A Complete Geometric Representation of Four-Player Weighted Voting Systems

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Abstract

The relatively new weighted voting theory applies to many important organizations such as the United States Electoral College and the International Monetary Fund. Various power indexes are used to establish a relationship between weights and influence; in 1965, the Banzhaf Power Index was used to show that areas of Nassau County were unrepresented in the county legislature. It is of interest to enumerate weighted voting systems, analyze paradoxes, and solve the “inverse problem” of constructing a voting system from a desired power distribution. These problems are usually addressed using the standard algebraic representation of weighted voting games consisting of a weight vector and a quota. Other ways of representing weighted voting games do exist, such as the set of minimum winning coalitions, an idea addressed in several papers. A newer idea, however, is the geometric representation. This representation contains all possible normalized $n$-player weighted voting games in a $(n - 1)$-simplex and thus acts as a complete representation of weighted voting games. The concept of the region, a portion of the simplex producing characteristically identical weighted voting systems, may greatly simplify analysis of weighted voting games. In this paper, four-player weighted voting games are completely solved using the geometric representation. The geometric representation will be shown to be a useful alternative to the algebraic representation.
1 Introduction

Weighted voting systems form the basis of many organizations, such as corporations, the International Monetary Fund, and the United States Electoral College. Hence, their analysis and exploration has many applications. A branch of game theory, known as voting power theory, includes the analysis of weighted voting systems. Most typically, \( n \)-player weighted voting systems are represented algebraically by an ordered \((n+1)\)-tuple consisting of a quota and the weights of the players. Though a player with more weight will never have less power than a player with less weight, the details of the power distribution created by weighted voting systems are best measured by the various power indices. Interestingly, very few indices can calculate the power distribution directly from the algebraic representation: usually, it is first converted into some other representation.

The algebraic representation, however, can be used to analyze weighted voting games in various other ways. Laruelle and Valenciano (2005) attempted to elucidate paradoxes of voting power through algebraic manipulation. They explain various counterintuitive results by exposing the algebra behind the manipulations which lead to them. Tolle (2003) produced a full analysis of four-player weighted voting games, though he only considered schemes where no player holds veto power. In his paper, he poses several questions. Cuttler et. al. (2005), Elkind et. al. (2009), and Keijzer et. al. (2010) all attempted to find an algorithm which generates feasible weighted voting systems, a discovery which would be able to answer Tolle’s “inverse problem”: given a desired power distribution, can a weighted voting system resulting in the desired power distribution be created? The algebraic representation of weighted voting games is concise and easy to display, so it is useful in analysis.

All reasonable power indices, however, will reveal that multiple weighted voting systems with different algebraic representations will result in the same power distribution. Besides explaining why most power indices cannot calculate voting power directly from the weight vector, this fact also reveals that many weighted voting systems are, in fact, equivalent because they generate the same set of winning and losing coalitions. Some papers such as Kirsch and Langner (2009) and Bishnu and Roy (2009) recognized that even more basic than the sets of winning and losing coalitions is the set of minimum winning coalitions. These two papers sought to find ways to directly calculate the power vector or rank the players in a weighted voting game using only the set of minimum winning coalitions, a representation of weighted voting games that is more revealing than the algebraic representation.

Other papers sought to use a geometric representation. Different papers such as Kirstein (2009) and Brown et. al. (2009) used geometric analysis to supplement their algebraic manipulations. The former focused on three-player scenarios to practically apply voting power theory to a dispute. The latter was a response to Tolle (2003), seeking to find an upper bound on the number of feasible power distributions in \( n \)-player weighted voting games. Though primarily employing algebraic techniques, the two papers also included geometric representations of weighted voting games.

One paper which employed a geometric representation of weighted voting games as its
primary tool was Jones (2009). Exploring the same topic as Laruelle and Valenciano (2005), Jones (2009) employed geometric analysis to shed light on paradoxes of voting power from a new angle. Though concerned primarily with three-player games, his representation of weighted voting games was very versatile: it can be generalized for any number of players and its nonexploitation of symmetry makes it a complete portrayal of n-player weighted voting games.

In section 2, this paper will present the geometric representation of weighted voting games, focusing on the concept of the region in section 3. Possible advantages over other representations of weighted voting games will be discussed in section 4. In sections 5 and 6, the geometric representation is used to completely present three player and four player weighted voting systems, respectively, with more detail about four player weighted voting systems in section 7.

2 Definitions and the Geometric Representation

The weighted voting games in this paper are games in which a set of players, \( S \), vote on a binary decision. Those in favor of the decision will form a coalition \( C \in \mathcal{P}(S) \), where \( \mathcal{P}(S) \) denotes the power set of \( S \). Certain coalitions will win, while others will lose. The set of winning coalitions is denoted \( W \). If \( C \in W \), then \( C \) is a winning coalition. Otherwise, \( C \) is losing.

Weighted voting games can be simple and/or proper. For this paper, only games which are both simple and proper will be considered. A voting game is simple if the following conditions hold:

1. \( \emptyset \notin W \)
2. \( S \in W \)
3. If \( A \subseteq B \) and \( A \in W \), then \( B \in W \)

In words, in a simple game, the null coalition of no players is losing, the grand coalition of all players is winning, and any superset of a coalition is winning (satisfying superadditivity). A voting game is proper if:

1. \( A \in W \rightarrow S/A \notin W \)

A simple proper voting game satisfies the condition that no two disjoint coalitions can be winning at once, a property important for practical applications.

Most often, a weighted voting system is implemented using a system consisting of a quota and a set of weights assigned to players. The weight of a coalition is equal to the sum of the weights of its members. A coalition is winning if and only if its weight is not less than the quota. An \( n \)-player weighted voting game is typically represented by \( [q; w_1, w_2, \ldots, w_n] \), where \( w_i, 1 \leq i \leq n \), is the weight of player \( i \) and \( q \) is the quota.
Much of voting power theory is devoted to the measuring of voting power; the power of player $i$, $p_i \in [0,1]$ reflects the influence player $i$ has over the outcome. A player with a voting power of 0 is a dummy with no influence over the outcome. A player with a voting power of 1 is a dictator whose sole decision determines the outcome.

Each voting power index, however, defines voting power using different parameters, and so these indices will almost always disagree on the ratio of power between pairs of players and sometimes even the power hierarchy of players. This paper will use the Banzhaf power index, introduced by Banzhaf in 1965.

The Banzhaf power index equates power with swing votes – if a coalition $C$ containing player $i$ is winning, yet coalition $C/\text{slash.}i$ is losing, then player $i$ is said to have a swing vote.

The only characteristic of weighted voting games significant in finding a player’s Banzhaf power is the set of minimum winning coalitions, since this set determines the distribution of swing votes. Hence, any modification to a weighted voting game that maintains the set of minimum winning coalitions will not change the power distribution – in fact, the two games will function identically. An infinite number of algebraic representations of weighted voting games will result in the same minimum winning coalitions, so all of those games are equivalent. This suggests that only a finite number of possible “different” weighted voting games exist for a finite number of players – a true statement. Several papers such as Brown et. al. (2009) have tried to find an upper bound for this number for finite values of $n$.

**Theorem:** There are only a finite number of possible “different” weighed voting games for $n$ players.

**Proof Sketch:** A game is defined by its set of minimum winning coalitions. Since the number of players is finite, the number of possible coalitions, $2^n$, is also finite. The number of sets of coalitions is also finite, $2^{2^n}$, which is a (very generous) upper bound on the number of possible sets of minimum winning coalitions. Since each distinct weighted voting game corresponds to one set of minimum winning coalitions, the number of distinct weighted voting games is also finite for an $n$-player weighted voting game for any finite $n$.

To decrease the number of possible algebraic representations while maintaining full flexibility, it becomes necessary to normalize the weighted voting game. Since the weighted voting games $[q;w_1,w_2,\ldots,w_i]$ and $[aq;aw_1,aw_2,\ldots,aw_i]$ are functionally identical for any positive $a$, it makes sense to make no distinction between these voting systems and pick one to represent them all. The normalization will scale the weights and quota so that the sum of the weights is 1 and the quota is in the interval $(\frac{1}{2},1]$.

From here, it is possible to convert a weighted voting game into its geometric representation. In the Geometric Representation of weighted voting systems with $n$-players, an $(n-1)$-simplex with unit height is used to represent all possible distributions of weights. The quota is assumed to be fixed at some value $q \in (\frac{1}{2},1]$. There will be a bijective mapping from all possible normalized weighted voting games with $n$ players and quota $q$ to points
contained on and in the \((n - 1)\)-simplex.

Now consider the \((n - 1)\)-simplex with unit height. By definition, the \((n - 1)\)-simplex has \(n\) faces of \((n - 2)\)-dimensions, which can be enumerated with the integers 1 through \(n\), each face representing one of the \(n\) players. A chosen point represents the weighted voting game where \(w_i\) (the normalized value) is equal to the length of the perpendicular segment from the chosen point to the face labeled with the integer \(i\). If a point is on a particular face \(i\), then that point represents a weighted voting game where \(w_i = 0\). The sum of the perpendicular segments from any point in the simplex with unit height to the faces is equal to 1, the sum of all the weights, as proven in theorem 1.

**Theorem 1:** The sum of the perpendicular segments from any point on the \(n\)-simplex with unit height to its \((n - 1)\)-hyperfaces is equal to 1.

**Proof:** Let the hypervolume (the concept analogous to volume for higher dimensions) of the \(n\)-simplex be \(V\). An \(n\)-simplex is essentially a regular \(n\)-dimensional pyramid. Then let the formula for calculating the hypervolume of the \(n\)-pyramid be expressed in the form \(CBh\), where \(C\) is some constant, \(B\) is the hypervolume of the \((n - 1)\)-dimensional base, and \(h\) is the height of the pyramid. Then let the hypervolume of each \((n - 1)\)-hyperface of the \(n\)-simplex be equal to \(S\). Then the hypervolume of the \(n\)-simplex as calculated by treating the entire simplex as one pyramid is \(V = CS(1) = CS\). However, the hypervolume of the \(n\)-simplex can also be calculated from an arbitrarily chosen point within the simplex. Draw lines extending from the arbitrarily chosen point to each vertex of the \(n\)-simplex, as well as the perpendicular segments (heights) from the chosen point to each \((n - 1)\)-hyperface. Then the \(n\)-simplex is divided into \(n\) smaller pyramids, each with a different hyperface as its base and the chosen point as the vertex. Let the length of the perpendicular segments be equal to \(a_1, a_2, \ldots, a_n\). Then the hypervolume of the \(n\)-simplex is calculated by summing the hypervolumes of each smaller pyramid.

\[
V = CSA_1 + CSA_2 + \cdots + CSA_n = CS(a_1 + a_2 + \cdots + a_n)
\]

Yet \(V = CS\), so it follows that \(a_1 + a_2 + \cdots + a_n = 1\).

The geometric representation of a weighted voting game displays it as a single point in a continuous space. Some change in weights or the quota will not change a weighted voting game since the set of minimum winning coalitions will remain unchanged. Naturally, a question arises: how much can a weighted voting system be altered without changing the set of minimum winning coalitions? To answer this question, the concept of the *region* is introduced.

### 3 The Properties of Regions

The most convenient property of the geometric representation of weighted voting games, besides its completeness, is the presence of the concept of the *region*. A *region* is defined to
be a set of points in the simplex – or equivalently a set of normalized weighted voting games – which return the same set of minimum winning coalitions, and thus are equivalent. A point is defined to be “in a region” if it returns the same set of minimum winning coalitions which characterize that region. Due to its properties, the region provides an extremely visual method for the analysis of weighted voting games. Properties of the region are proven in theorems 2, 3, 4, and 5.

**Theorem 2**: Regions are always convex. This means that if two points are in a region, then all points lying on the line segment with those two points as endpoints are also present in the region.

**Proof**: This proof requires a very useful lemma.

**Lemma 1**: If two weighted voting games in the same region with quotas \( a \) and \( b \) are \([a; w_{a1}, w_{a2} \ldots]\) and \([b; w_{b1}, w_{b2} \ldots]\), the following weighted voting game is also in the same region: \([af + b(1 - f); w_{a1}f + w_{b1}(1 - f), w_{a2}f + w_{b2}(1 - f) \ldots]\), where \(0 \leq f \leq 1\).

**Proof**: Examine an arbitrary winning coalition. Without loss of generality, let it be the coalition consisting of players 1, 2... and \( n \) (this can be done by permuting the players). By assumption,

\[
w_{a1} + w_{a2} + \cdots + w_{an} \geq a
\]

and

\[
w_{b1} + w_{b2} + \cdots + w_{bn} \geq b
\]

By multiplying the first inequality by \( f \) and the second by \( (1 - f) \), and then regrouping terms, the following is achieved:

\[
[w_{a1}f + w_{b1}(1 - f)] + [w_{a2}f + w_{b2}(1 - f)] + \ldots [w_{an}f + w_{bn}(1 - f)] \geq af + b(1 - f)
\]

This demonstrates that any coalition which is originally winning will remain winning. Similarly, it is possible to prove that any losing coalition will remain losing by replacing the “\( \geq \)” signs with “\( < \)” signs. Hence, the Lemma is true.

\[
\square
\]

Note: In other words, this lemma states that it is possible to generate a weighted voting game from the weighted average of two others, and that if the two parent games are in the same region, then so is the child.

After applying lemma 1 and setting \( a = b \), the following is derived – if any two points are in a region, then any point on the line connecting those two points is also in the region. Therefore, given any two points in a region, all points on the line segment connecting those two points are also in the region. Thus, all regions must be convex.

\[
\square
\]
**Theorem 3:** Regions are always both continuous and simple. This means that a region is comprised of adjacent points in space with no singularities.

**Proof:** This is a corollary to Theorem 2, since any discontinuities would contradict the convex property of regions.

**Theorem 4:** Regions are the equivalence classes for the points in the simplex under the equivalence relation “represents a weighted voting game with the same resulting minimum winning coalitions as”.

**Proof:** The relation stated above is reflexive, symmetric, and transitive; it can be equated with dropping balls into boxes. Therefore, regions form equivalence classes.

**Theorem 5:** Regions partition the simplex.

**Proof:** Corollary to Theorem 4.

The regions in the simplex are bounded by hyperplanes representing each possible coalition’s weight being equal exactly to the quota and by the faces of the simplex, which represent an individual’s weight being equal to 0. Thus, there are always a finite number of regions, whose shapes are dynamically related with the quota. As the quota changes, the shape of the regions change, since some points in the simplex originally belong to one region will now belong to a different region. It is possible to produce analysis by viewing how the shapes of regions change as the quota changes.

It is noticeable, however, that some regions only exist between certain values of the quota. This is usually due to one of two reasons. First, some regions are bounded between two parallel hyperplanes which move towards each other as the quota either decreases or increases. When the quota becomes too large or too small, the hyperplanes slide past each other, and thus the area bounded by the region shrinks to nothing. Second, a region may be bounded between several hyperplanes which come to intersect at one point at some value of the quota. At this point, the entire region consists of one point. If the planes then continue to slide past one another, then the region will disappear entirely.

In the analysis of three and four-player games in sections 5, 6, and 7, for ease of notation, the players will be labeled as A, B, and C for three-player games and A, B, C, and D for four-player games. A coalition will be notated as all of the players in the coalition’s labels concatenated together surrounded by square brackets. For example, the coalition with only players A and C is denoted [AC]. The weight of a coalition will be the name of the coalition with the brackets removed, so AC denotes the weight of coalition [AC], which is also equal to A + C.
All regions exist at some point in the quota interval \((\frac{1}{2}, 1]\), but not all regions exist during the entirety of that interval. In the complete analysis for four-player weighted voting games, it was important to notice which regions were bounded by a narrower quota interval than \((\frac{1}{2}, 1]\). In four-player weighted voting games, regions can have between 4 and 10 faces, each face representing a plane. It is curious how some regions’ shapes changed along with the quota, since as the quota changes, far-off but pertinent planes approached and intersected with the region to affect its shape.

In a region, every single coalition is labeled as either winning or losing, so in a way, a region is bounded by all hyperplanes in the simplex. However, when a stronger condition is met, such as \(A \geq q\), then weaker ones such as \(AB \geq q\) or \(ACD \geq q\) will become fulfilled automatically. Similarly, when a stronger condition such as \(BCD < q\) is met, then conditions such as \(B < q\) become unnecessary. Therefore, in describing the conditions for a region, it is only necessary to specify the minimum winning coalitions as winning and the maximal losing coalitions as losing.

By using the conditions limiting points within a region, a set of inequalities will be created. By finding which values of the quota satisfy all of the inequalities, it becomes possible to discover the “quota range” of a region. The “Quota Sandwich Theorem” may also be used to mitigate the problem of finding the quota range. It is stated and proved below.

**Theorem 6 (Quota Sandwich Theorem):** If a region exists when \(q = a\) and when \(q = b\), then the region exists in the interval \(q \in [a, b]\).

**Proof:** This proof requires lemma 1, reproduced below.

**Lemma 1:** If two weighted voting games in the same region with quotas \(a\) and \(b\) are \([a; w_{a1}, w_{a2}, \ldots]\) and \([b; w_{b1}, w_{b2}, \ldots]\), the following weighted voting game is also in the same region: \([af + b(1 - f); w_{a1}f + w_{b1}(1 - f), w_{a2}f + w_{b2}(1 - f), \ldots]\), where \(0 \leq f \leq 1\).

Now after applying Lemma 1, it can be seen that weighted voting games with all values of quota in the closed interval \([a, b]\) can be constructed.

4 Using Regions for Analysis

The most obvious advantage for using regions is to take advantage of their completeness. Since regions partition the simplex, every single point in the region belongs to a certain region. By looking at a simplex where the regions have been included, one can ascertain information regarding weighted voting games at the determined value of quota. It becomes possible to determine the answer to questions such as “given a fixed value of quota and a random weight vector, what is the probability of a certain power distribution occurring?”
The use of regions in the geometric representation of weighted voting games is also very powerful in dispelling paradoxes of voting power, as in Jones (2009). When the weight vector of a weighted voting game, its quota, or both change, no alteration will be made to the original weighted voting game unless it has moved to another region. When the weight vector is modified, the chosen point will move, and if it crosses a hyperplane, it will arrive in a new region, with new power distributions. Altering the quota produces a similar effect, since the location of the hyperplanes will become altered. Regions greatly simplify the analysis of weighted voting games because they represent a fixed number of states. A weighted voting game can be and must be in exactly one region, yet the number of regions is finite, fixing the number of possible states.

The quota, of course, has just as much influence on the geometric representation of weighted voting systems as the weight vector does, perhaps even more. Changing the weight vector amounts to changing the location of a point, which could place it in a different region, but changing the quota will change the shape and number of regions. Therefore, a better and more complete geometric representation could be made by including the quota as an additional dimension. See the next two sections for graphics. The prism created by adding an additional dimension to the geometric representation of all weighted voting games at a particular quota is powerful because of its completeness – geometric analysis using the prism may prove superior to algebraic manipulation.

Jones (2009) uses both the motion of planes representing change in the quota and the motion of the point representing the weight vector to explain paradoxes of weighted voting, but if the prism were used, then there would be no need for the motion of planes.

Tolle (2003) asks “for weighted voting systems of size \(n\), is there a formula in terms of \(n\) for the number of feasible power distributions?” and “with a complete enumeration of the power distributions feasible for weighted voting systems of size \(n\), can one efficiently generate a complete list of feasible power distributions for size \(n+1\) weighted voting systems?” Several papers have tried to answer Tolle’s questions, including Brown et. al. (2009), Cuttler et. al. (2005), Elkind et. al. (2005), and Keijzer et. al. (2010), though often, their definition of “feasible power distribution” was different from Tolle’s.

These papers either tried to establish various properties about the number of feasible power distributions for \(n\)-player weighted voting systems or presented an algorithm which would enumerate the complete set of feasible sets of minimum winning coalitions. The geometric representation combined with the concept of regions, however, might be able to simplify this analysis. Indeed, the geometric representations for \(n\)-player weighted voting games can be used to generate \((n+1)\)-player weighted voting games since the faces of a \((n)\)-simplex, representing \((n+1)\)-player weighted voting games are \((n-1)\)-simplexes, representing \(n\)-player weighted voting games. Instead of algebraic or set manipulation, it becomes necessary only to calculated the number of distinct regions bounded by hyperplanes in the prism representing the complete \(n\)-player weighted voting game. If such an enumeration were found, the “inverse problem” of discovering the weighted voting game returning a power distribution closest to the desired one would also be solved.
5 3-Player Weighted Voting Systems

Before proceeding to 4-Player Weighted Voting Systems, 3-Player Weighted Voting Systems must be analyzed. Much work was done on 3-player weighted voting systems by Kirstein (2009) and Jones (2009). The latter used a geometric analysis to reveal that a total of 11 different regions existed, a maximum of 10 of which could exist at any one given time.

The geometric representation of 3-player weighted voting systems at a specific quota are 2-simplexes, or equilateral triangles. There are two planes parallel to each side of the triangle, for a total of six; these planes partition the simplex into the regions. As the quota changes, the shape of the regions also change.

Region # | Minimum Winning Coalitions | Banzhaf Power Distribution | Quota Range
---|---|---|---
1 | [A] | [1, 0, 0] | ($\frac{1}{2} - 1$)
2 | [AB] | $[\frac{1}{2}, \frac{1}{2}, 0]$ | ($\frac{1}{2} - 1$)
3 | [AB][AC] | $[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$ | ($\frac{1}{2} - 1$)
4 | [AB][AC][BC] | $[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$ | ($\frac{1}{2} - \frac{2}{3}$)
5 | [ABC] | $[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$ | ($\frac{2}{3} - 1$)

Table 1: The Five Unique Region Types for 3-Player Weighted Voting Systems

Table 1 assumes that the weights for players A, B, and C satisfy $A \geq B \geq C$. By permuting the players, the first three region types produce 6 additional regions, so a total of 11 regions exist. At any given time, however, a maximum of 10 may exist since regions 4 and 5 exclude one another.

Regions 1, 2, and 3 exist at all values of the quota. More interesting is the relationship between regions 4 and 5, whose quota domains complement one another’s: when the quota is infinitesimally greater than $\frac{1}{2}$, region 4 begins as an upside-down triangle. When the quota is exactly $\frac{2}{3}$, region 4 has shrunk to a point at the center of the simplex. As the quota grows larger than $\frac{2}{3}$, region 5 replaces region 4, and it takes the shape of a triangle that grows to eventually fill the entire simplex when the quota is 1.

As previously stated, the representation can be made more complete by appending a third dimension, creating a prism.
Figure 1: The Geometric Representation for 3-Player Weighted Voting Systems

Each cross-section of this prism parallel to the bases is a simplex, which represents the weighted voting game with 3 players and the quota represented by the height of the cross-section – the top of the prism represents a quota of 1, and the bottom represents a quota of $\frac{1}{2}$.

Every single weighted voting game with 3 players can be represented as a point in the above prism; thus, it is the complete geometric representation of 3 player weighted voting systems.

6 4-Player Weighted Voting Systems: Overview

With four players, the 3-simplex is used, which is a tetrahedron with unit height. There are a total of 18 planes which bound the regions. They are represented in the following table.
Note that simplex coordinates, \((A, B, C, D)\), are being used, which label each point with the distance from that point to each face.

<table>
<thead>
<tr>
<th>Plane Type</th>
<th>Equation of Planes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Four Faces of Tetrahedron</td>
<td>(A = 0; B = 0; C = 0; D = 0)</td>
</tr>
<tr>
<td>1-P Coalitions</td>
<td>(A = q; B = q; C = q; D = q)</td>
</tr>
<tr>
<td>2-P Coalitions</td>
<td>(AB = q; AB = 1; AC = q; BC = q; BD = q; CD = q)</td>
</tr>
<tr>
<td>3-P Coalitions</td>
<td>(ABC = q; ABD = q; ACD = q; BCD = q)</td>
</tr>
</tbody>
</table>

Table 2: Types of Planes bounding Regions for 4-Player Weighted Voting Systems

An interesting observation is that the planes corresponding to the 1-P coalitions are parallel to the planes corresponding to the 3-P coalitions. This is due to the universal equation \(ABCD = 1\), which must always be satisfied. Thus, the 3-P coalitions could be represented as \(A = 1 - q; B = 1 - q; C = 1 - q; D = 1 - q\), which are equations producing planes parallel to those of the 1-P coalitions. These planes, in turn, are also parallel to the faces of the tetrahedron. The planes representing 2-P coalitions also form parallel pairs. The plane representing each coalition’s weight equaling the value of the quota is parallel to the plane corresponding to the coalition’s complement.

In figure 3 on the next page are nine 3-simplexes which are the geometric representation for 4-player weighted voting systems at various values of the quota. Although a more complete representation could be achieved by adding a fourth dimension to the simplex, such a figure would be difficult to display. The planes colored red are \(A = q, B = q, C = q,\) and \(D = q\). The green planes are \(AB = q, AC = q, AD = q, BC = q, BD = q,\) and \(CD = q\). The blue planes are \(ABC = q, ABD = q, ACD = q,\) and \(BCD = q\).

In four-player games, there are 80 different legal sets of minimum winning coalition, creating a total of 80 theoretically possible regions. Though all of these exist at some values of quota, not all of these regions, however, can exist at the same time. The maximum number of regions which can coexist under a fixed quota is 65. When \(q \in \left(\frac{1}{2}, \frac{2}{3}\right)\), the number of regions is 65. When \(q \in \left[\frac{2}{3}, 1\right)\), the number of regions is 53. When \(q = 1\), the number of regions is 15. If without of loss generality, it is assumed that the weighted voting games are canonical – this means that \(1 \geq A \geq B \geq C \geq D \geq 0\) – only 14 different regions are possible. These 14 regions represent each type of region, showing that only 14 different sets of minimum winning coalitions are possible where no two sets can be transformed to one another by permuting the players. These 14 regions are outlined in tables 3 and 4.
Figure 3: The Geometric Representation for 4-Player Weighted Voting Systems
Table 3: Regions in 4-Player Weighted Voting Games

<table>
<thead>
<tr>
<th>Region #</th>
<th>Minimum Winning Coalitions</th>
<th>Banzhaf Power Distribution</th>
<th>Quota Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[A]</td>
<td>[1, 0, 0]</td>
<td>(1/2, 1)</td>
</tr>
<tr>
<td>2</td>
<td>[AB]</td>
<td>[1/2, 1/2, 0, 0]</td>
<td>(1/2, 1)</td>
</tr>
<tr>
<td>3</td>
<td>[AB][AC]</td>
<td>[1/2, 1/2, 1/2, 0]</td>
<td>(1/2, 1)</td>
</tr>
<tr>
<td>4</td>
<td>[AB][AC][AD]</td>
<td>[1/3, 1/3, 1/3]</td>
<td>(1/2, 1)</td>
</tr>
<tr>
<td>5</td>
<td>[AB][ACD]</td>
<td>[1/3, 1/3, 1/3]</td>
<td>(1/2, 1)</td>
</tr>
<tr>
<td>6</td>
<td>[AB][AC][BC]</td>
<td>[1/2, 1/2, 1/2, 0]</td>
<td>(1/2, 1)</td>
</tr>
<tr>
<td>7</td>
<td>[AB][AC][BCD]</td>
<td>[1/2, 1/2, 1/2, 0]</td>
<td>(1/2, 1)</td>
</tr>
<tr>
<td>8</td>
<td>[AB][ACD][BCD]</td>
<td>[1/2, 1/2, 1/2, 1/2]</td>
<td>(1/2, 1)</td>
</tr>
<tr>
<td>9</td>
<td>[ABC]</td>
<td>[1/2, 1/2, 1/2, 0]</td>
<td>(1/2, 1)</td>
</tr>
<tr>
<td>10</td>
<td>[ABC][ABD]</td>
<td>[1/2, 1/2, 1/2, 0]</td>
<td>(1/2, 1)</td>
</tr>
<tr>
<td>11</td>
<td>[AB][AC][AD][BCD]</td>
<td>[1/2, 1/2, 1/2, 0]</td>
<td>(1/2, 1)</td>
</tr>
<tr>
<td>12</td>
<td>[ABC][ABD][ACD]</td>
<td>[1/2, 1/2, 1/2, 0]</td>
<td>(1/2, 1)</td>
</tr>
<tr>
<td>13</td>
<td>[ABC][ABD][ACD][BCD]</td>
<td>[1/2, 1/2, 1/2, 1/2]</td>
<td>(1/2, 1)</td>
</tr>
<tr>
<td>14</td>
<td>[ABCD]</td>
<td>[1/2, 1/2, 1/2, 1/2]</td>
<td>(1/2, 1)</td>
</tr>
</tbody>
</table>

Table 4: Quota Domains in 4-Player Weighted Voting Games

<table>
<thead>
<tr>
<th>Region #</th>
<th>Quota Domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[A]</td>
</tr>
<tr>
<td>2</td>
<td>[AB]</td>
</tr>
<tr>
<td>3</td>
<td>[AB][AC]</td>
</tr>
<tr>
<td>4</td>
<td>[AB][AC][AD]</td>
</tr>
<tr>
<td>5</td>
<td>[AB][ACD]</td>
</tr>
<tr>
<td>6</td>
<td>[AB][AC][BC]</td>
</tr>
<tr>
<td>7</td>
<td>[AB][AC][BCD]</td>
</tr>
<tr>
<td>8</td>
<td>[AB][ACD][BCD]</td>
</tr>
<tr>
<td>9</td>
<td>[ABC]</td>
</tr>
<tr>
<td>10</td>
<td>[ABC][ABD]</td>
</tr>
<tr>
<td>11</td>
<td>[AB][AC][AD][BCD]</td>
</tr>
<tr>
<td>12</td>
<td>[ABC][ABD][ACD]</td>
</tr>
<tr>
<td>13</td>
<td>[ABC][ABD][ACD][BCD]</td>
</tr>
<tr>
<td>14</td>
<td>[ABCD]</td>
</tr>
</tbody>
</table>
7 4-Player Weighted Voting Systems: Specifics of Selected Regions

The 14 regions are bounded by different planes as the quota changes, but four regions are particularly complex, being bounded by three different sets of planes in three different ranges for the quota.

7.1 Region 3: [AB][AC]

Region 3 never completely disappears, except for when the quota is 1. This region is characterized by the following bounds: \( AB/\geq AC/\geq, A < q, BCD < q, AD < q \). In this region, player D is a dummy, and player A has veto power. According to the Banzhaf power index, the power vector for this region is \( [\frac{3}{5}, \frac{1}{5}, \frac{1}{5}, 0] \).

![Figure 4: Shape of Regions for Region 3](image)

- (a) \( q = 0.55 \)
- (b) \( q = 0.64 \)
- (c) \( q = 0.75 \)

When the quota is between \( \frac{1}{2} \) and \( \frac{3}{5} \), the five planes intersect at six points, creating a triangular prism-like shape with non-parallel triangular faces. The base of this region rests on the plane \( D = 0 \); in fact, if the fourth player’s weight were 0, then this case would simplify into a three-player weighted voting game – hence, the shape of the base is the same as that of the region for a three-player weighted voting game: a trapezoid. When \( q = 0.55 \), this region occupies 1.05% of the entire simplex.

When the quota is \( \frac{2}{3} \), two of the points of intersection appear to join at one point, creating a triangular pyramid. When the quota is between \( \frac{3}{5} \) and \( \frac{2}{3} \), the five planes still intersect at six points, but in the other direction, so to speak. The intersection with the plane \( D = 0 \) is still a trapezoid. When \( q = 0.64 \), this region occupies 2.28% of the entire simplex.

When the quota exceeds \( \frac{2}{3} \), the region becomes a tetrahedron, since three points join together as one. The base becomes a triangle. The region remains in this shape until the
quota reaches 1, whereupon it disappears completely. When \( q = 0.75 \), this region occupies 0.781% of the entire simplex.

### 7.2 Region 5: [AB][ACD]

Region 5 is characterized by the following bounds: \( AC < q \), \( AB \geq q \), \( ACD \geq q \), \( BCD < q \). In this region, no player is a dummy but player 1 holds veto power. The Banzhaf power index weight vector is \( \left[ \frac{1}{2}, \frac{3}{10}, \frac{1}{10}, \frac{1}{10} \right] \).

![Figure 5: Shape of Regions for Region 5](image)

(a) \( q = 0.55 \)  
(b) \( q = 0.63 \)  
(c) \( q = 0.75 \)

When the quota is between \( \frac{1}{2} \) and \( \frac{3}{5} \), the region assumes the shape of a tetrahedron. When \( q = 0.55 \), this region occupies exactly 0.1% of the entire simplex.

If the quota exceeds \( \frac{3}{5} \) but not \( \frac{2}{3} \), then the region becomes a triangular prism-like shape, but the bases are not congruent or parallel. When \( q = 0.63 \), this region occupies 1.59% of the entire simplex.

If the quota exceeds \( \frac{2}{3} \), however, the region resumes the shape of a tetrahedron, shrinking to nothing when the quota becomes 1. When \( q = 0.75 \), this region occupies 0.781% of the entire simplex.

### 7.3 Region 12: [ABC][ABD][ACD]

Region 12 is characterized by these bounds: \( ACD \geq q \), \( BCD < q \), \( AB < q \). In this region, player A has veto power, and needs the support of any two of the other three players to form a winning coalition. The Banzhaf power index vector for this region is \( \left[ \frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5} \right] \).

This region comes into existence when the quota is \( \frac{3}{5} \). From this point until the quota is \( \frac{2}{3} \), the region is shaped like a tetrahedron. When \( q = 0.64 \), this region occupies 0.400% of
When the quota is between \( \frac{2}{3} \) and \( \frac{3}{4} \), the region becomes an interesting shape. It has seven faces, four triangular and three trapezoidal: it is as if the fustum of a triangular pyramid were glued to a tetrahedron. When \( q = 0.70 \), this region occupies 3.25\% of the entire simplex.

If the quota exceeds \( \frac{3}{4} \), then the region has six faces, all triangular. These meet in a double tetrahedron. The region disappears entirely when the quota is one. When \( q = 0.80 \), this region occupies 1.20\% of the entire simplex.

### 7.4 Region 13: \([ABC][ABD][ACD][BCD]\)

This region represents “majority rule” with four players. It is bounded by \( BCD > q \) and \( AB < q \). Though conceptually simple, its details are intricate: each player has equal power, but the regions have complicated shapes.

When the quota is between \( \frac{1}{2} \) and \( \frac{3}{4} \), the region is a cube. Each plane is parallel to one edge of the simplex. When \( q = 0.55 \), this region occupies 0.300\% of the entire simplex.

If the quota changes from less than \( \frac{3}{4} \) to more than it, the cube will be cut by four planes, each slicing off a corner; these corners form a tetrahedron. The result: a shape with 10 faces, including 4 triangles and 6 hexagons. When \( q = 0.65 \), this region occupies 4.98\% of the entire simplex.

When the quota exceeds \( \frac{2}{3} \), however, the 6 hexagonal faces shrink to nothing, and the four triangular faces meet, creating a perfect tetrahedron, upside-down relative to the simplex. This tetrahedron shrinks to nothing when the quota exceeds \( \frac{3}{4} \). When \( q = 0.70 \), this region occupies 0.800\% of the entire simplex.
8 Conclusion

The analysis of weighted voting games is a very broad topic. The use of the algebraic representation, while succinct, is sometimes inadequate and leaves many of the more intricate properties of weighted voting games obscured – there is no easy method to see the boundaries of a weighted voting game or to assess the finiteness of the number of distinct weighted voting games. The geometric representation, representing each weighted voting game as a point, assigns each distinct type of weighted voting game to a clearly bounded region within a simplex, allowing the analysis of other interesting properties of weighted voting games, such as shape. The geometric representation of weighted voting games was used for the analysis of four-player weighted voting games. The use of hyperplanes to limit these regions allows any changes in quota and weights to be clearly shown by the motion of points and hyperplanes. An even better but more difficult to display representation involves making the quota an additional dimension – changes in quota would then be represented only be the motion of a point. The completeness of the geometric representation makes it a very useful alternative to the standard algebraic representation.

References


