ORDINAL VERSUS CARDINAL VOTING RULES: A MECHANISM DESIGN APPROACH

SEMIN KIM

May 8, 2012

Abstract. We consider the performance and incentive compatibility of voting rules in a Bayesian environment with independent private values and at least three alternatives. It is shown that every Pareto efficient ordinal rule is incentive compatible under a symmetry assumption on alternatives. Furthermore, we prove that there exists an incentive compatible cardinal rule which strictly Pareto dominates any ordinal rule when the distribution of every agent's values is uniform.

Keywords: Ordinal rule, Pareto efficiency, Incentive compatibility, Bayesian mechanism design

I am greatly indebted to Yaron Azrieli for his thoughtful guidance. I am also grateful to PJ Healy, James Peck, Dan Levin, Caleb Cox and Jack Rouzer for their helpful comments and suggestions.
Department of Economics, The Ohio State University, 1945 North High street, Columbus, OH 43210, USA; kim.2607@osu.edu.

1. Introduction

The *ordinal* rule[1] has been extensively studied in social choice, strategic voting and mechanism design theory. The rule is a voting mechanism that depends only on ordinal information[2]; i.e. disregarding the intensities of the agents' preferences. In the literature, the problem of aggregation of preferences in a group decision has been studied within the class of ordinal rules. The objective of this paper is to apply a Bayesian mechanism design approach to explore the relationship between the classic notion of (weak)[3] Pareto efficiency and incentive compatibility of ordinal rules. Furthermore, this paper allows non-ordinal rules, which we call *cardinal* rules, and shows that there exists an incentive compatible cardinal rule which outperforms any ordinal rule.

We consider a Bayesian environment, where the preference over at least three possible alternatives is private information for an agent. In order to capture the intensities of the agents' preferences, values of agents are cardinal random variables which, we assume, are independent across agents *and neutral between alternatives*[4]; that is, no alternative is special in ex-ante perspective. To explain the main results of this paper, we need to be explicit regarding two notions: One, that a Social Choice Function (SCF) f is a mapping from value profiles to lotteries[5] over the set of alternatives; and two, the social welfare is measured in terms of ex-ante cardinal expected utilities driven by a SCF and is also used for the measure of the performance of the SCF.

First, we consider Pareto efficiency in the class of ordinal SCFs. A SCF f is Ordinally Pareto Efficient (OPE) if f is ordinal and there is no ordinal SCF that strictly increases the utilities of all agents from f. This paper shows that an OPE f is Incentive Compatible (IC) (Proposition 1); i.e., the usual conflict between Pareto efficiency and incentive compatibility does not exist in the class of ordinal rules. Note, this result obviously implies that there is no conflict between strong Pareto efficiency and incentive compatibility as well.

Even in the class of ordinal rules, the incentive compatibility problem has been well-known in the literature. In other words, an agent has an incentive to announce her

---

[1]Following the literature, we frequently use the term "rule" which refers to Social Choice Function (SCF) in this paper.

[2]This notion is the same as Apesteguia et al. (2011)

[3]In the literature, there are two notions of Pareto efficiency (weak Pareto efficiency and strong Pareto efficiency). Since we mainly consider the weak concept, we skip the term, "weak" from now.

[4]Specifically, values are identically and independently distributed (i.i.d.) between alternatives

[5]That is another reason why we consider cardinal values.

false preference if the false announcement can decrease the possibility that her least-liked alternative is chosen. However, Proposition 1 guarantees that there is no incentive compatibility problem in our environment if the rule satisfies the basic and appealing criteria, Pareto efficiency.

In the Bayesian mechanism design approach, both the design of an OPE f and the decision of agents depend on the distribution of agents' values. The form of the rule is similar to scoring rule where scores depend on the distribution. In addition, the assumption of neutral values between alternatives forces an OPE f to treat all alternatives symmetrically.[6] Given this form and property of the rule, an agent believes that the probabilities of each alternative being chosen under the rule before her announcement are the same and that her true announcement can increase her expected utility. Consequently, the agent reports her true preference.

Furthermore, beyond the class of ordinal rules, it might be more interesting to see the possibility of designing rules better than ordinal rules. Specifically, this paper shows that one can design an IC cardinal rule which strictly Pareto dominates any ordinal rule when the distribution of every agent's values is uniform[7] (Theorem 1). If we consider cardinal rules, Proposition 1 does not hold, which means that there exists tread-off relationship between Pareto efficiency and incentive compatibility. It becomes much more difficult to design a rule which both increases efficiency and satisfies the incentive constraint. However, Theorem 1 says that it is possible to design an IC cardinal rule superior to any ordinal rule in certain conditions.

In order to prove this main result, consider first a rule with a finer partition of agents' values than an ordinal partition which contains only ordinal information. This finer partition can allow the rule to use the information about the intensities of the agents' preference. Obviously, this added information helps to design a cardinal rule which generates higher utilities of agents than ordinal rules. Second, we can find a specific partition which balances the incentives of agents to be truthful or not.

In particular with 3 alternatives, we also show the connection between this new rule and well-known rules: *plurality, negative, Borda and approval* voting rules (Proposition 3). As in Myerson (2002), the general form of those voting rules is (A, B)-scoring rules, where each voter must choose a score that is a permutation of either $(1, A, 0)$ or $(1, B, 0)$,

[6]It is called "neutrality of rule" in the literature.

[7]We focus on symmetric agents in the sense that the value distribution is identical across agents. One advantage from this assumption is that the social welfare can be normalized to the utility of one agent - no interpersonal utility comparisons are needed to compare rules.

$(0 \leq A \leq B \leq 1)$. Proposition 3 shows that the new rule is closely related to specific (A, B)-scoring rules under the same assumptions in Theorem 1.

In addition, the method of using the finer partition and finding a condition for incentive compatibility in the proof of Theorem 1 is novel and worthy of attention. This method can be applicable to a more general distribution of agents' values to design a rule superior to any ordinal rule because the assumption of a uniform distribution of agents' values is a sufficient condition for the result.

This paper is organized as follows. The next section reviews related literature. In Section 3, we introduce the environment. Section 4 discusses the notions of Pareto efficiency of ordinal rules and incentive compatibility in a Bayesian environment. In Section 5, we show the existence of an IC cardinal rule superior to any ordinal rule. Section 6 discusses the connection to (A, B)-scoring rules, generalization to more than 3 alternatives and possible extension of Theorem 1.

2. Related literature

There is an extensive literature examining voting rules for the problem of aggregation of preferences and incentive compatibility. The celebrated Gibbard-Sattherthwaite theorem gave us a negative result with the strong incentive compatibility concept of strategy proofness. Under mild assumptions, only dictatorial SCFs are strategy proof. After that, much literature has confirmed the robustness of the theorem or tried to find a positive result with different perspectives such as changing of preference domain or weakening the concept of incentive compatibility.

Recently, there is a growing literature about a similar topic using a Bayesian mechanism design approach (e.g., Majumdar and Sen (2004), Schmitz and Tröger (2011), Jackson and Sonnenschein (2007)). A key insight in a Bayesian mechanism design approach is that the information of agents' value distribution is available. Jackson and Sonnenschein (2007) showed that incentive constraints can be overcome by linking decision problems using the insight. Moreover, if the information of *cardinal* value distribution is added (e.g., Apesteguia et al. (2011), Azrieli and Kim (2011), Barberà and Jackson (2006)), interesting results can be derived.

Apesteguia et al. (2011), which inspired the current paper, derived the form of utilitarian, maximin and maximax ordinal rules.[8] They used a Bayesian approach with information about almost the same cardinal value distribution as in this paper, which

[8]We introduce these rules in Section 3.

enabled them to derive the exact form of optimal ordinal rules. After the optimal ordinal rules are derived, the next question could be the incentive compatibility of the rules. Their focus was not on incentive compatibility. However, Proposition 1 in this paper can cover the issues of incentive compatibility of a wider class of rules including all of their rules. On the other hand, they focused on ordinal rules, which is a traditional approach. However, in Section 5, this paper naturally extends the class of SCFs from ordinal to cardinal rules and shows the possibility of designing an IC cardinal rule improving ordinal rules.

Majumdar and Sen (2004) analyzed the implications of weakening the incentive compatibility concept from strategy proofness to Ordinal Bayesian Incentive Compatibility (OBIC)[9]. They showed that a wide class of SCFs in uniform priors satisfies OBIC (Theorem 3.1 in their paper). It is close to Proposition 1 in this paper. However, their focus was not on the relationship between Pareto efficiency or social welfare and incentive compatibility of SCFs. Furthermore, while they used only ordinal information, we add the information about cardinal value distribution. Hence we can derive the form of ordinally Pareto efficient rules and prove that this class of rules is IC.

There are several papers that deal with voting rules with two alternatives. Lemma 1 in Azrieli and Kim (2011) implies that there is no incentive compatible rule which outperforms optimal ordinal rules. However, Azrieli and Kim (2011) shows that considering at least three alternatives leads to the possibility of improvement over optimal ordinal rules, which also motivates Theorem 1 in this paper. Schmitz and Tröger (2011) showed that the optimal rule among all anonymous and neutral rules in symmetric and neutral environments is the majority rule, which is indeed an ordinal rule.

Lastly, many papers on voting rules did not allow monetary transfers even though they apply the mechanism design approach (Börgers and Postl (2009), Miralles (2011)). The main reason is that we often see monetary transfers which are infeasible or excluded for ethical reasons, such as child placement in public schools, transplant organs to patients, collusion in markets, task allocation in organizations, etc. The current paper follows these previous studies in the sense that monetary transfers are not allowed.

[9]In fact, if we consider only ordinal rules, OBIC is almost the same as the incentive compatibility in our setting.

3. Environment

Consider a standard Bayesian environment with private values. The set of agents is $N = \{1, 2, ..., n\}$, and the set of alternatives[10] is $L = \{a, b, c\}$. Agents' valuations of the alternatives are real-valued random variables. We assume that values are independent across agents and *i.i.d. between alternatives* (neutral alternatives). For each $i \in N$ and $l \in L$, $\hat{v}_i^l$ denotes a continuous random variable representing the value of the alternative l for agent i. Let $V_i^l \subseteq \mathbb{R}$ be a support of the alternative l for agent i, and μ_i on V_i^l be the distribution of $\hat{v}_i^l$. Let $\hat{v}_i = (\hat{v}_i^a, \hat{v}_i^b, \hat{v}_i^c)$ be a random vector representing the value for all alternatives of agent i, and $v_i = (v_i^a, v_i^b, v_i^c)$ be a realized value of $\hat{v}_i$. Let $V_i = \left(V_i^a, V_i^b, V_i^c\right) \subseteq \mathbb{R}^3$ be a support of $\hat{v}_i$, and $V = V_1 \times V_2 \times .. \times V_n$ be a set of value profiles.

A Social Choice Function (SCF) is a measurable mapping[11] $f : V \to \triangle(L)$. For example, $f(v) = (\frac{1}{2}, \frac{1}{2}, 0)$ means that at value profile v, a or b are chosen with probability $\frac{1}{2}$. Let F be the set of all SCFs. For every agent i and SCF f , $U_i(f) = \mathbb{E}\left(\hat{v}_i \cdot f(\hat{v})\right)$ denotes the (ex-ante) expected utility of agent i under f.[12] Finally, the following is standard in the mechanism design literature:

Definition 1. A SCF f is *Incentive Compatible (IC)* if truth-telling is a Bayesian equilibrium of the direct revelation mechanism associated with f. In our environment, this means that

for all $i \in N$ and all $v_i, v_i' \in V_i$,

$$v_i \cdot \left(\mathbb{E}\left(f(v_i, \hat{v}_{-i})\right) - \mathbb{E}\left(f(v_i', \hat{v}_{-i})\right)\right) \geq 0$$

4. Ordinal Pareto Efficiency and Incentive compatibility

In this section, we restrict our attention to ordinal SCFs, which is traditional in the literature. The only information used for determining the value of an ordinal SCF is ordinal information; i.e. disregarding the intensities of the agents' preferences.

The primary objective of this section is to show non-existence of the usual conflict between Pareto efficiency and incentive compatibility within our environment. For this objective, we need more definitions.

[10]We focus on 3 alternatives for simplicity. The discussion of generalization to more than 3 alternatives is in Section 6.
[11]For every finite set X , $\triangle(X)$ denotes the set of probability measure on X.
[12]$x \cdot y$ denotes the inner product of the vector x and y.

Let P_i^{ORD} be the ordinal partition of V_i into six sets.

$$V_i^{abc} = \{v_i \in V_i \mid v_i^a \geq v_i^b \geq v_i^c\},\ V_i^{acb} = \{v_i \in V_i \mid v_i^a > v_i^c \geq v_i^b\}$$
$$V_i^{bca} = \{v_i \in V_i \mid v_i^b \geq v_i^c > v_i^a\},\ V_i^{bac} = \{v_i \in V_i \mid v_i^b > v_i^a \geq v_i^c\}$$
$$V_i^{cab} = \{v_i \in V_i \mid v_i^c \geq v_i^a > v_i^b\},\ V_i^{cba} = \{v_i \in V_i \mid v_i^c > v_i^b \geq v_i^a\}$$

Thus, P_i^{ORD} reflects the ordinal types[13] of agent i over alternatives. Let $P^{ORD} = P_1^{ORD} \times .. \times P_n^{ORD}$ be the corresponding product partition of V.

Definition 2. A SCF f is *ordinal* if it is P^{ORD}-measurable.

The set of all ordinal SCFs is denoted by F^{ORD}.

We consider (ex-ante)[14] Pareto efficiency in the class of ordinal SCFs.

Definition 3. A SCF $f \in F^{ORD}$ is *Ordinally Pareto Efficient* (OPE) if there is no $g \in F^{ORD}$ such that $U_i(g) > U_i(f)$ for every agent $i \in N$.

The following proposition is the main result of this section.

Proposition 1. *If a SCF f is OPE, then it is IC.*

Proof. Assume f is OPE. We claim that there are numbers $\{\lambda_i\}_{i\in N}$, $\lambda_i \geq 0$ such that f maximizes the social welfare$\sum_{i\in N} \lambda_i U_i(g)$ among all functions $g \in F^{ORD}$. The utility possibility set of F^{ORD} , i.e., $\left\{(U_1, ..., U_n) \in \mathbb{R}^n : \ U_i \leq U_i\left(f\right) \ for \ i = 1, ..., n, \ f \in F^{ORD}\right\}$ is convex. For convex utility possibility sets, it is well known that Pareto efficiency can equivalently be represented by maximization of linear combinations of utilities with $\{\lambda_i\}_{i\in N}$, $\lambda_i \geq 0$ and $\lambda = (\lambda_1, ..., \lambda_n) \neq 0$ [Mas-colell et al. (1995), page 560, Holmström and Myerson (1983) , page 1805]. Fix $\{\lambda_i\}_{i\in N}$, it follows that $f = arg \max\limits_{g\in F^{ORD}} \sum_{i\in N} \lambda_i U_i(g)$. Then, consider the utilitarian social welfare, $W_\lambda(f) = \sum_{i\in N} \lambda_i U_i\left(f\right)$, i.e., the weighted sum of expected utilities of all the agents. The next two lemmas are useful for the proof.

Lemma 1. *Given λ, if a SCF $f \in F^{ORD}$ is a maximizer of $W_\lambda\left(g\right)$ in F^{ORD} if and only if it satisfies*

$$Supp\left(f(v)\right) \ \subseteq \ \underset{l\in L}{argmax} \sum_{i\in N} w_i^l(v_i) \ for \ \mu - almost \ every \ v$$

[13]Since the ties have zero probability and do not affect the result of this paper, we can put ties into the sets. Thus, the usual assumption of strict preference is not necessary.
[14]From now, we skip the term "ex-ante" since we deal only with ex-ante efficiency.

where

$$w_i^l(v_i) = \lambda_i \mathbb{E}\left(\hat{v}_i^l \mid P^{ORD}\right) \in \mathbb{R},\ l \in L.$$

Proof of Lemma 1)[15] Given λ and P^{ORD} and for every $g \in F^{ORD}$ we have

$$\begin{aligned} W_\lambda(g) &= \mathbb{E}\left(\sum_{i\in N} \lambda_i \hat{v}_i \cdot g(\hat{v})\right) = \mathbb{E}\left[\mathbb{E}\left(\sum_{i\in N} \lambda_i \hat{v}_i \cdot g(\hat{v}) \middle| P^{ORD}\right)\right] \\ &= \mathbb{E}\left[g(\hat{v}) \cdot \lambda_i \mathbb{E}\left(\sum_{i\in N} \hat{v}_i \middle| P^{ORD}\right)\right] \end{aligned}$$

Thus, a SCF g is a maximizer of W_λ in F^{ORD} if and only if it satisfies

$$Supp\,(g(v)) \subseteq \underset{l\in L}{argmax}\, \lambda_i \mathbb{E}\left(\sum_{i\in N} \hat{v}_i^l \middle| P^{ORD}\right) \quad for\ \mu - almost\ every\ v$$

The above is precisely the same condition of Lemma 1.

Note that the optimal weight $w_i^l(v_i)$ depends on the welfare weight λ_i, the agent's announcement and the value distribution conditional on the ordinal partition. It is convenient to consider the optimal weight vector $w_i(v_i) = \lambda_i \mathbb{E}\left(\hat{v}_i \mid P^{ORD}\right) \in \mathbb{R}^3$. The OPE f chooses the alternative with the maximum total weight based on the sum of optimal weight vectors of all agents. The following example of f is helpful to understand the rule.

Example. Consider $\hat{v}_1^l$ is uniformly distributed on $[0,1]$ and $\hat{v}_2^l$ is uniformly distributed on $[0,2]$ for $l = \{a,b,c\}$. Given $\lambda_1 = \lambda_2 = 1$, f is a maximizer of $W_\lambda$in F^{ORD}. If agent 1 and 2 announce $v_1 \in V_1^{abc}$, $v_2 \in V_2^{bca}$, then $w_1(v_1) = \left(\frac{3}{4}, \frac{2}{4}, \frac{1}{4}\right)$, $w_2(v_2) = \left(\frac{1}{2}, \frac{3}{2}, 1\right)$ and $f = (0,1,0)$ because $w_1 + w_2 = (\frac{5}{4}, \frac{8}{4}, \frac{5}{4})$.

Let $\sigma : L \to L$ be a permutation of L and $x = \left(x^a, x^b, x^c\right) \in \mathbb{R}^3$. We denote by x^σ the vector $\left(x^{\sigma(a)}, x^{\sigma(b)}, x^{\sigma(c)}\right)$.

Lemma 2. *If $x = (x^a, x^b, x^c)$, $y = (y^a, y^b, y^c) \in \mathbb{R}^3$ are such that $x^a \geq x^b \geq x^c$ and $y^a \geq y^b \geq y^c$, then $x \cdot y \geq x \cdot y^\sigma$ for any permutation σ.*

The proof of Lemma 2 is trivial and is therefore omitted.

[15]The proof is a generalization of the first part in the proof of Proposition 1 in Azrieli and Kim (2011).

With these lemmas, we can check the incentive compatibility of f. It is sufficient to consider only $v_i \in V_i^{abc}$ instead of all ordinal types because of neutrality of the rule from i.i.d. assumption. Pick $v_i \in V_i^{abc}$, $v_i' \in V_i$. Note that $w_i(v_i') = w_i(v_i)^\sigma$ for some σ. Specifically, the i.i.d. assumption generates the same coordinates in the optimal weight vector, but different order of the coordinates according to the order of agent's value announcement. Because of this property and the way of determining alternatives of f explained above in Lemma 1, we can observe[16] $\mathbb{E}\left(f(v_i', \hat{v}_{-i})\right) = \mathbb{E}\left(f(v_i, \hat{v}_{-i})\right)^\sigma$ and $\mathbb{E}\left(f(v_i, \hat{v}_{-i})\right)^a \geq \mathbb{E}\left(f(v_i, \hat{v}_{-i})\right)^b \geq \mathbb{E}\left(f(v_i, \hat{v}_{-i})\right)^c$. Then, by Lemma 2, $v_i \cdot \left(\mathbb{E}\left(f(v_i, \hat{v}_{-i})\right) - \mathbb{E}\left(f(v_i', \hat{v}_{-i})\right)\right) \geq 0$. It means that f is IC.□

The optimal rules based on different welfare functions are of some interest. Apesteguia et al. (2011) described the optimal ordinal rules based on (purely)[17] utilitarian, maximax and maximin welfare functions under similar but stronger assumptions[18] than in this paper. The maximin welfare function evaluates an alternative in terms of the expected utility of the worst-off agent, disregarding any other expected utility, i.e., $W_{MN}(f) = min_{i \in N} U_i(f)$. In contrast to the maximin, the maximax concentrates on the best-off agent, i.e., $W_{MM}(f) = max_{i \in N} U_i(f)$. Obviously, all of their rules are ordinally Pareto efficient. Their focus was not on incentive compatibility, but Proposition 1 shows that the incentive constraints for all of their rules are satisfied anyway.

Note, as a result of Lemma1, we can find an important feature of the set of OPE rules. The set of OPE rules is in the class of asymmetric scoring rules where it allows asymmetric scores (even some zero scores) across agents. Regarding the class of rules, Apesteguia et al. (2011) showed that the optimal rule based on maximax or maximin welfare function under their environment may not be a standard scoring rule which allows only the same score across agents but the approximation of the rule is a standard scoring rule. The finding from Lemma 1 complements their analysis.

5. A superior incentive compatible cardinal rule

Can we design an IC rule which strictly Pareto dominates any ordinal rule? The main result of this section is to answer this question in the affirmative, at least in some

[16]In the above example, we can calculate that $\mathbb{E}\left(f(v_1, \hat{v}_2)\right) = \left(\frac{5}{12}, \frac{4}{12}, \frac{3}{12}\right)$. If agent 1 announces $v_1' \in V_1^{bca}$, then $\mathbb{E}\left(f(v_1', \hat{v}_2)\right) = \left(\frac{3}{12}, \frac{5}{12}, \frac{4}{12}\right)$. Note the value and order of coordinates in $\mathbb{E}\left(f(v_1, \hat{v}_2)\right)$ and $\mathbb{E}\left(f(v_1', \hat{v}_2)\right)$.

[17]The welfare weights $\lambda_i = 1$ for every $i \in N$.

[18]Specifically, Apesteguia ea al. (2011) considered only the case of identical value distribution across agents.

environments. The key is to direct our attention to cardinal rules. Consider any finite measurable partition P_i that divides V_i. Let $P = (P_1 \times ... \times P_n)$ be the corresponding partition product of V. Let F^P denote the set of P-measurable SCFs. Given a partition P, we say that a SCF $f \in F^P$ is P-Utilitarian Rule if $f \in \underset{g \in F^P}{argmax}\, W(g)$ where $W(g) = W_\lambda(g)$ with $\lambda_i = 1$ for every $i \in N$. The following proposition generalizes Lemma 1 of the previous section in more general partitions, but with the restriction of λ.[19]

Proposition 2. *A SCF f is a P-Utilitarian Rule if and only if it satisfies*

$$Supp\,(f(v)) \subseteq \underset{l \in L}{argmax} \sum_{i \in N} w_i^l(v_i) \quad for\ \mu - almost\ every\ v$$

where

$$w_i^l(v_i) = \mathbb{E}\left(\hat{v}_i^l \mid P\right) \in \mathbb{R},\ l \in L$$

The proof of this proposition is omitted because it is the same as the proof of Lemma 1 except the change of partition and λ. Proposition 2 is used in the proof of the next theorem which is the main result of this section.

Theorem 1. *Let $\underline{\theta} < \overline{\theta}$ be two numbers. Assume that for each $i \in N$ and $l \in L$, $\hat{v}_i^l$ has a uniform distribution on $\left[\underline{\theta}, \overline{\theta}\right]$. Then, there exists an IC cardinal rule that strictly Pareto dominates any ordinal rule.*

Proof. We concentrate on a uniform distribution on $[0, 1]$ because the the extended uniform distribution on $\left[\underline{\theta}, \overline{\theta}\right]$ does not affect the result.[20] The proof consists of 4 steps. Step 1 introduces a special family of partitions $\left\{P^\beta\right\}_{\beta \in (0,1)}$ and a P^β-Utilitarian Rule when $\hat{v}_i^l$ has a uniform distribution on $[0, 1]$. In Step 2, we find a condition on β for incentive compatibility of the rule. Step 3 shows that there exists a rule which satisfies the condition on β. In Step 4, we prove that the new rule strictly Pareto dominates any ordinal rule.

Step 1) Let P_i^β be a partition of each subset in P_i^{ORD} into two sets. For example, the set V_i^{abc} is partitioned into the two sets $V_i^{abc^H}(\beta)$ and $V_i^{abc^L}(\beta)$ according to the partition coefficient $\beta \in (0, 1)$.

$V_i^{abc^H}(\beta) = \{v_i \in V_i^{abc} \mid v_i^b \geq \beta v_i^a + (1 - \beta)\, v_i^c\}$

[19] We can generalize Lemma 1 without the restriction of λ. But, the restriction is simply useful for the next theorem which focuses on the comparison of rules.

[20] The extended distribution results in only the linear transform of the weight vector. See footnote 21.

$V_i^{abc^L}(\beta) = \{v_i \in V_i^{abc} \mid v_i^b < \beta v_i^a + (1-\beta)\, v_i^c\}$

Similarly, $V_i^{acb^H}, V_i^{acb^L}, ..., V_i^{cab^H}, V_i^{cab^L}$ are defined.

Let f_β be a P^β-Utilitarian Rule where ties are broken by uniform distribution over the set of maximizers. For simplicity, we use type-based notations. Consider the type set which has 12 types, $T_i = \left\{abc^H, abc^L, acb^H, acb^L, bac^H, ..., cab^L\right\}$. Also, consider a type function $t_i^\beta(v_i)$ which maps a value of each agent to the corresponding type $t_i \in T_i$. For example, $t_i^\beta(v_i) = abc^H$ if $v_i \in V_i^{abc^H}(\beta)$. Given β, let $\nu_i^\beta(t_i) = \mu_i\left(\left\{v_i : t_i^\beta(v_i) = t_i\right\}\right)$. Then, we can identify each SCF f_β with a g_β: $T \to \triangle(L)$ by $g_\beta(t) = f_\beta\left(t^\beta(v)\right)$.

Let $w^H(\beta) = \mathbb{E}\left(\hat{v}_i \mid t_i^\beta(v_i) = abc^H\right)$, $w^L(\beta) = \mathbb{E}\left(\hat{v}_i \mid t_i^\beta(v_i) = abc^L\right)$,

$P^H(\beta) = \mu_i\left(\left\{v_i : t_i^\beta(v_i) = abc^H\right\}\right)$ and $P^L(\beta) = \mu_i\left(\left\{v_i : t_i^\beta(v_i) = abc^L\right\}\right)$

When $\hat{v}_i^l$ has a uniform distribution on $[0,1]$, by simple calculation,

$w^H(\beta) = \left(\frac{3}{4}, \frac{2+\beta}{4}, \frac{1}{4}\right)$, $w^L(\beta) = \left(\frac{3}{4}, \frac{1+\beta}{4}, \frac{1}{4}\right)$, $P^H(\beta) = \frac{1-\beta}{6}$ *and* $P^L(\beta) = \frac{\beta}{6}$.[21]

Step 2) Because of the definition of P-Utilitarian Rule and i.i.d. assumption between alternatives, g_β is neutral. Hence, it is sufficient to consider only the case, $t_i^\beta(v_i) = abc^H$ and $t_i^\beta(v_i)' = abc^L$ to examine incentive compatibility.

Let $(A(\beta), B(\beta), C(\beta)) = \mathbb{E}\left(g_\beta\left(t_i, \hat{t}_{-i}\right)\right)$ *for* $t_i = abc^H$,

and $(A'(\beta), B'(\beta), C'(\beta)) = \mathbb{E}\left(g_\beta\left(t_i', \hat{t}_{-i}\right)\right)$ *for* $t_i' = abc^L$.

Denote by $h(\beta) = (A(\beta) - A'(\beta)) + \beta\,(B(\beta) - B'(\beta))$ the balance function of g_β.

Claim. If $h(\beta) = 0$, then g_β is IC.

Proof of claim) Assume $h(\beta) = 0$, for $v_i \in V_i^{abc^H}(\beta)$ and $v_i' \in V_i^{abc^L}(\beta)$, then $t_i = abc^H$ and $t_i' = abc^L$,

$$v_i \cdot \left(\mathbb{E}\left(g_\beta\left(t_i, \hat{t}_{-i}\right)\right) - \mathbb{E}\left(g_\beta\left(t_i', \hat{t}_{-i}\right)\right)\right)$$
$$= (v_i^a - v_i^c)\,(A(\beta) - A'(\beta)) + (v_i^b - v_i^c)\,(B(\beta) - B'(\beta))$$
$$\geq^{22} (v_i^a - v_i^c)\,[(A(\beta) - A'(\beta)) + \beta\,(B(\beta) - B'(\beta))]$$
$$= (v_i^a - v_i^c) h(\beta) = 0.$$

Similarly,

$$v_i' \cdot \left(\mathbb{E}\left(g_\beta\left(t_i', \hat{t}_{-i}\right)\right) - \mathbb{E}\left(f_\beta\left(t_i, \hat{t}_{-i}\right)\right)\right)$$

[21] In *a* uniform distribution on $[\underline{\theta}, \overline{\theta}]$, $w^H(\beta) = (\overline{\theta} - \underline{\theta})\left(\frac{3}{4}, \frac{2+\beta}{4}, \frac{1}{4}\right) + \underline{\theta}(1,1,1)$, $w^L(\beta) = (\overline{\theta} - \underline{\theta})\left(\frac{3}{4}, \frac{1+\beta}{4}, \frac{1}{4}\right) + \underline{\theta}(1,1,1)$.

[22] Considering $w^H(\beta)$ and $w^L(\beta)$, obviously, $B(\beta) \geq B'(\beta)$ and $A(\beta) \leq A'(\beta)$ for any $\beta \in (0,1)$.

$\geq (v_i'^a - v_i'^c)\,(-h(\beta^*)) = 0.$

Consider remaining incentive constraints regarding other ordinal type announcements. For $t_i'' = O^H,\ t_i''' = O^L$ where $O \in \{acb, bac, bca, cab, cba\}$,

$\mathbb{E}\left(g_\beta\left(t_i'', \hat{t}_{-i}\right)\right) = \mathbb{E}\left(g_\beta\left(t_i, \hat{t}_{-i}\right)\right)^\sigma$ for some σ. similarly,
$\mathbb{E}\left(g_\beta\left(t_i''', \hat{t}_{-i}\right)\right) = \mathbb{E}\left(g_\beta\left(t_i', \hat{t}_{-i}\right)\right)^\sigma$ for some σ.

Note, still, $A(\beta) \geq B(\beta) \geq C(\beta)$ and $A'(\beta) \geq B'(\beta) \geq C'(\beta)$.
By Lemma 2 of the previous section and the above argument,

$v_i \cdot \mathbb{E}\left(g_\beta\left(t_i, \hat{t}_{-i}\right)\right) \geq v_i \cdot \mathbb{E}\left(g_\beta\left(t_i'', \hat{t}_{-i}\right)\right),$
$v_i \cdot \mathbb{E}\left(g_\beta\left(t_i, \hat{t}_{-i}\right)\right) \geq v_i \cdot \mathbb{E}\left(g_\beta\left(t_i', \hat{t}_{-i}\right)\right) \geq v_i \cdot \mathbb{E}\left(g_\beta\left(t_i''', \hat{t}_{-i}\right)\right)$, and
$v_i' \cdot \mathbb{E}\left(g_\beta\left(t_i', \hat{t}_{-i}\right)\right) \geq v_i' \cdot \mathbb{E}\left(g_\beta\left(t_i''', \hat{t}_{-i}\right)\right),$
$v_i' \cdot \mathbb{E}\left(g_\beta\left(t_i', \hat{t}_{-i}\right)\right) \geq v_i' \cdot \mathbb{E}\left(g_\beta\left(t_i, \hat{t}_{-i}\right)\right) \geq v_i' \cdot \mathbb{E}\left(g_\beta\left(t_i'', \hat{t}_{-i}\right)\right).$

Step 3) We proceed with a series of claims about $h(\beta)$ to show the existence of an IC cardinal rule. For ease of notation, we use a weight vector, $w_i(\beta, t_i) = \mathbb{E}\left(\hat{v}_i \mid t_i^\beta(v_i) = t_i\right)$.

Claim 1. $\lim_{\beta\to 0} h(\beta) < 0$ and $\lim_{\beta\to 1} h(\beta) > 0$.

Proof of claim 1) First, we observe that $A(\beta) - A'(\beta) \leq 0$ and $B(\beta) - B'(\beta) \geq 0$ for any $\beta \in (0,1)$ because $w_i(\beta, t_i) - w_i(\beta, t_i') = \left(0, \frac{1}{4}, 0\right)$. In other words, the change of one's weights from H type to L type at fixed others' type profile weakly increases the probability that a or c is chosen and decreases the probability that b is chosen.

For $\lim_{\beta\to 0} h(\beta) = \lim_{\beta\to 0} A(\beta) - A'(\beta) < 0$, we can always find the case that $g_\beta(t_i, t_{-i}) = (\frac{1}{2}, \frac{1}{2}, 0)$ or $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ when every one's type is H type.[23] If only the type of agent i is changed to L type, then $g_\beta(t_i', t_{-i}) = (1,0,0)$ or $\left(\frac{1}{2}, 0, \frac{1}{2}\right)$. Since every other types are H types ($\lim_{\beta\to 0} P^H(\beta) = \frac{1}{6}$), $\lim_{\beta\to 0} A(\beta) - A'(\beta) < 0$

For $\lim_{\beta\to 1} h(\beta) = \lim_{\beta\to 1} C'(\beta) - C(\beta) > 0$,[24] Similarly, we can always find the case that $g_\beta(t_i', t_{-i}) = (0, \frac{1}{2}, \frac{1}{2})$ or $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ when every one's type is L type. If only the type of agent i is changed to H type, then $g_\beta(t_i, t_{-i}) = (0,1,0)$. Since every other types are L types ($\lim_{\beta\to 1} P^L(\beta) = \frac{1}{6}$), $\lim_{\beta\to 1} C'(\beta) - C(\beta) > 0$.

With Claim 1, if $h(\beta)$ is continuous on $(0,1)$, we can easily find a β^* such that $h(\beta^*) = 0$ by the intermediate value theorem. Then, g_{β^*} is an IC cardinal rule, $f^*\left(t^{\beta^*}(v)\right) = g_{\beta^*}(t)$.

[23] For example, for $n = 2$, $t_1 = abc^H$ and $t_2 = bac^H$. For $n = 3$, $t_1 = abc^H$, $t_2 = bca^H$ and $t_3 = cab^H$. Generally, for any n, we can combine above two profiles to find the case.

[24] $h(\beta) = (A(\beta) - A'(\beta)) + \beta\,(B(\beta) - B'(\beta)) = (1-\beta)\,(A(\beta) - A'(\beta)) + \beta\,(C'(\beta) - C(\beta))$

However, there may not exist a β^* such that $h(\beta^*) = 0$ because of the possible discontinuity of $h(\beta)$. Given $t_i = abc^H$, a decision of $g_\beta(t_i, t_{-i})$ at each t_{-i} depends on sum of weight vectors. Let $W^H(\beta, t_{-i}) = \left((w^H(\beta)) + \sum_{j\neq i} w_j(\beta, t_j)\right)$ be the sum of weight vectors. We can express $A(\beta)$ with this function.

$$
\begin{aligned}
A(\beta) \quad = \quad & \nu_{-i}^{\beta}\left(\left\{t_{-i} : W^H(\beta, t_{-i})^a > W^H(\beta, t_{-i})^b \;\; and \; W^H(\beta, t_{-i})^c\right\}\right) + \\
& \frac{1}{2}\nu_{-i}^{\beta}\left(\left\{t_{-i} : \; W^H(\beta, t_{-i})^a = W^H(\beta, t_{-i})^b > W^H(\beta, t_{-i})^c\right\}\right) + \\
& \frac{1}{2}\nu_{-i}^{\beta}\left(\left\{t_{-i} : \; W^H(\beta, t_{-i})^a = W^H(\beta, t_{-i})^c > W^H(\beta, t_{-i})^b\right\}\right) + \\
& \frac{1}{3}\nu_{-i}^{\beta}\left(\left\{t_{-i} : \; W^H(\beta, t_{-i})^a = W^H(\beta, t_{-i})^b = W^H(\beta, t_{-i})^c\right\}\right)
\end{aligned}
$$

$B(\beta)$ and $C(\beta)$ can be similarly expressed. Also, $A(\beta)', B(\beta)'$and $C(\beta)'$ can be expressed with $W^L(\beta, t_{-i})$ and $w^L(\beta)$.

Note that the decision of the rule g_β is determined by the value order of $W^H(\beta, t_{-i})^a$, $W^H(\beta, t_{-i})^b$ and $W^H(\beta, t_{-i})^c$ at each t_{-i}. If there is no change of decision of the rule on any t_{-i} as β is changed, then $A(\beta), B(\beta)$ and $C(\beta)$ are continuous in β because $P^H(\beta)$, $P^L(\beta)$ and $w_i(\beta, t_i)$ are continuous in β. However, as β is changed, the value order of coordinates of $W^H(\beta, t_{-i})$ can be changed as well. We can observe in the above expression of $A(\beta)$ that this feature can cause a jump of $A(\beta)$ at some βs, which means the possible discontinuity of $A(\beta)$. Eventually, it could generate discontinuity of $h(\beta)$.

Even in this case, however, we can construct a slightly different rule which becomes an IC cardinal rule. Define the set $D = \{\beta \in (0,1) : \; h(\beta) \; is \, discontinuous \, at \, \beta\}$.

Claim 2. The set D is finite.

Proof of Claim 2) By the above argument, the only candidates of elements in D are some βs that change value order of coordinates of $W^H(\beta, t_{-i})$ or $W^L(\beta, t_{-i})$. The sum of weight function $W^H(\beta, t_{-i})^l$ is increasing linear function in β for $l \in \{a, b, c\}$. Hence, fixed t_{-i}, some of the three functions are identical or meet at most three times in $\beta \in (0,1)$. The similar argument applies to $W^L(\beta, t_{-i})$ and for every finite t_{-i}. Hence, D is finite.

Claim 3. If there does not exist a β such that $h(\beta) = 0$, then there is a $\hat{\beta} \in D$ such that $\underset{\beta\to\hat{\beta}-}{h(\beta)} \cdot \underset{\beta\to\hat{\beta}+}{h(\beta)} \leq 0$.

Proof of Claim 3) Suppose not (for all $\hat{\beta} \in D$, $\underset{\beta\to\hat{\beta}-}{h(\beta)} \cdot \underset{\beta\to\hat{\beta}+}{h(\beta)} > 0$), we have two cases. The first case is that there exists $\hat{\beta}_1, \hat{\beta}_2 \in D$ such that $\hat{\beta}_1 < \hat{\beta}_2$ and $\underset{\beta\to\hat{\beta}_1+}{h(\beta)} \cdot \underset{\beta\to\hat{\beta}_2-}{h(\beta)} < 0$. By Claim 2 and the intermediate value theorem, there is a $\beta \in (\hat{\beta}_1, \hat{\beta}_2)$ such that $h(\beta) = 0$, which contradicts the assumption of non-existence of such β. The second case is that for any $\hat{\beta}_1, \hat{\beta}_2$, $\underset{\beta\to\hat{\beta}_1+}{h(\beta)} \cdot \underset{\beta\to\hat{\beta}_2-}{h(\beta)} > 0$. By Claim 1 and 2, $h(\beta)$ is continuous on some$\left(0, \hat{\beta}_1\right)$ for $\underset{\beta\to\hat{\beta}_1-}{h(\beta)} > 0$ or on some $\left(\hat{\beta}_2, 1\right)$ for $\underset{\beta\to\hat{\beta}_2+}{h(\beta)} < 0$. By Claim 1 and the intermediate value theorem, there is a $\beta \in (0,1)$ such that $h(\beta) = 0$.

Claim 3 provides a way to design an IC cardinal rule with $\hat{\beta}$ by the appropriate convex combination of two rules; the balance function at $\hat{\beta}$ of one rule is $\underset{\beta\to\hat{\beta}+}{h(\beta)}$ and the other is $\underset{\beta\to\hat{\beta}-}{h(\beta)}$. By Claim 3, fix$\hat{\beta}$, we can design a rule, $g^+(t)$ based on fixed $P^{\hat{\beta}}$ but with a different weight vector, $w_i\left(\hat{\beta}+\epsilon, t_i\right)$. With sufficiently small $\epsilon > 0$ such that $\hat{\beta}+\epsilon \notin D$, the decisions of $g^+(t)$ are different from those of $g_{\hat{\beta}}(t)$ only in some of tie cases of $g_{\hat{\beta}}(t)$. That is because any $\hat{\beta} \in D$ involves tie cases where $W^H\left(\hat{\beta}, t_{-i}\right)^l = W^H\left(\hat{\beta}, t_{-i}\right)^{l'}$ or $W^L\left(\hat{\beta}, t_{-i}\right)^l = W^L\left(\hat{\beta}, t_{-i}\right)^{l'}$ *for* $l \neq l' \in K$. Also, with the same $P^{\hat{\beta}}$, we can obtain the balance function of $g^+(t)$, $h^+\left(\hat{\beta}\right) = \underset{\beta\to\hat{\beta}+}{h(\beta)}$. Because of the same partition $P^{\hat{\beta}}$ and the different decisions only in tie cases of $g_{\hat{\beta}}(t)$, $g^+(t)$ is still a maximizer of W_λ in $F^{P^{\hat{\beta}}}$. Similar but oppositely, we can design a rule, $g^-(t)$ based on $P^{\hat{\beta}}$ but with a $w_i\left(\hat{\beta}-\epsilon, t_i\right)$ and $h^-\left(\hat{\beta}\right) = \underset{\beta\to\hat{\beta}-}{h(\beta)}$.

Finally, consider a rule $\tilde{g}_\alpha = \alpha g^+ + (1-\alpha) g^-$, where[25] $\alpha = \frac{-h^-\left(\hat{\beta}\right)}{h^+\left(\hat{\beta}\right) - h^-\left(\hat{\beta}\right)} \in [0,1]$ such that the balance function of $\tilde{g}_\alpha$, $\tilde{h}_\alpha\left(\hat{\beta}\right) = 0$. Then, $\tilde{g}_\alpha$ is an IC cardinal rule. In order to compare with ordinal rules, we will use the identical function, $f^*\left(t^{\hat{\beta}}(v)\right) = \tilde{g}_\alpha(t)$ in the next step.

Step 4) Let $f_{OUR} \in F^{ORD}$ be a P^{ORD}-Utilitarian Rule. We prove the strict Pareto dominance of f^* over f_{OUR} by showing that f^* has a higher social welfare than f_{OUR} because the identical value distribution across agents makes $W(f) = NU_i(f)$. From the way of construction of f^* (finer partition than f_{OUR}), $W(f^*) \geq W(f_{OUR})$ (equality holds when $f^*(v) = f_{OUR}(v)$ for μ-almost every $v \in V$). However, we can simply find a set of v (non-zero measure) such that $f^*(v) \neq f_{OUR}(v)$ if we see the cases in the proof of Claim 1. In those cases where only the type of agent i is H type or L type, each $f^*(v)$ is

[25]Recall $h^+\left(\hat{\beta}\right) \cdot h^-\left(\hat{\beta}\right) \leq 0$ from Claim 3

different at v and v' while $f_{OUR}(v)$ is the same because v and v' are in the same ordinal type set in P^{ORD}. These cases guarantee the difference of social welfare between f^* and f_{OUR}, it follows that $W(f^*) > W(f_{OUR})$. □

6. Discussion

6.1. **Connection to (A, B)-scoring rule.** The next proposition connects the new rule f^* to the well-known rules: *plurality, negative, Borda and approval* voting rules. As in Myerson (2002), the general form of those voting rules for three candidates is the (A, B)-scoring rule, where each voter must choose a score that is a permutation of either $(1, B, 0)$ or $(1, A, 0)$. That is, the voter can give a maximum of 1 point to one candidate, A or B $(0 \leq A \leq B \leq 1)$ to some other candidate and a minimum of 0 to the remaining candidate. The specific (A, B)-scoring rules are widely used in practice and theory. The case (A, B)=$(0, 0)$ is *plurality* voting rule, where each voter can support a single candidate. The case, (A, B)=$(1, 1)$ is *negative* voting rule, where each voter can oppose a single candidate. The case (A, B)=$(0.5, 0.5)$ is *Borda* voting rule, where each voter can give candidates a completely ranked score. These rules can be classified to ordinal rules because the information about ordinal preference is sufficient to implement the rules. However, (A, B)=$(0, 1)$, *approval* voting rule where each voter can support or oppose a group of candidate requires more than the information about ordinal preference. This property of the approval voting is similar to P^{β}-Utilitarian Rule and is one of reasons why approval voting can be more efficient than other ordinal rules, which is close to Theorem 1. The following proposition clearly shows the relationship between P^{β}- Utilitarian Rules and (A, B)-scoring rules.

Proposition 3. *Under the same assumption in Theorem 1, a rule is a P^{β}- Utilitarian Rule if and only if it is a (A, B) −scoring rule with $A = \frac{\beta}{2}$ and $B = \frac{1+\beta}{2}$.*[26]

[26]Proposition 3 implies that a (A, B)-scoring rule with $A = B - \frac{1}{2}$ is a P^{β}-Utilitarian Rule. It guarantees the efficiency of the rule. However, it may be more important to find an IC (A, B)-scoring rule among efficient (A, B)-scoring rules. In this sense, Theorem 1 is useful. Consider Step 3 in the proof of Theorem 1. In the case of continuous $h(\beta)$, the new rule f^* is obviously IC (A, B)-scoring rule with the normalization of the weight vector. But, in the other case, the new rule f^* does not look like a (A, B)-scoring rule because of the convex combination of two rules. However, the new rule f^* has the same decisions of $f_{\hat{\beta}}$ except some of tie cases. It implies that the new rule is still an IC (A, B)-scoring rule with $A = \frac{\hat{\beta}}{2}$, $B = \frac{1+\hat{\beta}}{2}$ but with a different tie breaking rule from $f_{\hat{\beta}}$.

Proof. Let f_β be a P^β-Utilitarian Rule with $\lambda_i = 1$ for all $i \in N$.[27]Since we already discussed the argument between a uniform distribution $[\underline{\theta}, \overline{\theta}]$ and $[0,1]$, it is sufficient to show the connection between f_β in a uniform distribution on $[0,1]$ and a (A,B)-scoring rule. Simply, the weight vectors $w^L(\beta) = \left(\frac{3}{4}, \frac{1+\beta}{4}, \frac{1}{4}\right)$ and $w^H(\beta) = \left(\frac{3}{4}, \frac{2+\beta}{4}, \frac{1}{4}\right)$ can be normalized to $(1, A, 0) = \left(1, \frac{\beta}{2}, 0\right)$ and $(1, B, 0) = \left(1, \frac{1+\beta}{2}, 0\right)$ because the coordinates of $w^L(\beta)$ and $w^H(\beta)$ are the same except $w^L(\beta)^b$ and $w^H(\beta)^b$. This normalization does not affect the decision of f_β. Thus f_β with the normalized weight is a (A,B)-scoring rule with $A = \frac{\beta}{2}$ and $B = \frac{1+\beta}{2}$. With this normalization, the proof of converse is straightforward and is therefore omitted. □

6.2. **Generalization to more than 3 alternatives.** With more than 3 alternatives, the argument for Proposition 1 does not change. However, for Theorem 1, we need a simple modification. For $i \in N$, let $r_k(v_i)$ denote the kth ranked alternative in v_i, $1 \leq k \leq\mid L \mid$ and, for simple notations, let $v_i^{[k]} = v_i^{r_k(v_i)}$ denote the value of the kth ranked alternative in v_i. Denote by $\bar{t}_i(v_i) = r_1(v_i) r_2(v_i) ... r_{|L|}(v_i)$ the ordered type function of agent i. Let P_i^β be a partition of each subset in P_i^{ORD} into two sets.

$$V^{\bar{t}_i^H}(\beta) = \{v_i \in V^{\bar{t}_i} \mid v_i^{[2]} \geq \beta v_i^{[1]} + (1-\beta) v_i^{[3]}\}$$
$$V_i^{\bar{t}_i^L}(\beta) = \{v_i \in V^{\bar{t}_i} \mid v_i^{[2]} < \beta v_i^{[1]} + (1-\beta) v_i^{[3]}\}$$

where $\beta \in (0,1)$ is a partition coefficient.

Also, consider a modified rule $\bar{f}_\beta$:

$$\bar{f}_\beta(v) = \begin{cases} f_\beta(v) & if\ Supp(f_\beta(v)) \subseteq \bigcap_{i \in N} \{r_1(v_i) r_2(v_i), r_3(v_i)\} \\ f_{OUR}(v) & otherwise \end{cases}$$

Let $\left(A_1(\beta), A_2(\beta), ..., A_{|L|}(\beta)\right) = \mathbb{E}\left(\bar{f}_\beta(v_i, \hat{v}_{-i})\right)\ for\ v_i \in V^{t_i^{\bar{H}}}$,
and $\left(A'_1(\beta), A'_2(\beta), ..., A'_{|L|}(\beta)\right) = \mathbb{E}\left(\bar{f}_\beta(v'_i, \hat{v}_{-i})\right)\ for\ v'_i \in V^{t_i^{\bar{L}}}$.
Note, by the property of $\bar{f}_\beta$, $A_k(\beta) = A'_k(\beta)$ for $k \geq 4$.
From the incentive constraint, we can derive the balance function of $\bar{f}_\beta$.

For $v_i \in V^{t_i^{\bar{H}}}$ and $v'_i \in V^{t_i^{\bar{L}}}$,

$$\begin{aligned} & v_i \cdot \left(\mathbb{E}\left(\bar{f}_\beta(v_i, \hat{v}_{-i})\right) - \mathbb{E}\left(\bar{f}_\beta(v'_i, \hat{v}_{-i})\right)\right) \\ &= v_i^{[1]}\left(A_1(\beta) - A'_1(\beta)\right) + v_i^{[2]}\left(A_2(\beta) - A'_2(\beta)\right) + v_i^{[2]}\left(A_3(\beta) - A'_3(\beta)\right) \\ &= \left(v_i^{[1]} - v_i^{[3]}\right)\left[\left(A_1(\beta) - A'_1(\beta)\right) + \beta\left(A_2(\beta) - A'_2(\beta)\right)\right] \end{aligned}$$

[27]Here, we do not need a restriction of tie breaking rule.

We get $\bar{h}_\beta = [(A_1(\beta) - A_1'(\beta)) + \beta\,(A_2(\beta) - A_2'(\beta))]$. Then, for incentive compatibility part, we can find a rule with similar arguments in the proof of Theorem 1. For efficiency part, the new rule f^* derived from $\bar{f}_\beta\,(v)$ is not a maximizer of W in F^{P^β}. However, still $W\,(f^*) > W\,(f_{OUR})$ because of the definition of $\bar{f}_\beta\,(v)$.

6.3. **Possible extension of Theorem 1 in more general distributions.** The extension of Theorem 1 in more general distributions is desirable because the assumption of a uniform distribution of values is restrictive. Even though the assumption is restrictive, the method to design a new rule in the proof of Theorem 1 is not restrictive. The key idea of the method is to use finer partitions than the ordinal partition and find a special partition for incentive compatibility. Given a different value distribution, we can apply the method to design an IC cardinal rule superior to any ordinal rule because the assumption is a sufficient condition for Theorem 1.[28] Furthermore, an extension of Theorem 1 is possible:

Assume $N \geq 3$, if v_i^l has a distribution such as

$$\lim_{\beta\to 0}\left(w^L(\beta)^a - w^L(\beta)^b\right) > \lim_{\beta\to 0}\left(w^H(\beta)^a - w^H(\beta)^b\right) \text{ or}$$
$$\lim_{\beta\to 1}\left(w^H(\beta)^b - w^H(\beta)^c\right) > \lim_{\beta\to 1}\left(w^L(\beta)^b - w^L(\beta)^c\right),$$

then there exists an IC cardinal rule which strictly Pareto dominates any ordinal rule.

There are numerous distributions which satisfy the above condition of value distribution.[29] However, the problem for the formal extension of Theorem 1 in more general distribution is that it is not simple for me to characterize an intuitive family of distribution which satisfies the condition.

6.4. **Application to other fields.** The method in the proof of Theorem 1 can be also applicable to other fields such as random assignment and matching field. This method based on the partition approach involves an assumption of cardinal preferences and a Bayesian environment, which is, however, surprisingly rare in those fields. Similarly to Theorem 1, the method can be used to design a cardinal mechanism superior to ordinal mechanisms which are widely used and studied in those fields. It remains a possible direction of future research.

[28]Examples and the proof of the following extension are available upon request.

[29]Since the weight vector is directly related to the value distribution, the above condition is a fundamental condition of value distribution.

References

[1] Apesteguia, J., Ballester, M.A. and Ferrer, R. (2011), On the justice of decision rules, *Review of economic studies* 78(1), 1-16. Miralles, A. (2011), Cardinal Bayesian allocation mechanisms without transfers, *Journal of Economic Theory* , doi:10.1016/j.jet.2011.11.002

[2] Azrieli, Y. and Kim, S. (2011), Pareto efficiency and weighted majority rules, working paper.

[3] Barberà, S. and Jackson, M. O. (2006), On the weights of nations: Assiging voting weights in a heterogeneous union, *The Journal of Political Economy* **114**, 317-339.

[4] Börgers, T. and Postl, P. (2009), Efficient compromising, *Journal of Economic Theory* **144**, 2057-2076.

[5] Holmström, B., and Myerson, R. B. (1983), Efficient and durable decision rules with incomplete information, *Econometrica,* **51**, 1799-1819.

[6] Gibbard, A. (1973), Manipulation of Voting Schemes: A General Result, *Econometrica,* **41**, 587-601.

[7] Jackson, M. O. and Sonnenschein, H.F. (2007), Overcoming incentive constraints by linking decisions, *Econometrica* **75**, 241-257.

[8] Majumdar, D. and Sen, A. (2004), Ordinally Bayesian incentive compatible voting rules. *Econometrica*, 72(2), 523-540.

[9] Mas-colell, A., Whinston, M. D. and Green, J. R. (1995), Microeconomic Theory, Oxford University Press.

[10] Miralles, A. (2011), Cardinal Bayesian allocation mechanisms without transfers, *Journal of Economic Theory* , doi:10.1016/j.jet.2011.11.002

[11] . Myerson, R. B. (2002), Comparison of Scoring Rules in Poisson Voting Games, *Journal of Economic Theory*, 103 (1), 219-251.

[12] Satterthwaite, M. A. (1975), Strategy-proofness and arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions, *Journal of Economic Theory*, 10(2), 187-217.

[13] Schmitz, P. W. and Tröger, T. (2011), The (sub-)optimality of the majority rule, *Games and Economic Behavior*, forthcoming.