N)$. Then all players are cleared by everybody at stage $t + 1$, since player $k$ has at least two neighbors. Second, assume $N \setminus N(k) \in m_{t+1}^k(N)$. Since player $k$’s neighbors have more than two neighbors, they are cleared by all players at stage $t + 1$. Obviously, so are players other than $k$’s neighbors. Third, assume $\{N^{-k}, N \setminus N(k)\} \notin m_{t+1}^k(N)$. As before, $m_{t+1}^k$ is not taken into account. Therefore, in any case, $X_{i}^{t+1} = \emptyset$ for each player $i$ in $N$. \hfill \Box

References


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