Optimal VCG Mechanisms to Assign Multiple Tasks*

Jung S. You
Rice University, Houston, TX, USA
E-mail: jsyou10@gmail.com, jsyou@rice.edu

Revised March 15, 2011

Abstract

We develop optimal VCG mechanisms in order to assign identical economic "bads" (for example, costly tasks) to agents. The optimal non-deficit VCG mechanism achieves asymptotic budget balance, yet the non-deficit requirement is incompatible with reasonable welfare bounds. If we omit the non-deficit requirement, individual rationality greatly changes the relationship between burning a budget surplus and financing a budget deficit. Allowing a slight deficit, the optimal individually rational VCG mechanism becomes asymptotically budget balanced more easily. Such a phenomenon cannot be found in the case of assigning economic "goods."

Keywords VCG mechanisms; Worst case analysis; Individual rationality; Asymptotic budget balance

JEL Classification D44; D79

1 Introduction

There exist m undesirable objects (or "bads") which are identical and which need to be allocated to n, $n > m$ strategic agents. Each agent is required to take at most an object. Cries of NIMBY greeting waste disposal facilities represents a problem of allocating economic bads (Kunreuther and Easterling (1996)). For the problem of assigning economic bads, the seminal Vickrey-Clarke-Groves (VCG) mechanisms (Generalized Vickrey Auction) achieve both allocative efficiency and incentives by way of money transfer. They are uniquely characterized by strategy-proofness[1] and allocative efficiency (Green and Laffont (1977, 1979), Suijs (1996), Holmstrom (1979)).

However, it is not possible for VCG mechanisms to be budget balanced at all valuation profiles (Green and Laffont (1979)). If there is a budget surplus, then it needs to be burnt by a social planner in order to preserve the incentive compatibility of a VCG mechanism. In case of budget deficit, the social planner must finance the mechanism. Interpreting any budget imbalance as a mechanism implementation cost, our aim is to design VCG mechanisms that minimize the budget imbalance.[2]

*I am very grateful to Hervé Moulin for his guidance and support. I thank Mingyu Guo for his kind answers to my questions.

[1]A mechanism is said to be *strategy-proof* if truth telling is a dominant strategy for every agent.

[2]Parkes et al. (2001) and Faltings (2005) construct budget balanced mechanisms forsaking efficiency or strategy-proofness.

If we weaken the incentive criterion from dominant strategy, we can use Bayesian assumptions for the distribution of utility functions, and therefore calculate the expected budget imbalance (Bailey (1997)).[3] For the problem of provisioning public goods, Deb, Gosh and Seo (2002), Green et al.(1976) and Green and Laffont (1979) give the asymptotic behavior of the expected budget imbalance under the pivotal mechanism (Vickrey Auction), and Zhou (2007) provides the same for the problem of private good exchange.

On the other hand, if we wish to maintain dominant strategy incentive compatibility, we assume no prior and approach the problem using the worst case analysis. Moulin (1986) as well as Deb and Seo (1998) investigate the pivotal mechanism in the worst scenario for a public good provision problem, and Moulin and Shenker (2001) do the same for a cost sharing problem. Goldberg et al.(2001, 2006) and Aggarwal et al.(2005) as well as Hartline and McGrew (2005) design worst case profit maximizing mechanisms.

Favoring the prior-free above Bayesian approach, we will adopt the worst case analysis. Moulin (2009, 2010) measure the performance of a VCG mechanism by the largest absolute budget imbalance relative to efficient surplus over all valuation profiles and call this performance index *efficiency loss* of the VCG mechanism. Considering that a social planner has to deal differently with budget surplus and budget deficit, however, we will use two performance indices differentiating between budget surplus and budget deficit.

Our *worst case relative surplus* (WCRS) (*worst case relative deficit* (WCRD)) is defined as the largest amount of money burned (money supplied) relative to efficient surplus over all valuation profiles, respectively. By using these indices WCRS and WCRD, we can see the different effectiveness of money-burning and money-supplying (subsidizing) acts in reducing a mechanism implementation cost relative to the created benefit. We will observe that when a social planner subsidizes a VCG mechanism, the mechanism achieves approximate budget balance more easily than in case of no subsidization.

When comparing the performances of any two VCG mechanisms, the preferable values of both WCRS and WCRD are the lesser ones, so we will adopt the concept of dominance and define the *optimal frontier* as the set of undominated pairs of WCRS and WCRD. A VCG mechanism is said *optimal* when its pair of WCRS and WCRD belongs to the optimal frontier.[4] In our framework, when a social planner is capable of both discarding a budget surplus and subsidizing a mechanism, he makes tradeoffs within the set of optimal VCG mechanisms.

The main results are presented in Section 3. We not only compute optimal VCG mechanisms in the problem of allocating bads, but also conduct basic tests of fairness as well.[5] For the basic fairness tests, we will adopt unanimity upper bound and individual rationality. If a mechanism guarantees each participant a net loss smaller than the loss he would experience under random assignment, the mechanism satisfies *unanimity upper bound*. A mechanism is said to be *individually rational* if participation in the mechanism brings each agent a smaller net loss than the loss he would experience in an anarchistic state where everyone performs one task on his own. We will show that when we require mechanisms to satisfy individual rationality, the optimal mechanisms for "goods" and the optimal mechanisms for "bads" behave very differently. This point is explained in Section 3.2 and

[3] In both the public good provision problem and the bilateral trading problem, there exists no budget balanced mechanism that is Bayesian-incentive compatible, efficient, and individually rational (Laffont and Maskin (1979) and Myerson and Satterthwaite (1983)).

[4] Apt et al.(2008) and Guo and Conitzer (2008a) use dominance to define optimal mechanisms. A VCG mechanism dominates another if it always charges less payment against each agent.

[5] As an application of VCG mechanisms to the assignment problem of identical economic bads, Porter, Shoham and Tennenholtz (2004) provide an equity test called k-fairness and develop a 3-Fair mechanism. Moulin (2010) discusses tradeoffs between efficiency and k-fairness. He constructs a VCG mechanism which guarantees each participant a fair share of the q^{th} highest valuation and minimizes the efficiency loss in the allocation problem of a single object.

Appendix A2.

The most relevant articles to our problem have been written by Moulin (2009) and Guo and Conitzer (2009). They investigate the problem of assigning multiple "goods" and develop optimal VCG mechanisms[6] using the worst case analysis. The resulting optimal VCG mechanisms significantly improve upon the previous in Cavallo (2006).[7]

For the problem of assigning economic goods, Moulin (2009) makes two interesting points. The first being that the optimal efficiency loss of any non-deficit VCG mechanism is strictly smaller than the optimal efficiency loss of any individually rational and non-deficit VCG mechanism. Thus individual rationality plays a role when $m \geq 2$. Both indices converge exponentially fast to zero in n if the scarcity ratio $\frac{m}{n}$ is less than $\frac{1}{2}$, and as $\frac{1}{\sqrt{n}}$ if $\frac{m}{n} \simeq \frac{1}{2}$. Their behavior, however, is quite different if $\frac{m}{n} > \frac{1}{2}$. The optimal efficiency loss, excluding individual rationality, still converges fast to zero in n, while the optimal efficiency loss under individual rationality does not converge to zero in n.

Secondly, Moulin (2009) points that whether or not deficit is allowed does not make an essential difference in total optimal efficiency loss. The optimal efficiency loss of any VCG mechanism (allowing deficit) is about one-half (saying exactly, between $\frac{1}{2+\frac{1}{n-1}}$ and $\frac{1}{2}$) of the optimal efficiency loss of any non-deficit VCG mechanism. He conjectures that this property still holds true even if individual rationality is imposed.

On the other hand, Guo and Conitzer (2009) use the worst ratio of absolute budget imbalance to the budget surplus of the pivotal mechanism to measure performance. Although their design goal is different from the goal in Moulin (2009), their non-deficit optimal mechanism is the same as the non-deficit and individually rational optimal mechanism in Moulin (2009). Individual rationality is irrelevant in Guo and Conitzer (2009), since their non-deficit optimal mechanism remains the same even if we impose individual rationality.

In addition, the optimal efficiency loss of any non-deficit VCG mechanism in Guo and Conitzer (2009) equals the optimal efficiency loss of any non-deficit and individually rational VCG mechanism in Moulin (2009). This demonstrates that the non-deficit optimal VCG mechanism in the former fails asymptotic budget balance altogether when the scarcity ratio is greater than one-half. In addition, according to Guo and Conitzer (2009), when $m = n - 1$, the pivotal mechanism will always be optimal among all VCG mechanisms. This is undesirable since the efficiency loss of the pivotal mechanism is always greater than 1. In addition, similarly to Moulin's findings, allowing deficit does not essentially change the optimal efficiency loss for Guo and Conitzer (2009) either.[8]

In the problem of assigning "bads", we show that the performance measurement suggested by Guo and Conitzer (2009) fails to be in use for all m, $m < n$. If we measure the performance of a mechanism and find the optimal mechanism according to the standards in Guo and Conitzer (2009), it rarely redistributes the surplus of the pivotal mechanism for every m, $m \geq 2$. For $m = 1$, the pivotal mechanism is optimal, therefore there is no redistribution. Thus, the optimal mechanism is far from achieving Guo and Conitzer's original objective of redistributing the surplus of the pivotal mechanism. We can predict that this optimal mechanism will have a large efficiency loss since the pivotal mechanism generates the largest efficiency loss among all non-deficit and individually rational VCG mechanisms. This point is shown in detail in Appendix A1 (Proposition 2 and Proposition 3).

[6]They define an optimal VCG mechanism as a VCG mechanism whose index is the smallest over all (linear) VCG mechanisms.

[7]Cavallo (2006) constructs a VCG mechanism to redistribute some of the payment back to the agents in a way that will not affect incentives. For the instance of a single object auction, Cavallo's mechanism redistributes to agent i $\frac{1}{n}$ times the second highest bid among bids other than his own bid.

[8]The optimal loss with no deficit λ_G and the optimal loss allowing deficit μ_G in Guo and Conitzer (2009) relate as follows: for $m \leq n-2$, $\frac{\mu_G}{\lambda_G} = \frac{1}{2-\lambda_G}$ and $\frac{\mu_G}{\lambda_G}$ converges to $\frac{1}{2}$ in n given m.

In Section 3.1, we compute the optimal WCRS $\lambda^*_{n,m}$ of any non-deficit VCG mechanism and its corresponding optimal mechanism for all m and n. For both $m = 1$ and $m = n-1$, the worst case surplus in the optimal mechanism never exceeds $\frac{n}{2^{n-3}}$ of efficient surplus (Theorem 1.1 and Theorem 1.3). For m, $2 \le m \le n-2$, the optimal WCRS of any non-deficit VCG mechanism vanishes fast at exponential speed in n: $\lambda^*_{n,m} \simeq \frac{n^m}{m!2^{n-2}}$ (Theorem 1.2). This tells that a performance index whose denominator is efficient surplus works well for the problem of assigning "bads" as well as for the problem of assinging "goods." In addition, similarly to the problem of assigning economic goods, Theorem 1.4 shows that whether we require non-deficit property or not, has no bearing on the total optimal worst case budget imbalance in the problem of assigning economic bads as well.

Section 3.2, however, shows that the non-deficit property is incompatible with preliminary tests of welfare bounds. Proposition 1 shows that the unanimity upper bound test fails under the non-deficit constraint. The non-deficit constraint also makes the pivotal mechanism the uniquely optimal individually rational VCG mechanism. Interestingly, if the non-deficit constraint is abandoned, individual rationality becomes greatly significant to our problem. We compute the optimal frontier of any individually rational VCG mechanism and the corresponding optimal mechanisms for all n and m.

Theorem 2.1 shows that when assigning a single bad, there exist only two optimal individually rational mechanisms. This result differs from the case of multiple bads in which we can find an infinite number of optimal individually rational mechanisms. For the case of a single bad, the pivotal mechanism is optimal (and non-deficit), but generates infinite WCRS. In contrast, another optimal VCG mechanism does not generate any budget surplus and its WCRD is 1.

Theorem 2.3 shows that to assign multiple bads, $m \ge 3$, we can find the optimal frontier of any individually rational VCG mechanisms: $\lambda^*_{n,m}/A(n,m)+\mu^*_{n,m}/B(n,m) = 1$ where $A(n,m) > B(n,m)$ for all n and m. The asymptotic behavior of the ratio $B(n,m)/A(n,m)$ such that $B(n,m)/A(n,m) \simeq \frac{n^{m-1}}{(m-1)m!2^{n-2}}$ implies that as more agents participate, a very minute amount of deficit can replace unit surplus. By allowing a slight deficit, we can almost achieve budget balanced VCG mechanisms more easily. This result stands in stark contrast to the outcome of assigning economic goods. For the case of economic goods, regardless of individual rationality, unit surplus can only be replaced with unit deficit (Proposition 4). Theorem 2.2 also provides similar results for $m = 2$.

All proofs are gathered in Appendix A3.

2 The Model

Let $N = \{1, \cdots, n\}$ be the set of agents. m of the n agents should perform m identical tasks together. The tasks are undesirable, and thus, they are economic "bads" which are costly to agents. Every agent is equally responsible and is liable for at most one task. It is assumed that $1 \le m \le n-1$ (if $n = m$, everyone performs a task) and that a monetary transfer occurs.

Each agent i, $i \in N$ can perform a task with cost c_i, which is private information. Performing a task causes agent i disutility c_i. Let $c = (c_1, c_2, \cdots, c_n)$. Given a cost profile $c \in \mathcal{R}^N_+$, the vector $c^* \in \mathcal{R}^N_+$ is the permutation of c whose coordinates are arranged increasingly:

$$c^{*1} \le c^{*2} \le \cdots \le c^{*n}.$$

Let $c_{-i} = (c_1, \cdots, c_{i-1}, c_{i+1}, \cdots, c_n)$. We denote by $(c_{-i})^{*k}$ the kth lowest cost among $c_1, \cdots, c_{i-1}$, $c_{i+1}, \cdots, c_n$. Given a cost profile $c \in \mathcal{R}^N_+$, *efficient cost* for performing m tasks is the minimal cost $\tau_m(c) = \sum_{k=1}^m c^{*k}$.

VCG mechanisms assign tasks to a subset of m agents whose total cost to perform m tasks together is minimal. And each VCG mechanism is defined by n arbitrary real-valued functions t_i on $\mathcal{R}_+^{N\backslash\{i\}}$. The function $t_i(c_{-i})$ represents a monetary transfer from agent i to the mechanism given a cost profile c. Agent i's net disutility V_i in a VCG mechanism is written as:

$$V_i(c) = \tau_m(c) + t_i(c_{-i}) \text{ for all } c \in \mathcal{R}_+^N.$$

Every VCG mechanism is *efficient* since an allocation determined by the mechanism always minimizes the total cost to perform m tasks.[9] It is *strategy-proof* since every agent is always better off when he reveals his private information truthfully. Holmstrom (1979) proves that VCG mechanisms are the only strategy-proof and allocatively efficient mechanisms in our model.

However, VCG mechanisms cannot be budget balanced for all preference profiles (Green and Laffont (1979)). We use Δ to denote the budget imbalance of a VCG mechanism as follows:

$$\Delta(c) = \sum_{i\in N} V_i(c) - \tau_m(c) = (n-1)\tau_m(c) + \sum_{i\in N} t_i(c_{-i}).$$

Given a cost profile $c \in \mathcal{R}_+^N$, if $\Delta(c) = 0$, then we have a *balanced budget*, if $\Delta(c) > 0$, a *budget surplus* exists, and if $\Delta(c) < 0$, a *budget deficit* is indicated.

Among VCG mechanisms, the pivotal mechanism (the Vickrey auction) is a benchmark mechanism (Green and Laffont (1979)). In the pivotal mechanism, each agent i's net disutility equals "efficient cost to perform m tasks - efficient cost to perform $(m-1)$ tasks with agent i ignored." If agent i is ignored, other agents force agent i to perform one task and allocate residual $(m-1)$ tasks efficiently among themselves. This implies $t_i(c_{-i}) = -\tau_{m-1}(c_{-i})$. Thus, the net disutility under the pivotal mechanism is written as:

$$V_i^p(c) = \tau_m(c) - \tau_{m-1}(c_{-i}) \text{ for all } i \text{ and } c. \tag{1}$$

We can simplify equation (1) as $V_i^p(c) = c_i$ if $c_i \le c^{*(m-1)}$ or $V_i^p(c) = c^{*m}$ if $c_i \ge c^{*m}$. Given cost profile $c \in \mathcal{R}_+^N$, the pivotal mechanism generates a budget surplus of:

$$ps(c) = \sum_{i\in N} V_i^p - \tau_m(c) = (n-m)c^{*m}.$$

Whether a mechanism under our consideration generates budget surplus or not, it is convenient to write the function $t_i(c_{-i})$ as $t_i(c_{-i}) = -\tau_{m-1}(c_{-i}) - r(i;c_{-i})$, where $r(i;c_{-i})$ is a *redistribution scheme* for agent i. Thus, the general form of VCG mechanisms is given as:

$$V_i(c) = \tau_m(c) - \tau_{m-1}(c_{-i}) - r(i;c_{-i}) = V_i^p(c) - r(i;c_{-i}) \text{ for all } c \in \mathcal{R}_+^N.$$

Our VCG mechanisms ask the social planner to first run the pivotal mechanism. Then, the social planner distributes a suitable rebate to each agent if there is a budget surplus, or charges agents of additional tax if there is a deficit. We rewrite the budget imbalance of a VCG mechanism with a redistribution scheme r as:

$$\Delta(c,r) = ps(c) - \sum_{i=1}^{n} r(i;c_{-i}) = (n-m)c^{*m} - \sum_{i=1}^{n} r(i;c_{-i}).$$

[9]The objects go to the agents with the highest valuations in the case of (desirable) "goods", the lowest disutilities in the case of bads.

Interpreting budget imbalance as an implementation cost, we will adopt the worst case analysis to measure the performance of any VCG mechanism. The worst case performance index of a mechanism will be defined as the largest implementation cost relative to implementation gain over all cost profiles. That is, it is defined as the largest budget surplus or deficit relative to a meaningful measure of "efficient surplus" over all cost profiles. Since the source of budget imbalance is different between budget surplus and budget deficit, we differentiate between two performance measurements, *worst case relative surplus* and *worst case relative deficit*.

Drawing on the concept of opportunity cost, we notice that implementing a VCG mechanism actually saves costs when performing tasks. To perform tasks, a VCG mechanism will spend the efficient cost while a random assignment, as the primitive benchmark, will spend average cost. The saved cost garnered by the VCG mechanism is the difference between the average cost and the efficient cost. Thus, we define *efficient surplus* (es) as follows:[10]

$$es(c) = \frac{m}{n} c_N - \tau_m(c)$$

where $c_N = \sum_{i \in N} c_i$.

With efficient surplus defined as above, we now define *worst case relative surplus* (WCRS) when $\Delta(c, r) > 0$ as the following number $\lambda_{n,m}$, $0 < \lambda_{n,m} \leq \infty$:

$$\lambda_{n,m}(r) = \sup_{c \in \mathcal{R}_+^N} \frac{\Delta(c, r)}{es(c)}$$

for the case of n agents and m objects. Likewise, *worst case relative deficit* (WCRD) when $\Delta(c, r) < 0$ is defined as the following number $\mu_{n,m}$, $0 < \mu_{n,m} \leq \infty$:

$$\mu_{n,m}(r) = \sup_{c \in \mathcal{R}_+^N} -\frac{\Delta(c, r)}{es(c)}$$

for the case of n agents and m objects. If $\Delta(c, r) > 0$ ($\Delta(c, r) < 0$) and $es(c) = 0$ given a cost profile $c \in \mathcal{R}_+^N$, we set $\lambda_{n,m}(r) = \infty$ ($\mu_{n,m}(r) = \infty$) conventionally. For the convenience of discussion, we may drop n and m from $\lambda_{n,m}$ and $\mu_{n,m}$.

Rewriting the above definitions in terms of linear programming, a relative surplus is bounded by λ and the absolute value of a relative deficit is bounded by μ, that is:

$$\text{if } \Delta(c, r) > 0, \ \frac{\Delta(c, r)}{es(c)} \leq \lambda \ \text{ and } \ \text{if } \Delta(c, r) < 0, \ -\frac{\Delta(c, r)}{es(c)} \leq \mu.$$

This two-way worst case constraint is written as:

$$ps(c) - \lambda \cdot es(c) \leq \sum_{i \in N} r(i; c_{-i}) \leq ps(c) + \mu \cdot es(c) \ \text{ for all } c \in \mathcal{R}_+^N. \tag{2}$$

A VCG mechanism with redistribution scheme r has two dimensional performance indices, λ and μ. Thus, to compare any two VCG mechanisms, we use a concept of dominance. Firstly, a pair of (λ, μ) is said to be *feasible* if it satisfies constraint (2). Let Λ be the set of all feasible (λ, μ) pairs. Then, for two pairs (λ', μ') and (λ, μ) in Λ, if $\lambda' \geq \lambda$ with $\mu' > \mu$ holds or $\lambda' > \lambda$ with $\mu' \geq \mu$ holds, then (λ, μ) *dominates* (λ', μ'). When $\lambda' > \lambda$ and $\mu' > \mu$, (λ, μ) *strictly dominates* (λ', μ'). If a pair (λ^*, μ^*) in Λ is not dominated by any pairs in Λ, the pair is said to be *optimal*. We denote the set

[10] Remark 2 discusses other possible measurements of efficient surplus.

of all optimal pairs by using $\partial\Lambda$ and call $\partial\Lambda$ the ***optimal frontier***. A VCG mechanism is said to be ***optimal*** if its redistribution scheme r^* generates an optimal pair (λ^*, μ^*) in $\partial\Lambda$.

Remark 1 Moulin (2009, 2010) and Guo and Conitzer (2008b, 2009) used worst case analysis, but their indices do not distinguish budget surplus from budget deficit. Their optimal mechanisms try to minimize the absolute value of the largest relative budget imbalance. By contrast, our definitions of optimal frontier and optimality are more inclusive. Our definitions can reveal the interesting tradeoffs between budget surplus and budget deficit. As we will show later in Section 3.2, for the case of economic goods, deficit and surplus are one to one tradable. However, for the case of bads, this interchangeability of deficit with surplus is no longer symmetric when we impose individual rationality.

Remark 2 Another natural estimator of efficient surplus is the spread between maximal cost (worst case outcome under fully uninformed decision) and minimal cost (efficient cost). Using this estimator, Moulin (2010) performs the worst-case analysis when the object is a single costly task and a deficit is not allowed. The corresponding index of worst case relative surplus is smaller due to an increase in the denominator. It is, however, difficult to write a general formula for the optimal worst case relative surplus using that estimator when $m \geq 2$.

Our benchmark, the pivotal mechanism does not redistribute anything, that is, $r(i; c_{-i}) = 0$, and does not generate deficit. Its worst case relative surplus is computed as follows:

$$\begin{aligned}\lambda_{n,m}(0) &= \sup_{c \in \mathcal{R}_+^N} \frac{(n-m)c^{*m}}{\frac{m}{n}\left[\sum_{i=1}^{m} c^{*i} + \sum_{i=m+1}^{n} c^{*i}\right] - \sum_{i=1}^{m} c^{*i}} \\ &= \sup_{c \in \mathcal{R}_+^N} \frac{(n-m)c^{*m}}{\frac{m-n}{n}\sum_{i=1}^{m-1} c^{*i} + \frac{m-n}{n} c^{*m} + \frac{m}{n}\sum_{i=m+1}^{n} c^{*i}} = \frac{n}{m-1}.\end{aligned}$$

The last equality holds since the worst case occurs when $c^{*1}, \cdots, c^{*(m-1)}$, $c^{*(m+1)}, \cdots, c^{*n}$ are as small as possible. By setting $c^{*1} = \cdots = c^{*(m-1)} = 0$ and $c^{*m} = c^{*(m+1)} = \cdots = c^{*n}$, we find the worst case relative surplus of the pivotal mechanism. If $m = 1$, the pivotal mechanism has infinite worst case relative surplus. Given m, $m \geq 2$, its $\lambda_{n,m}$ is increasing in n. Since $\lambda_{m+1,m} = 1 + \frac{2}{m-1}$, the smallest worst case relative surplus in n is already greater than 1. With this, the implementation cost of the pivotal mechanism is too large compared to the benefit it creates. Therefore, the pivotal mechanism is not desirable, so we will construct nonzero redistribution schemes.

3 Main Results

We denote by $\binom{n}{k}$ the binomial coefficient. The notation $f(n) \simeq g(n)$ means $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$. The notation $r^*_{n,m}$ denotes the optimal redistribution scheme when there are n agents and m objects. Likewise, $\lambda^*_{n,m}$ and $\mu^*_{n,m}$ denote the optimal WCRS and the optimal WCRD for the case of n agents and m objects, respectively.

Section 3.1 presents optimal VCG mechanisms when deficit is not allowed.

3.1 Optimal Non-Deficit VCG Mechanisms

The social planner is not required to create financial inflow, that is, $\mu = 0$, so the redistribution scheme should be designed to satisfy the following *non-deficit* constraint:

Non-Deficit (ND): given r, $\Delta(c,r) \geq 0$ for all $c \in \mathcal{R}_+^N$.

Theorem 1.1 *Let $m = 1$ and $n \geq 3$. the optimal worst case relative surplus of any non-deficit VCG mechanism is given as:*

$$\lambda_{n,1}^* = \frac{n-1}{2^{n-2}-1}.$$

The following linear redistribution scheme defines an optimal mechanism:

$$\begin{aligned}
r_{3,1}^*(c_{-i}) &= (c_{-i})^{*1} - \frac{1}{3}(c_{-i})^{*2};\\
r_{4,1}^*(c_{-i}) &= (c_{-i})^{*1} - \frac{1}{4}(c_{-i})^{*2};\\
r_{5,1}^*(c_{-i}) &= (c_{-i})^{*1} - \frac{23}{105}(c_{-i})^{*2} + \frac{1}{21}(c_{-i})^{*3} - \frac{1}{35}(c_{-i})^{*4};
\end{aligned}$$

and for n, $n \geq 6$,

$$r_{n,1}^*(c_{-i}) = \sum_{k=1}^{3} \alpha_k^*(c_{-i})^{*k} + \sum_{k=4}^{n-2} \beta_k^*(c_{-i})^{*k} + \omega_{n-1}^*(c_{-i})^{*(n-1)}$$

where

$$\alpha_1^* = 1,\ \alpha_2^* = \frac{n^2 - 2^{n-2}n - 2n + 2}{(2^{n-2}-1)(n-2)n},\ \alpha_3^* = -\frac{n^2 - 2^{n-1} - 3n + 4}{(n-2)(n-3)(2^{n-2}-1)},$$

$$\begin{aligned}
\beta_k^* &= \frac{\lambda_{n,1}^*}{n} + \frac{\lambda_{n,1}^*}{n-1} \cdot \frac{\sum_{j=1}^{k-2}\binom{n-2}{j}}{\binom{n-2}{k-1}} - \frac{1}{\binom{n-2}{k-1}} \quad \textit{if } k \textit{ is even};\\
\beta_k^* &= -\frac{\lambda_{n,1}^*}{(n-k)} - \frac{\lambda_{n,1}^*}{n-1}\frac{\sum_{j=1}^{k-3}\binom{n-2}{j}}{\binom{n-2}{k-1}} + \frac{1}{\binom{n-2}{k-1}} \quad \textit{if } k \textit{ is odd};
\end{aligned}$$

$$\begin{aligned}
\omega_{n-1}^* &= -\frac{1}{n\left(2^{n-2}-1\right)} \quad \textit{if } n \textit{ is odd};\\
\omega_{n-1}^* &= 0 \quad \textit{if } n \textit{ is even}.
\end{aligned}$$

Remark 3 If the spread between maximal cost and efficient cost is used as an estimator of efficient surplus, the optimal WCRS of any non-deficit VCG mechanism for $m = 1$, $n \geq 3$ is $\lambda_{n,1}^* = \frac{n-1}{2^{n-1}-1}$ when n is odd, and $\lambda_{n,1}^* = \frac{n-1}{2^{n-1}-2}$ when n is even (Moulin (2010)). As we mentioned in Section 2, this index is smaller than our optimal WCRS.

Theorem 1.2 *For $2 \leq m \leq n-2$,* (i) *the optimal worst case relative surplus of any non-deficit VCG mechanism is given as:*

$$\lambda_{n,m}^* = \frac{(n-m)\binom{n-1}{m-1}}{(n-m)\sum_{k=0}^{m-2}\binom{n-2}{k} + m\sum_{k=m}^{n-2}\binom{n-2}{k}}$$

and (ii) *for a fixed* m,

$$\lambda^*_{n,m} \simeq \frac{n^m}{m!2^{n-2}}.$$

We provide the optimal redistribution schemes corresponding to Theorem 1.2 in Appendix A.3 (Corollary 1 and Corollary 2).

Theorem 1.3 *For* $n \geq 3$ *and* $m = n-1$, *the optimal worst case relative surplus of any non-deficit VCG mechanism is written as:*

$$\lambda^*_{n,n-1} = \frac{n-1}{2^{n-2}-1}.$$

Notice that the optimal WCRS for $m = n-1$ is the same as the optimal WCRS for $m = 1$.

Now we wonder what would be the result if we abandon non-deficit requirement. Theorem 1.4 shows that the total optimal worst case relative budget imbalance of any VCG mechanism is almost the same as the optimal worst case relative surplus of any non-deficit VCG mechanism. This implies that a unit of budget surplus is interchangeable with a unit of budget deficit.

Theorem 1.4 *Let* $n \geq 3$. *The optimal frontier of any VCG mechanism for* $m \leq n-2$ *is given as follows:*

$$\frac{\lambda^*_{n,m}}{A(n,m)} + \frac{\mu^*_{n,m}}{B(n,m)} = 1$$

where

$$A(n,m) = \frac{\binom{n-1}{m-1}}{\sum_{k=0}^{m-2}\binom{n-2}{k} + \frac{m}{n-m}\sum_{k=m}^{n-2}\binom{n-2}{k}}, \quad B(n,m) = \frac{\binom{n-1}{m-1}}{\sum_{k=0}^{m-3}\binom{n-2}{k} + \frac{m}{n-m}\sum_{k=m-1}^{n-2}\binom{n-2}{k}}.$$

In addition, $A(n,n-1) = \frac{n-2}{2^{n-2}-1}$ *and* $B(n,n-1) = \frac{n-2}{2^{n-2}}$. *For any* m, *we have*

$$A(n,m) \simeq B(n,m).$$

Remark 4 Moulin (2009, 2010) call the index ζ the *efficiency loss* of a mechanism:

$$\zeta_{n,m} = \sup_{c \in \mathcal{R}^N_+} \frac{|\Delta(c,r)|}{es(c)}.$$

Using this index, Theorem 1.4 can be restated as follows: among all VCG mechanisms, the smallest efficiency loss $\zeta^{\#}_{n,m}$ satisfies $\zeta^{\#}_{n,m} \simeq \frac{1}{2}\lambda^*_{n,m}$ for all n and m where $\lambda^*_{n,m}$ is the optimal worst case relative surplus of any non-deficit VCG mechanism. Moulin (2009) presents the comparable result for the case of economic "goods."

Even if we discard non-deficit constraint, and request the social planner to finance the mechanism, there is no essential change in the total optimal worst case relative budget imbalance. The optimal worst case relative surplus under a non-deficit mechanism is equally split into worst case relative surplus and worst case relative deficit.

We will see, however, in the following section, this almost one-to-one tradeoff between surplus and deficit no longer holds under individual rationality, when assigned objects are not desirable.

3.2 Optimal Individually Rational VCG Mechanisms

In this section, we will restrict our discussion to anonymous mechanisms.

Anonymity (AN): A VCG mechanism with the redistribution scheme r is *anonymous* if $r(i; c_{-i}) = r(c_{-i})$ for all $i \in N$.

A natural benchmark mechanism, random assignment is not only strategy-proof but also simple to implement. However, it is not efficient, whereas every VCG mechanism is efficient. We demand that a VCG mechanism guarantees each participant a net disutility smaller than the net disutility he would experience under random assignment:

Unanimity Upper Bound (UUB): $V_i(c) \leq \frac{m}{n} c_i$ for all $i \in N$ and $c \in \mathcal{R}^N_+$.

Unfortunately, this test is not compatible with the non-deficit property in our model.[11]

Proposition 1 *There exists no anonymous linear VCG mechanism that satisfies unanimity upper bound and non-deficit.*[12]

A weaker constraint for unanimity upper bound is individual rationality. Individual rationality implies that participation in the mechanism will cost any agent less than or equal to what it would cost them if they were to perform the task alone.

Individual Rationality (IR): $V_i \leq c_i$ for all $i \in N$.

We provide the optimal VCG mechanisms for $m = 1$ below. Theorem 2.1 shows that for the case of $m = 1$, $n \geq 2$, the pivotal mechanism is the unique optimal anonymous VCG mechanism that satisfies individual rationality and non-deficit. Together with Proposition 1, this implies that non-deficit requirement is very restrictive, and therefore makes implementing VCG mechanisms unattractive. When there is a single bad, we cannot improve upon the pivotal mechanism that has an infinite worst case relative surplus. Therefore, we will investigate VCG mechanisms that allow for a budget deficit.

Theorem 2.1 *For the case of $m = 1$, $n \geq 2$, there are two optimal anonymous and individually rational VCG mechanisms. One is the pivotal mechanism whose $\lambda^*_{n,1} = \infty$ and $\mu^*_{n,1} = 0$. For the other, $\mu^*_{n,1} = 1$ with $\lambda^*_{n,1} = 0$, and its linear redistribution scheme is $r^*_{n,1}(c_{-i}) = \frac{n-1}{n}(c_{-i})^{*1}$ for all $i \in N$.*

Remark 5 According to Theorem 2.1, there are only two extreme pairs of $\mu^*_{n,1}$ and $\lambda^*_{n,1}$ for the case of $m = 1$. The pivotal mechanism has infinite worst case relative surplus, and therefore generates no deficit. This phenomenon is unique for the case of $m = 1$, while there are infinitely

[11] Moulin (2010) proves a similar but more universal point for the case of a single bad. For $m = 1$, the unanimity upper bound test fails under the non-deficit constraint for general strategyproof mechanisms.

[12] A linear VCG mechanism is a VCG mechanism whose redistribution scheme is defined by a vector of constants:

$$r(i; c_{-i}) = a^i_0 + a^i_1(c_{-i})^{*1} + \cdots + a^i_{n-1}(c_{-i})^{*(n-1)}.$$

many pairs of $\lambda^*_{n,m}$ and $\mu^*_{n,m}$ for $m \geq 2$. In addition, as the other optimal mechanism has $\mu^*_{n,1} = 1$ with $\lambda^*_{n,1} = 0$ (generating no surplus), its optimal worst case relative deficit is relatively small, compared to the infinite worst case relative surplus of the pivotal mechanism. This implies that by allowing deficit, we can minimize a budget imbalance more effectively. We will observe that this property holds true for $m \geq 2$ in the following Theorem 2.2 and Theorem 2.3:

Theorem 2.2 *For the case of $m = 2$, $n \geq 3$, the optimal frontier of any individually rational VCG mechanism is given as follows:*

$$\frac{\lambda^*_{n,2}}{A(n,2)} + \frac{\mu^*_{n,2}}{B(n,2)} = 1$$

where

$$A(n,2) = n - 1 \quad \textit{and} \quad B(n,2) = \frac{\binom{n-1}{2}}{2^{n-2} - 1}.$$

$B(n,2)$ is strictly decreasing in n and $B(n,2) \simeq \frac{n^2}{2^{n-1}}$.

Remark 6 The function $P(n,2) = B(n,2)/A(n,2)$ is strictly decreasing in n. As the number of agents increases, deficit becomes much more inexpensive than surplus to achieve budget balance. For instance $P(3,2) = 0.5$ implies that unit surplus can be replaced with 0.5 unit deficit when there are three agents. Computing $P(4,2) = 0.33$, $P(5,2) = 0.21$, and $P(6,2) = 0.13$, we observe that when more agents participate, the shrinking deficit can replace unit surplus.

Here we illustrate the optimal redistribution schemes corresponding to Theorem 2.2. If $\mu^*_{n,2} = 0$ (non-deficit), the optimal redistribution scheme is $r^*(c_{-i}) = \frac{n-2}{n}(c_{-i})^{*1}$. For the opposite case, $\lambda^*_{n,2} = 0$ (deficit only), the redistribution scheme of the optimal individually rational VCG mechanism is given as follows:

$$r^*(c_{-i}) = \sum_{k=1}^{6} \alpha^*_k (c_{-i})^{*k} + \sum_{k=7}^{n-1} \beta^*_k (c_{-i})^{*k}$$

where

$$\alpha^*_1 = 0; \; \alpha^*_2 = 1 - \frac{2\binom{n-1}{2}}{n(2^{n-2}-1)}; \; \alpha^*_3 = \frac{2\left[\binom{n-1}{2} - (2^{n-2}-1)\right]}{(2^{n-2}-1)(n-3)};$$
$$\alpha^*_4 = -\frac{12\binom{n-1}{4} + 9\binom{n}{3} - 3n(2^{n-2}-1)}{n(2^{n-2}-1)\binom{n-3}{2}};$$
$$\alpha^*_5 = \frac{4\left[\binom{n-1}{2} + \binom{n-1}{4} - (2^{n-2}-1)\right]}{(2^{n-2}-1)\binom{n-3}{3}};$$
$$\alpha^*_6 = -\frac{2\binom{n-1}{2}}{(n-5)(2^{n-2}-1)} - \frac{2\binom{n-1}{2}\binom{n-1}{3}}{n(2^{n-2}-1)} - \frac{5\left[\binom{n-1}{2} - (2^{n-2}-1)\right]}{\binom{n-3}{4}(2^{n-2}-1)};$$

$$\beta^*_k = -\frac{2\binom{n-1}{2}\sum_{l=k}^{n-2}\binom{n-2}{l}}{(2^{n-2}-1)\binom{n-1}{k-1}(n-k)} \quad \textit{if } k \textit{ is odd};$$
$$\beta^*_k = -\frac{2\binom{n-1}{2}}{(2^{n-2}-1)(n-k+1)} - \frac{2\binom{n-1}{2}\left[\frac{k}{n}\binom{n-1}{k-2} - \sum_{l=k-3}^{n-2}\binom{n-2}{l}\right]}{(2^{n-2}-1)\binom{n-1}{k-1}(n-k)} \quad \textit{if } k \textit{ is even}.$$

In addition, the optimal redistribution schemes for any $\mu^*_{n,2} > 0$ are provided in Appendix A.3 (Corollary 3).

Theorem 2.3 *For m, $3 \leq m \leq n-1$, the optimal frontier of any anonymous and individually rational VCG mechanism is given as:*

$$\frac{\lambda^*_{n,m}}{A(n,m)} + \frac{\mu^*_{n,m}}{B(n,m)} = 1$$

where

$$A(n,m) = \frac{\binom{n-1}{m-1}}{\sum_{k=0}^{m-2}\binom{n-2}{k}} \simeq \frac{n}{m-1};$$

$$B(n,m) = \frac{\binom{n-1}{m-1}}{\sum_{k=0}^{m-3}\binom{n-2}{k} + \frac{m}{n-m}\sum_{k=m-1}^{n-2}\binom{n-2}{k}} \simeq \frac{n^m}{m!2^{n-2}}.$$

Remark 7 We conjecture that for a fixed m, the function $P(n,m) = B(n,m)/A(n,m)$ is strictly decreasing in n as is $P(n,2)$. This implies that the more agents participate, the smaller deficit that results can replace unit surplus. Because $P(n,m) \simeq \frac{n^{m-1}}{m(m-2)!2^{n-2}}$, more participation enables this replacement to be effective: the deficit becomes much more inexpensive than surplus as the number of agents increases. This behavior is not present in the problem of assigning economic goods. As Moulin (2009) discusses, individual rationality does not affect the relationship between surplus and deficit. That is to say, for the case of economic goods, unit surplus can be replaced only with unit deficit regardless of individual rationality. This point is shown in Appendix A2.

Remark 8 Recall that the WCRS of the pivotal mechanism is $\frac{n}{m-1}$. $A(n,m) \simeq \frac{n}{m-1}$ in Theorem 2.3 tells us that the optimal mechanism for $m \geq 3$ converges to the pivotal mechanism if deficit is not allowed. Again, the worst case relative surplus of the pivotal mechanism increases as more agents participate and its implementation cost always exhausts the entirety of efficient surplus.

4 Conclusion

Contrary to expectations, individual rationality significantly changes the characteristics of optimal mechanisms when facing the problem of assigning bads. Additionally, we need to run further equity tests on our optimal mechanisms. Although we provide a partial answer in Appendix A1, a more systematic analysis of the relationship between different performance measures could raise interesting questions.

Appendices

A1. Alternative Measure in Guo and Conitzer (2009)

We will illustrate that the alternative performance measure in Guo and Conitzer (2009) does not work in the problem of assigning bads. According to Guo and Conitzer (2009), the index is defined as:

$$\eta_{n,m}(r) = \sup_{c \in \mathcal{R}_+^N} \frac{|\Delta(c, r)|}{ps(c)}.$$

The *optimal "GC" mechanism* is a linear VCG mechanism with a redistribution scheme r^* that generates $\eta^*_{n,m} = \eta_{n,m}(r^*) \leq \eta_{n,m}(r)$ for any linear redistribution scheme r. The following propositions show that this measure is inappropriate since its optimal mechanism cannot even achieve its original goal.

Proposition 2 below presents the optimal "GC" mechanism for "bads" and the corresponding index. Proposition 3 proves that the "GC" optimality fails to achieve its original objective.

Proposition 2 *The optimal non-deficit linear "GC" mechanism for "bads" has the index:*

$$\eta^*_{n,m} = \frac{\binom{n-1}{m-1}}{\sum_{j=0}^{m-1} \binom{n-1}{j}}.$$

If $m = 1$, the mechanism redistributes nothing. For $m \geq 2$, its redistribution scheme r is written as $r^*(c_{-i}) = \sum_{k=1}^{m-1} a^*_k (c_{-i})^{*k}$. If m is odd, $a^*_k = (-1)^k a_k$ and if m is even, $a^*_k = (-1)^{k-1} a_k$. Here we write:

$$a_k = \frac{(n-m)\sum_{j=0}^{k-1}\binom{n-1}{j}}{k\binom{n-1}{k}}\eta^*_{n,m} = \frac{(n-m)\binom{n-1}{m-1}\sum_{j=0}^{k-1}\binom{n-1}{j}}{k\binom{n-1}{k}\sum_{j=0}^{m-1}\binom{n-1}{j}}.$$

Proof The worst case constraint is as follows:

$$\eta_{n,m} \geq 1 - \frac{\sum_{i=1}^{n} r(i; c_{-i})}{(n-m)c^{*m}}.$$

The non-deficit and worst case constraints are written together as:

$$(n-m)c^{*m} \geq \sum_{i \in N} r(i; c_{-i}) \geq (1 - \eta_{n,m})(n-m)c^{*m}.$$

Again, the system of inequalities is symmetric across all variables, so we will construct a symmetric redistribution scheme $r(c_{-i})$. We can write:

$$\sum_{i \in N} r(c_{-i}) = na_0 + (n-1)a_1 \cdot c^{*1} + (a_1 + (n-2)a_2)c^{*2} + (2a_2 + (n-3)a_3)c^{*3} + \cdots$$
$$+ ((n-3)a_{n-3} + 2a_{n-2})c^{*(n-2)} + ((n-2)a_{n-2} + a_{n-1})c^{*(n-1)} + (n-1)a_{n-1} \cdot c^{*n}.$$

Step 1: We first show that the non-deficit and worst case constraints imply $a_m = a_{m+1} = \cdots = a_{n-1} = 0$ and $a_0 = 0$. For cost profile $c^{*1} = c^{*2} = \cdots = c^{*n} = 0$, non-deficit and worst case constraints imply $0 \geq na_0 \geq 0$, that is, $a_0 = 0$. For a cost profile $c^{*1} = c^{*2} = \cdots = c^{*(n-1)} = 0$ and $c^{*n} = 1$, the two constraints imply $0 \geq (n-1)a_{n-1} \geq 0$, so $a_{n-1} = 0$. For a cost profile $c^{*1} = \cdots = c^{*(n-2)} = 0$ and $c^{*(n-1)} = 1$, the constraints imply $0 \geq (n-2)a_{n-2} \geq 0$, so $a_{n-2} = 0$. Likewise, we can conclude that $a_m = \cdots = a_{n-1} = 0$.

Then, the non-deficit and worst case constraints are written as:

$$\begin{aligned}0 \geq\ & (n-1)a_1 \cdot c^{*1} + (a_1 + (n-2)a_2)c^{*2} + (2a_2 + (n-3)a_3)c^{*3} + \cdots \\ & + ((m-2)a_{m-2} + (n-m+1)a_{m-1})c^{*(m-1)} + ((m-1)a_{m-1} + (n-m)(-1))c^{*m}\end{aligned}$$

and

$$\begin{aligned}0 \leq\ & (n-1)a_1 \cdot c^{*1} + (a_1 + (n-2)a_2)c^{*2} + (2a_2 + (n-3)a_3)c^{*3} + \cdots \\ & + ((m-2)a_{m-2} + (n-m+1)a_{m-1})c^{*(m-1)} + ((m-1)a_{m-1} + (n-m)(-1+\eta_{n,m}))c^{*m}.\end{aligned}$$

Applying Lemma 1, we transform the original optimization problem into a linear program. We aim to minimize $\eta_{n,m}$ satisfying the non-deficit and worst case constraints as follows:

$$\begin{aligned}(n-m) \geq\ & (m-1)a_{m-1} \geq (n-m)(1-\eta_{n,m}) \\ (n-m) \geq\ & (m-2)a_{m-2} + na_{m-1} \geq (n-m)(1-\eta_{n,m}) \\ (n-m) \geq\ & (m-3)a_{m-3} + n(a_{m-2} + a_{m-1}) \geq (n-m)(1-\eta_{n,m}) \\ & \vdots \\ (n-m) \geq\ & a_1 + n(a_2 + a_3 + \cdots + a_{m-1}) \geq (n-m)(1-\eta_{n,m}) \\ (n-m) \geq\ & n(a_1 + a_2 + a_3 + \cdots + a_{m-1}) \geq (n-m)(1-\eta_{n,m}).\end{aligned}$$

Step 2: Let a redistribution scheme $\hat{r}(c_{-i}) = \sum_{k=1}^{m-1} \hat{a}_k (c_{-i})^{*k}$ generate $\hat{\eta}_{n,m}$. Suppose that $\hat{\eta}_{n,m} \leq \eta^*_{n,m}$. Let $\hat{x}_k = \sum_{j=k}^{m-1} \hat{a}_j$ and let $x^*_k = \sum_{j=k}^{m-1} a^*_j$ for $k = 1, \cdots, m-1$. If m is odd, observe that

$$\begin{aligned}(m-1)x^*_{m-1} &= (n-m)(1-\eta^*_{n,m}) \\ (m-2)x^*_{m-2} + (n-m+2)x^*_{m-1} &= (n-m) \\ (m-3)x^*_{m-3} + (n-m+3)x^*_{m-2} &= (n-m)(1-\eta^*_{n,m}) \\ (m-4)x^*_{m-4} + (n-m+4)x^*_{m-3} &= (n-m) \\ &\vdots \\ 2x^*_2 + (n-2)x{*}_3 &= (n-m)(1-\eta^*_{n,m}) \\ x^*_1 + (n-1)x^*_2 &= (n-m) \\ nx^*_1 &= (n-m)(1-\eta^*_{n,m}).\end{aligned}$$

Since the redistribution scheme $\hat{r}$ satisfies non-deficit and worst case constraints, we have $(m-1)\hat{x}_{m-1} \geq (n-m)(1-\hat{\eta}_{n,m})$. In addition, we have $\hat{\eta}_{n,m} \leq \eta^*_{n,m}$ and $(m-1)x^*_{m-1} = (n-m)(1-\eta^*_{n,m})$. We can then conclude $\hat{x}_{m-1} \geq x^*_{m-1}$. The constraints also give $(n-m) \geq (m-2)\hat{x}_{m-2} + (n-m+2)\hat{x}_{m-1}$, and the previous observation gives $(m-2)x^*_{m-2} + (n-m+2)x^*_{m-1} = (n-m)$. With $\hat{x}_{m-1} \geq x^*_{m-1}$, we conclude $\hat{x}_{m-2} \leq x^*_{m-2}$. Applying the same logic from the third to the $(m-1)$th constraints and observation, we know $\hat{x}_{m-3} \geq x^*_{m-3}$, $\hat{x}_{m-4} \leq x^*_{m-4}$, $\cdots$, $\hat{x}_1 \leq x^*_1$ (the direction of inequality is alternating). Finally, the m^{th} constraints give $n\hat{x}_1 \geq (n-m)(1-\hat{\eta}_{n,m})$ and the observation gives $(n-m)(1-\eta^*_{n,m}) = nx^*_1$, so $\hat{x}_1 \geq x^*_1$. Concluding $\hat{x}_1 = x^*_1$ and $\hat{\eta}_{n,m} = \eta^*_{n,m}$, we have $\hat{x}_i = x_i^*$ for $i = 1, \cdots, m-1$, and this implies that $\hat{a}_k = a_k^*$ for $k = 1, \cdots, m-1$. Therefore, $\eta_{n,m}^*$ is optimal, and r^* is a unique optimal redistribution scheme. ■

Proposition 3

(i) *For m fixed, $\eta_{n,m}^*$ increases in n and it converges to 1.*

(ii) *For n fixed, $\eta_{n,m}^*$ decreases in m.*

(iii) *For m fixed, the largest ratio of budget imbalance to efficient surplus (efficiency loss) of the optimal "GC" mechanism diverges in n if m is even: $\eta_{n,m}^* \simeq n$ and it is infinite if m is odd.*

Proof

(i) We define $h(n) = \frac{\binom{n-1}{m-1}}{\sum_{j=0}^{m-1}\binom{n-1}{j}}$. For $k \geq m+1$, we have

$$\begin{aligned} h(k+1) - h(k) =& \frac{\binom{k}{m-1}\sum_{j=0}^{m-1}\binom{k-1}{j} - \binom{k-1}{m-1}\sum_{j=0}^{m-1}\binom{k}{j}}{\sum_{j=0}^{m-1}\binom{k}{j}\cdot\sum_{j=0}^{m-1}\binom{k-1}{j}} \\ =& \frac{\binom{k-1}{m-1}}{(k-m+1)\sum_{j=0}^{m-1}\binom{k}{j}\cdot\sum_{j=0}^{m-1}\binom{k-1}{j}}\sum_{j=0}^{m-1}(m-1-j)\binom{k}{j} > 0. \end{aligned}$$

which implies that h is increasing in n. Finally, $\lim_{n\to\infty}\eta_{n,m}^* = 1$. This is because $\binom{n-1}{m-1} \simeq \frac{n^{m-1}}{(m-1)!}$ and $\sum_{j=0}^{m-1}\binom{n-1}{j} \simeq \frac{n^{m-1}}{(m-1)!}$.

(ii) Proposition 2 states that the optimal "GC" mechanism generates $\eta_{n,m}^* = \frac{\binom{n-1}{m-1}}{\sum_{j=0}^{m-1}\binom{n-1}{j}} = 1 - \frac{\sum_{j=0}^{m-2}\binom{n-1}{j}}{\sum_{j=0}^{m-1}\binom{n-1}{j}}$. We fix n and define $l(m) = \frac{\sum_{j=0}^{m-2}\binom{n-1}{j}}{\sum_{j=0}^{m-1}\binom{n-1}{j}}$ for $2 \leq m \leq n-1$. $l(2) = \frac{1}{n}$ and $l(3) = \frac{2n}{n^2-n+2}$ which leads $l(3) - l(2) = \frac{n(n+1)-2}{n(n(n-1)+2)} > 0$. For $2 \leq k \leq n-2$, we have

$$\begin{aligned} l(k+1) - l(k) =& \Big(\sum_{j=0}^{k-1}\binom{n-1}{j}\Big)^2 - \Big(\sum_{j=0}^{k}\binom{n-1}{j}\Big)\Big(\sum_{j=0}^{k-2}\binom{n-1}{j}\Big) \\ =& \Big(\sum_{j=0}^{k-1}\binom{n-1}{j}\Big)\Big(-\binom{n-1}{k} + \binom{n-1}{k-1}\Big) + \binom{n-1}{k}\binom{n-1}{k-1} \\ =& \frac{(n-1)!}{(k-1)!(n-k-1)!}\Big(-\frac{1}{k} + \frac{1}{n-k}\Big)\Big(\sum_{j=0}^{k-1}\binom{n-1}{j}\Big) + \binom{n-1}{k}\binom{n-1}{k-1} \\ =& \frac{(n-1)!}{k!(n-k)!}\Big((-n+2k)\sum_{j=0}^{k-1}\binom{n-1}{j} + k\binom{n-1}{k}\Big) \\ =& \frac{(n-1)!}{k!(n-k)!}\Big(k\sum_{j=0}^{k}\binom{n-1}{j} + (-n+k)\sum_{j=0}^{k-1}\binom{n-1}{j}\Big). \end{aligned}$$

Define $L(k) = k\sum_{j=0}^{k}\binom{n-1}{j} + (-n+k)\sum_{j=0}^{k-1}\binom{n-1}{j}$. Then, we have

$$\begin{aligned}
L(k+1)-L(k) =&(k+1)\sum_{j=0}^{k+1}\binom{n-1}{j} + (-n+k+1)\sum_{j=0}^{k}\binom{n-1}{j}\\
&-k\sum_{j=0}^{k}\binom{n-1}{j} - (-n+k)\sum_{j=0}^{k-1}\binom{n-1}{j}\\
=&k\binom{n-1}{k+1} + \sum_{j=0}^{k+1}\binom{n-1}{j} + \binom{n-1}{k}(-n+k) + \sum_{j=0}^{k}\binom{n-1}{j}\\
=&-\binom{n-1}{k} + 2\sum_{j=0}^{k}\binom{n-1}{j} > 0
\end{aligned}$$

and thus, $L(k)$ is increasing in k. With $L(2) > 0$, $L(k)$ is positive for any $k \geq 2$, so $l(k)$ increases in k. Therefore, $\eta^*_{n,m}$ decreases in m.

(iii) Suppose m is even. Then, $l \cdot a_l + (n-l-1)\cdot a_{l+1} = \frac{(-1)^l(n-m)\binom{n-1}{m-1}}{\sum_{j=0}^{m-1}\binom{n-1}{j}}$ for $0 \leq l \leq m-2$ and $(m-1)a_{m-1} = (n-m)\frac{\sum_{j=0}^{m-2}\binom{n-1}{j}}{\sum_{j=0}^{m-1}\binom{n-1}{j}}$. Since we have

$$\sum_{i\in N} r^*(c_{-i}) = \sum_{l=0}^{m-2}\frac{(-1)^l(n-m)\binom{n-1}{m-1}}{\sum_{j=0}^{m-1}\binom{n-1}{j}}c^{*(l+1)} + (n-m)\frac{\sum_{j=0}^{m-2}\binom{n-1}{j}}{\sum_{j=0}^{m-1}\binom{n-1}{j}}c^{*m},$$

the efficiency loss of the optimal "GC" mechanism is

$$\lambda = \frac{(n-m)\binom{n-1}{m-1}}{\sum_{j=0}^{m-1}\binom{n-1}{j}} \cdot \sup_{c\in\mathcal{R}^N_+}\frac{n\big[\sum_{\substack{k=2\\ even}}^{m} c^{*k} - \sum_{\substack{k=1\\ odd}}^{m-1} c^{*k}\big]}{m\sum_{k=m+1}^{n} c^{*k} - (n-m)\sum_{k=1}^{m} c^{*k}}.$$

Observing that the ratio increases as $c^{*(m+1)}, c^{*(m+2)}, \cdots, c^{*n}$ decrease, we write

$$\sup_{c\in\mathcal{R}^N_+}\frac{n\big[\sum_{\substack{k=2\\ even}}^{m} c^{*k} - \sum_{\substack{k=1\\ odd}}^{m-1} c^{*k}\big]}{m\sum_{k=m+1}^{n} c^{*k} - (n-m)\sum_{k=1}^{m} c^{*k}} = \sup_{c\in\mathcal{R}^N_+}\frac{n\big[c^{*m} + \sum_{\substack{k=2\\ even}}^{m-2} c^{*k} - \sum_{\substack{k=1\\ odd}}^{m-1} c^{*k}\big]}{(n-m)\big[(m-1)c^{*m} - \sum_{\substack{k=1\\ odd}}^{m-1} c^{*k} - \sum_{\substack{k=2\\ even}}^{m-2} c^{*k}\big]}.$$

Notice that given $c^{*1}, c^{*3}, \cdots, c^{*(m-1)}$, the ratio increases as $c^{*2}, c^{*4}, \cdots, c^{*(m-2)}$ increase. Thus, the expression is written as

$$\begin{aligned}
&\sup_{c\in\mathcal{R}^N_+}\frac{n\big[c^{*m} + \sum_{\substack{k=3\\ odd}}^{m-1} c^{*k} - \sum_{\substack{k=1\\ odd}}^{m-1} c^{*k}\big]}{(n-m)\big[(m-1)c^{*m} - \sum_{\substack{k=1\\ odd}}^{m-1} c^{*k} - \sum_{\substack{k=3\\ odd}}^{m-1} c^{*k}\big]}\\
=&\sup_{c\in\mathcal{R}^N_+}\frac{n(c^{*m} - c^{*1})}{(n-m)\big[(m-1)c^{*m} - c^{*1} - 2\sum_{\substack{k=3\\ odd}}^{m-1} c^{*k}\big]}\\
=&\sup_{c\in\mathcal{R}^N_+}\frac{n(c^{*m} - c^{*1})}{(n-m)\big[(m-1)c^{*m} - c^{*1} - (m-2)c^{*m}\big]} = \frac{n}{n-m}.
\end{aligned}$$

The second last equality holds since the ratio increases as $c^{*3}, c^{*5}, \cdots, c^{*(m-1)}$ increase. Thus, $\lambda = \frac{n\binom{n-1}{m-1}}{\sum_{j=0}^{m-1}\binom{n-1}{j}}$. We know that $n\binom{n-1}{m-1} \simeq \frac{n^m}{(m-1)!}$ and $\sum_{j=0}^{m-1}\binom{n-1}{j} \simeq \frac{n^{m-1}}{(m-1)!}$. Therefore, $\lambda \simeq n$ if m is even. Similarly, if m is odd, $m \geq 3$,

$$\begin{aligned}
\lambda &= \frac{(n-m)}{\sum_{j=0}^{m-1}\binom{n-1}{j}} \sup_{c\in\mathcal{R}_+^N} \frac{n\Big\{\big[\binom{n-1}{m-1} + 2\sum_{j=0}^{m-2}\binom{n-1}{k}\big]c^{*m} + \binom{n-1}{m-1}\big[\sum_{\substack{k=1\\ odd}}^{m-2} c^{*k} - \sum_{\substack{k=2\\ even}}^{m-1} c^{*k}\big]\Big\}}{m\sum_{k=m+1}^{n} c^{*k} - (n-m)\sum_{k=1}^{m} c^{*k}} \\
&= \frac{(n-m)}{\sum_{j=0}^{m-1}\binom{n-1}{j}} \sup_{c\in\mathcal{R}_+^N} \frac{n\Big\{\big[\binom{n-1}{m-1} + 2\sum_{j=0}^{m-2}\binom{n-1}{k}\big]c^{*m} + \binom{n-1}{m-1}\big[\sum_{\substack{k=1\\ odd}}^{m-2} c^{*k} - \sum_{\substack{k=2\\ even}}^{m-1} c^{*k}\big]\Big\}}{(n-m)\big[(m-1)c^{*m} - \sum_{\substack{k=1\\ odd}}^{m-2} c^{*k} - \sum_{\substack{k=2\\ even}}^{m-1} c^{*k}\big]}.
\end{aligned}$$

The last equality holds since the ratio increases as $c^{*(m+1)}, \cdots, c^{*n}$ decrease. Observing that the ratio increase as $c^{*1}, c^{*3}, \cdots, c^{*(m-2)}$ increase, we write

$$\begin{aligned}
\lambda &= \frac{(n-m)}{\sum_{j=0}^{m-1}\binom{n-1}{j}} \sup_{c\in\mathcal{R}_+^N} \frac{n\big[\binom{n-1}{m-1} + 2\sum_{j=0}^{m-2}\binom{n-1}{j}\big]c^{*m}}{(n-m)\big[(m-1)c^{*m} - 2\sum_{\substack{k=2\\ even}}^{m-1} c^{*k}\big]} \\
&= \frac{(n-m)}{\sum_{j=0}^{m-1}\binom{n-1}{j}} \sup_{c\in\mathcal{R}_+^N} \frac{n\big[\binom{n-1}{m-1} + 2\sum_{j=0}^{m-2}\binom{n-1}{j}\big]c^{*m}}{(n-m)\big[(m-1)c^{*m} - (m-1)c^{*m}\big]} = \infty.
\end{aligned}$$

The second last equality holds since the ratio increases as $c^{*2}, c^{*4}, \cdots, c^{*(m-1)}$ increase. If $m = 1$, we know the pivotal is optimal and its efficiency loss is infinite. ■

The statement (i) points out that the pivotal mechanism becomes optimal as the number of agents increases. The optimal mechanism fails to redistribute any of the budget surplus of the pivotal mechanism. The statement (iii) shows that the efficiency loss of the optimal "GC" mechanism diverges in n or is infinite. Therefore, throughout this paper, we insist that we measure the performance of a mechanism by a worst case ratio whose denominator is efficient surplus.

A2. Comparison with the Case of Economic "Goods"

We will show that individual rationality does not affect the relationship between worst case relative surplus and worst case relative deficit for the case of "goods." To show this, we first introduce the parallel model for "goods."

Each agent $i \in N$ demands at most one unit and has private valuation $a_i \in \mathcal{R}_+$. Given a valuation profile $a = (a_1, \cdots, a_n) \in \mathcal{R}_+^N$, the vector a^* is the permutation of a whose coordinates are arranged accordingly:

$$a^{*1} \geq a^{*2} \geq \cdots \geq a^{*n}.$$

We let $a_{-i} = (a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n)$ and denote by $(a_{-i})^{*k}$ the kth highest valuation among $a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n$. Given a valuation profile $a \in \mathcal{R}_+^N$, we denote the maximal value by $v_m(a) = \sum_{k=1}^{m} a^{*k}$. The net utility of agent i in a VCG mechanism is written as:

$$U_i(a) = v_m(a) - h_i(a_{-i}) = v_m(a) - v_m(a_{-i}) + r(i; a_{-i}) = U_i^p(a) + r(i; a_{-i})$$

where $U_i^p(a)$ denotes agent i's net utility under the pivotal mechanism (Vickrey auction). The budget imbalance of a VCG mechanism is written as:

$$\begin{aligned}\Delta(a;r) &= v_m(a) - \sum_{i\in N} U_i(a) = v_m(a) - \sum_{i\in N}\{U_i^p(a) + r(i;a_{-i})\}\\ &= ps(a) - \sum_{i\in N} r(i;a_{-i})\end{aligned}$$

where $ps(a)$ denotes the budget surplus generated by the pivotal mechanism. Notice that $ps(a) = ma^{*(m+1)}$.

For the case of economic "goods", *individual rationality* is expressed in the following way:

$$U_i(a) \geq 0 \;\Leftrightarrow\; v_m(a) \geq h_i(a_{-i})$$

for all $i \in N$ and this is the same as $r(i;a_{-i}) \geq 0$ for all $i \in N$.

Efficient surplus is now defined similarly to the case of economic bads. A VCG mechanism generates greater value than the value created by random assignment. Thus, efficient surplus as the additional gain from implementing a VCG mechanism is written as $es(a) = v_m(a) - \frac{m}{n} a_N$ where $a_N = \sum_{i\in N} a_i$.

The two-way worst case constraint to identify feasible pairs of (λ, μ) is the same as (2) and is rewritten as follows:

$$ma^{*(m+1)} - \lambda\left(v_m(a) - \frac{m}{n}a_N\right) \leq \sum_{i\in N} r(i;a_{-i}) \leq ma^{*(m+1)} + \mu\left(v_m(a) - \frac{m}{n}a_N\right). \tag{3}$$

From constraint (3) together with $r(i;a_{-i}) \geq 0$ for all $i \in N$, we can compute the optimal frontier of individually rational VCG mechanisms.

For the sake of contrast, we will only consider the case of a single "good." Individual rationality does not change the one-to-one tradeoff between surplus and deficit in this case, while it causes extremely asymmetric tradeoff for the case of a single "bad."

Proposition 4 The optimal frontier of any individually rational VCG mechanism for the case of a single "good", $n \geq 3$ is given as:

$$\frac{\lambda^*_{n,1}}{A_{n,1}} + \frac{\mu^*_{n,1}}{B_{n,1}} = 1$$

where $A_{n,1} = \frac{n-1}{2^{n-2}-1}$ and $B_{n,1} = \frac{n-1}{2^{n-2}}$. Then, $\frac{B_{n,1}}{A_{n,1}} \simeq 1$ indicates almost one-to-one tradeoff between surplus and deficit.

Proof For $m = 1$, $v_m(a) = a^{*1}$ and $\sum_{i\in N} v_m(a_{-i}) = a^{*2} + (n-1)a^{*1}$. The basis of the space $\Gamma = \{a^* \in \mathcal{R}^n_+ : \; a^{*1} \geq \cdots \geq a^{*n}\}$ consists of e^k's, $0 \leq k \leq n$ such that $e^k = (1,\cdots,1,0,\cdots,0) \in \mathcal{R}^n_+$ where $\sum_{i\in N} e^k_i = k$. Likewise, we define $\epsilon^k = (1,\cdots,1,0,\cdots,0) \in \mathcal{R}^{n-1}_+$ where $\sum_{i\in N} \epsilon^k_i = k$. Let $r_m(\epsilon^k) = \rho_k$ for $0 \leq k \leq n-1$. Applying e^k for $0 \leq k \leq n$ to constraint (3) and the individual rationality condition, we have $\rho_0 = 0$ for $k = 0$. For $k = 1$, $v_m(a) = 1$ and $\sum_{i\in N} v_m(a_{-i}) = n-1$ gives

$$0 \leq (n-1)\rho_1 \leq \mu\left(1 - \frac{1}{n}\right),$$

and for $k = n$, we have

$$\max\{0, 1-\lambda\} \leq n\rho_{n-1} \leq 1+\mu.$$

For $n-1 \geq k \geq 2$, $v_m(a) = 1$ and $\sum_{i\in N} v_m(a_{-i}) = n$ gives

$$\max\{0, 1-\lambda+\frac{1}{n}k\lambda\} \leq k\rho_{k-1} + (n-k)\rho_k \leq 1+\mu-\frac{k}{n}\mu.$$

Assuming $1-\lambda+\frac{k}{n}\lambda \geq 0$ for $k \geq 2$, we have $0 \leq X_1 \leq \mu\left(1-\frac{1}{n}\right)$ and $1+\left(\frac{k}{n}-1\right)\lambda \leq X_k \leq 1+\left(1-\frac{k}{n}\right)\mu$ for $n \geq k \geq 2$. The hyperplane argument gives us:

$$\mu(n-1) + \sum_{\substack{k=3 \\ odd}} \binom{n}{k}\left(1+\left(1-\frac{k}{n}\right)\mu\right) \geq \sum_{\substack{k=1 \\ odd}} \binom{n}{k} X_k = \sum_{\substack{k=2 \\ even}} \binom{n}{k} X_k \geq \sum_{\substack{k=2 \\ even}} \binom{n}{k}\left(1+\left(\frac{k}{n}-1\right)\lambda\right).$$

Arranging again, we rewrite this as:

$$\lambda(2^{n-2}-1) + \mu 2^{n-2} \geq n-1.$$

We can easily check that $\lambda^*_{n,1} \leq \frac{n}{n-2}$ to satisfy individual rationality. ■

A3. Proofs

We will use notations as follows:

$$B_s^{t,t'} = \sum_{k=t}^{t'} \binom{s}{k}, \quad B_s^t = B_s^{t,s}, \quad B_s^{\to t} = B_s^{0,t}$$

$$\sum_{\substack{j=2 \\ even}} x_k = x_2 + x_4 + \cdots \quad \text{and} \quad \sum_{\substack{j=1 \\ odd}} x_k = x_1 + x_3 + \cdots .$$

Lemma 1

(i) *$b_1c_1 + \cdots + b_nc_n \leq 0$ for $0 \leq c_1 \leq \cdots \leq c_n$ if and only if $\sum_{j=k}^n b_j \leq 0$ for $k = 1, \cdots, n$.*

(ii) *$b_1c_1 + \cdots + b_nc_n \geq 0$ for $0 \leq c_1 \leq \cdots \leq c_n$ if and only if $\sum_{j=k}^n b_j \geq 0$ for $k = 1, \cdots, n$.*

Proof (i) Let $d_1 = c_1$, $d_2 = c_2 - c_1$, $\cdots$, $d_n = c_n - c_{n-1}$. Then, $b_1c_1 + \cdots + b_nc_n \leq 0$ for $0 \leq c_1 \leq \cdots \leq c_n$ if and only if $b_nd_n + (\sum_{j=n-1}^n b_j)d_{n-1} + (\sum_{j=n-2}^n b_j)d_{n-2} + \cdots + (\sum_{j=1}^n b_j)d_1 \leq 0$ for all $d_i \geq 0$, $i \in N$. Setting for each $i \in N$, $d_i = 1$ and $d_j = 0$ for all $j \in N$, $j \neq i$, we have the statement proven. (ii) can be proven in the same way. ■

Proof of Theorem 1.2

Statement (i)

Firstly, we will show the statement for the case of m, $4 \leq m \leq n-2$.

<u>Case 1</u>. *m is odd:*

The worst case constraint is as follows:

$$\lambda \geq \frac{(n-m)c^{*m} - \sum_{i=1}^{n} r(i;c_{-i})}{\frac{m}{n}\sum_{i=1}^{n} c_i - \sum_{i=1}^{m} c^{*i}}.$$

The non-deficit and worst case constraints are characterized by a system of linear inequalities as follows:

$$(n-m)c^{*m} \geq \sum_{i=1}^{n} r(i;c_{-i}) \geq (n-m)c^{*m} - \lambda\big(\frac{m}{n}\sum_{i=1}^{n} c_i - \sum_{i=1}^{m} c^{*i}\big).$$

In the inequalities above, both sides of $\sum_{i=1}^{n} r(i;c_{-i})$ is symmetric in all variables. If every $r(i;c_{-i})$ for $i \in N$ satisfies all inequalities, we can construct a symmetric scheme $\bar{r}$ meeting the inequalities. The symmetric scheme is written as $\bar{r}(c_{-i}) = \frac{1}{n!}\sum_{i\in N,\pi\in\Pi} r(i;c_{-i}^{\pi})$ where Π is the set of all permutations of $N \setminus \{i\}$ and c_{-i}^{π} results from permuting the coordinates of c_{-i} accordingly. Therefore, it is natural to restrict our discussion to symmetric redistribution schemes. $r(i;c_{-i})$ will be denoted by $r(c_{-i})$ from now on.

Let $e^{n-k} = (0,0,\cdots,0,1,1,\cdots,1)$, $e^{n-k} \in \mathcal{R}^n$ for $k = 0,1,\cdots,n$ where $\sum_{i=1}^{n}(e^{n-k})_i = n-k$. Let $\epsilon^{n-1-k} = (0,0,\cdots,0,1,1,\cdots,1)$, $\epsilon^{n-1-k} \in \mathcal{R}^{n-1}$ for $k = 0,1,\cdots,n-1$ where $\sum_{i=1}^{n-1}(\epsilon^{n-1-k})_i = n-1-k$. Define $\rho_k = r(\epsilon^{n-1-k})$ for $k = 0,\cdots,n-1$. The set $\{e^0, e^1,\cdots,e^n\}$ is a basis of C, $C = \{c \in R_+^n \mid c_1 \leq c_2 \leq \cdots \leq c_n\}$. Each $c^* \in C$ is uniquely written as a linear combination of elements of the basis.

Since $c_{-i} = (\epsilon^{n-1} - \epsilon^{n-2})(c_{-i})^{*1} + (\epsilon^{n-2} - \epsilon^{n-3})(c_{-i})^{*2} + \cdots + (\epsilon^3 - \epsilon^2)(c_{-i})^{*(n-3)} + (\epsilon^2 - \epsilon^1)(c_{-i})^{*(n-2)} + \epsilon^1(c_{-i})^{*(n-1)}$, the redistribution scheme is written as $r(c_{-i}) = (\rho_0 - \rho_1)(c_{-i})^{*1} + (\rho_1 - \rho_2)(c_{-i})^{*2} + \cdots + (\rho_{n-3} - \rho_{n-2})(c_{-i})^{*(n-2)} + \rho_{n-2}(c_{-i})^{*(n-1)}$.

Recall that $ps(c) = (n-m)c^{*m}$ and $es(c) = \frac{m}{n}\sum_{i\in N} c_i - \sum_{i=1}^{m} c^{*i}$. For a cost profile e^{n-k}, we notice that if $0 \leq k \leq m-1$, $es(c) = \frac{k}{n}(n-m)$ and $ps(c) = n-m$ and if $m \leq k \leq n$, $es(c) = \frac{m}{n}(n-k)$ and $ps(c) = 0$.

Now we will apply e^{n-k} for various k's. When $k = 0$, the non-deficit and worst case constraints are written as $n-m \leq n\rho_0 \leq n-m$, so $\rho_0 = \frac{n-m}{n}$. When $k = n$, the two constrains are written as $0 \leq n\rho_{n-1} \leq 0$, so $\rho_{n-1} = 0$. Applying e^{n-k} for other k, $1 \leq k \leq n-1$, the non-deficit and worst case constraints are written as follows:

$$\begin{aligned}
(n-m)\left(1-\frac{1}{n}-\frac{\lambda}{n}\right) &\le (n-1)\rho_1 \le (n-m)\left(1-\frac{1}{n}\right)\\
(n-m)\left(1-\frac{2}{n}\lambda\right) &\le 2\rho_1+(n-2)\rho_2 \le n-m\\
(n-m)\left(1-\frac{3}{n}\lambda\right) &\le 3\rho_2+(n-3)\rho_3 \le n-m\\
&\vdots\\
(n-m)\left(1-\frac{m-1}{n}\lambda\right) &\le (m-1)\rho_{m-2}+(n-m+1)\rho_{m-1} \le n-m\\
-\frac{m(n-m)}{n}\lambda &\le m\rho_{m-1}+(n-m)\rho_m \le 0\\
-\frac{m(n-m-1)}{n}\lambda &\le (m+1)\rho_m+(n-m-1)\rho_{m+1} \le 0\\
&\vdots\\
-\frac{2m}{n}\lambda &\le (n-2)\rho_{n-3}+2\rho_{n-2} \le 0\\
-\frac{m}{n}\lambda &\le (n-1)\rho_{n-2} \le 0.
\end{aligned}$$

We will use the notations M and ρ as follows:

$$M = \begin{pmatrix}
(n-1) & 0 & 0 & \cdots & 0 & 0\\
2 & (n-2) & 0 & \cdots & 0 & 0\\
0 & 3 & (n-3) & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & 3 & 0\\
0 & 0 & 0 & \cdots & (n-2) & 2\\
0 & 0 & 0 & \cdots & 0 & (n-1)
\end{pmatrix} \text{ and } \rho = \begin{pmatrix}\rho_1\\ \rho_2\\ \vdots\\ \rho_{n-3}\\ \rho_{n-2}\end{pmatrix}.$$

Then, M is a $(n-1)\times(n-2)$ matrix and $\rho \in \mathcal{R}^{n-2}$. Using the notations of M and ρ, the central part of above inequalities is written as the follows: $(M\rho)_1 = (n-1)\rho_1$, $(M\rho)_2 = 2\rho_1+(n-2)\rho_2$, $(M\rho)_3 = 3\rho_2+(n-3)\rho_3$, $\cdots$, $(M\rho)_{n-2} = (n-2)\rho_{n-3}+2\rho_{n-2}$ and $(M\rho)_{n-1} = (n-1)\rho_{n-2}$.

By computing the null space of the transposed M, we find the hyperplane of $\mathcal{R}^{n-1}$ as the range of M. For X in the range of M, the hyperplane is presented as

$$\binom{n}{1}X_1 + \binom{n}{3}X_3 + \cdots = \binom{n}{2}X_2 + \binom{n}{4}X_4 + \cdots$$

and the last term $\binom{n}{n-1}X_{n-1}$ appears in either side depending on whether n is odd or even. The no deficit and worst case constraints imply that

$$(n-m)\left(1-\frac{1}{n}-\frac{\lambda}{n}\right) \le X_1 \le (n-m)\left(1-\frac{1}{n}\right),$$

and

$$(n-m)\left(1-\frac{k}{n}\lambda\right) \le X_k \le n-m$$

for $2 \leq k \leq m-1$. And

$$-\lambda \frac{m(n-k)}{n} \leq X_k \leq 0$$

for $m \leq k \leq n-1$.

When n is odd, the non-deficit and worst case constraints imply that

$$\begin{aligned}
&(n-m)\Bigg[\sum_{\substack{k=1\\ odd}}^{m-2}\binom{n}{k}-1\Bigg] \geq \\
&\binom{n}{1}X_1+\sum_{\substack{k=3\\ odd}}^{m-2}\binom{n}{k}X_k+\sum_{\substack{k=m\\ odd}}^{n-2}\binom{n}{k}X_k=\sum_{\substack{k=2\\ even}}^{m-1}\binom{n}{k}X_k+\sum_{\substack{k=m+1\\ even}}^{n-1}\binom{n}{k}X_k \\
&\geq (n-m)\sum_{\substack{k=2\\ even}}^{m-1}\binom{n}{k}\left(1-\frac{k}{n}\lambda\right)-\lambda\sum_{\substack{k=m+1\\ even}}^{n-1}\binom{n}{k}\frac{m}{n}(n-k).
\end{aligned}$$

Then, we have

$$\lambda \geq \frac{(n-m)\Big[\sum_{\substack{k=0\\ even}}^{m-1}\binom{n}{k}-\sum_{\substack{k=1\\ odd}}^{m-2}\binom{n}{k}\Big]}{(n-m)\sum_{\substack{k=2\\ even}}^{m-1}\frac{k}{n}\binom{n}{k}+\sum_{\substack{k=m+1\\ even}}^{n-1}\frac{m}{n}(n-k)\binom{n}{k}}.$$

Likewise, when n is even, the non-deficit and worst case constraints imply that

$$\begin{aligned}
&(n-m)\Bigg[\sum_{\substack{k=1\\ odd}}^{m-2}\binom{n}{k}-1\Bigg] \geq \\
&\binom{n}{1}X_1+\sum_{\substack{k=3\\ odd}}^{m-2}\binom{n}{k}X_k+\sum_{\substack{k=m\\ odd}}^{n-1}\binom{n}{k}X_k=\sum_{\substack{k=2\\ even}}^{m-1}\binom{n}{k}X_k+\sum_{\substack{k=m+1\\ even}}^{n-2}\binom{n}{k}X_k \\
&\geq (n-m)\sum_{\substack{k=2\\ even}}^{m-1}\binom{n}{k}\left(1-\frac{k}{n}\lambda\right)-\lambda\sum_{\substack{k=m+1\\ even}}^{n-2}\binom{n}{k}\frac{m}{n}(n-k).
\end{aligned}$$

Then, we have

$$\lambda \geq \frac{(n-m)\Big[\sum_{\substack{k=0\\ even}}^{m-1}\binom{n}{k}-\sum_{\substack{k=1\\ odd}}^{m-2}\binom{n}{k}\Big]}{(n-m)\sum_{\substack{k=2\\ even}}^{m-1}\frac{k}{n}\binom{n}{k}+\sum_{\substack{k=m+1\\ even}}^{n-2}\frac{m}{n}(n-k)\binom{n}{k}}.$$

The optimal efficiency loss is written as follows:

$$\lambda^*_{n,m} = \frac{(n-m)\Big[\sum_{\substack{k=0\\ even}}^{m-1}\binom{n}{k}-\sum_{\substack{k=1\\ odd}}^{m-2}\binom{n}{k}\Big]}{(n-m)\sum_{\substack{k=2\\ even}}^{m-1}\frac{k}{n}\binom{n}{k}+\sum_{\substack{k=m+1\\ even}}^{\tilde{n}}\frac{m}{n}(n-k)\binom{n}{k}}.$$

From $\sum_{j=0}^{k}(-1)^j\binom{n}{j} = (-1)^k\binom{n-1}{k}$ for k, $0 \le k \le n-1$, we write

$$\sum_{\substack{k=0\\ even}}^{m-1}\binom{n}{k} - \sum_{\substack{k=1\\ odd}}^{m-2}\binom{n}{k} = \sum_{k=0}^{m-1}(-1)^k\binom{n}{k} = (-1)^{m-1}\binom{n-1}{m-1}.$$

From $k\binom{n}{k} = n\binom{n-1}{k-1}$, we write

$$\begin{aligned}\sum_{\substack{k=2\\ even}}^{m-1} k\binom{n}{k} &= n\sum_{\substack{k=2\\ even}}^{m-1}\binom{n-1}{k-1} = n\sum_{\substack{k=1\\ odd}}^{m-2}\binom{n-1}{k}\\ &= n\sum_{\substack{k=1\\ odd}}^{m-2}\left[\binom{n-2}{k} + \binom{n-2}{k-1}\right] = n\left[\sum_{\substack{k=1\\ odd}}^{m-2}\binom{n-2}{k} + \sum_{\substack{k=0\\ even}}^{m-3}\binom{n-2}{k}\right] = n\sum_{k=0}^{m-2}\binom{n-2}{k}\end{aligned}$$

and

$$\begin{aligned}\sum_{\substack{k=m+1\\ even}}^{\tilde{n}}\frac{m}{n}(n-k)\binom{n}{k} &= \frac{m}{n}\left[n\sum_{\substack{k=m+1\\ even}}^{\tilde{n}}\binom{n}{k} - \sum_{\substack{k=m+1\\ even}}^{\tilde{n}} k\binom{n}{k}\right] = m\left[\sum_{\substack{k=m+1\\ even}}^{\tilde{n}}\binom{n}{k} - \sum_{\substack{k=m+1\\ even}}^{\tilde{n}}\binom{n-1}{k-1}\right]\\ &= m\left[\sum_{\substack{k=m+1\\ even}}^{\tilde{n}}\binom{n}{k} - \sum_{\substack{k=m\\ odd}}^{\tilde{n}-1}\binom{n-1}{k}\right]\\ &= m\left\{\sum_{\substack{k=m+1\\ even}}^{\tilde{n}}\left[\binom{n-1}{k} + \binom{n-1}{k-1}\right] - \sum_{\substack{k=m\\ odd}}^{\tilde{n}-1}\left[\binom{n-2}{k} + \binom{n-2}{k-1}\right]\right\}\\ &= m\left[\sum_{k=m}^{\tilde{n}}\binom{n-1}{k} - \sum_{k=m-1}^{\tilde{n}-1}\binom{n-2}{k}\right].\end{aligned}$$

Therefore, the optimal efficiency loss is written as

$$\lambda^*_{n,m} = \frac{(n-m)\binom{n-1}{m-1}}{(n-m)\sum_{k=0}^{m-2}\binom{n-2}{k} + m\left[\sum_{k=m}^{\tilde{n}}\binom{n-1}{k} - \sum_{k=m-1}^{\tilde{n}-1}\binom{n-2}{k}\right]}.$$

<u>*Case 2*</u>*. m is even:*

When n is odd, the non-deficit and worst case constraints imply that

$$\begin{aligned}&n(n-m)\left(1-\frac{1}{n}-\frac{\lambda}{n}\right) + (n-m)\sum_{\substack{k=3\\ odd}}^{m-1}\binom{n}{k}\left(1-\frac{k}{n}\lambda\right) - \lambda\sum_{\substack{k=m+1\\ odd}}^{n-2}\binom{n}{k}\frac{m}{n}(n-k)\\ &\le \binom{n}{1}X_1 + \sum_{\substack{k=3\\ odd}}^{m-1}\binom{n}{k}X_k + \sum_{\substack{k=m+1\\ odd}}^{n-2}\binom{n}{k}X_k = \sum_{\substack{k=2\\ even}}^{m-2}\binom{n}{k}X_k + \sum_{\substack{k=m\\ even}}^{n-1}\binom{n}{k}X_k\\ &\le (n-m)\sum_{\substack{k=2\\ even}}^{m-2}\binom{n}{k}.\end{aligned}$$

Then, we have

$$\lambda \geq \frac{(n-m)\Big[\sum_{\substack{k=1\\ odd}}^{m-1}\binom{n}{k} - \sum_{\substack{k=0\\ even}}^{m-2}\binom{n}{k}\Big]}{(n-m)\sum_{\substack{k=1\\ odd}}^{m-1}\frac{k}{n}\binom{n}{k} + \sum_{\substack{k=m+1\\ odd}}^{n-2}\frac{m}{n}(n-k)\binom{n}{k}}.$$

Likewise, when n is even, the non-deficit and worst case constraints imply that

$$\begin{aligned}
&n(n-m)\Big(1-\frac{1}{n}-\frac{\lambda}{n}\Big) + (n-m)\sum_{\substack{k=3\\ odd}}^{m-1}\binom{n}{k}\Big(1-\frac{k}{n}\lambda\Big) - \lambda\sum_{\substack{k=m+1\\ odd}}^{n-1}\binom{n}{k}\frac{m}{n}(n-k)\\
&\leq \binom{n}{1}X_1 + \sum_{\substack{k=3\\ odd}}^{m-1}\binom{n}{k}X_k + \sum_{\substack{k=m+1\\ odd}}^{n-1}\binom{n}{k}X_k = \sum_{\substack{k=2\\ even}}^{m-2}\binom{n}{k}X_k + \sum_{\substack{k=m\\ even}}^{n-2}\binom{n}{k}X_k\\
&\leq (n-m)\sum_{\substack{k=2\\ even}}^{m-2}\binom{n}{k}.
\end{aligned}$$

Then, we have

$$\lambda \geq \frac{(n-m)\Big[\sum_{\substack{k=1\\ odd}}^{m-1}\binom{n}{k} - \sum_{\substack{k=0\\ even}}^{m-2}\binom{n}{k}\Big]}{(n-m)\sum_{\substack{k=1\\ odd}}^{m-1}\frac{k}{n}\binom{n}{k} + \sum_{\substack{k=m+1\\ odd}}^{n-1}\frac{m}{n}(n-k)\binom{n}{k}}.$$

The optimal efficiency loss is written as follows:

$$\lambda^*_{n,m} = \frac{(n-m)\Big[\sum_{\substack{k=1\\ odd}}^{m-1}\binom{n}{k} - \sum_{\substack{k=0\\ even}}^{m-2}\binom{n}{k}\Big]}{(n-m)\sum_{\substack{k=1\\ odd}}^{m-1}\frac{k}{n}\binom{n}{k} + \sum_{\substack{k=m+1\\ odd}}^{\bar{n}}\frac{m}{n}(n-k)\binom{n}{k}}.$$

We write $\sum_{\substack{k=1\\ odd}}^{m-1}\binom{n}{k} - \sum_{\substack{k=0\\ even}}^{m-2}\binom{n}{k} = -\sum_{k=0}^{m-1}(-1)^k\binom{n}{k} = (-1)^m\binom{n-1}{m-1} = \binom{n-1}{m-1}$ since m is even. And we write

$$\begin{aligned}
\frac{n-m}{n}\sum_{\substack{k=1\\ odd}}^{m-1}k\binom{n}{k} &= (n-m)\sum_{\substack{k=1\\ odd}}^{m-1}\binom{n-1}{k-1} = (n-m)\sum_{\substack{k=0\\ even}}^{m-2}\binom{n-1}{k}\\
&= (n-m)\Big[1+\sum_{\substack{k=2\\ even}}^{m-2}\Big[\binom{n-2}{k}+\binom{n-2}{k-1}\Big]\Big] = (n-m)\sum_{k=0}^{m-2}\binom{n-2}{k}
\end{aligned}$$

and

$$\begin{aligned}
\frac{m}{n}\sum_{\substack{k=m+1\\ odd}}^{\bar{n}}(n-k)\binom{n}{k} &= m\Big[\sum_{\substack{k=m+1\\ odd}}^{\bar{n}}\binom{n}{k} - \sum_{\substack{k=m+1\\ odd}}^{\bar{n}}\binom{n-1}{k-1}\Big] = m\Big[\sum_{\substack{k=m+1\\ odd}}^{\bar{n}}\binom{n}{k} - \sum_{\substack{k=m\\ even}}^{\bar{n}-1}\binom{n-1}{k}\Big]\\
&= m\Big\{\sum_{\substack{m+1\\ odd}}^{\bar{n}}\Big[\binom{n-1}{k}+\binom{n-1}{k-1}\Big] - \sum_{\substack{k=m\\ even}}^{\bar{n}-1}\Big[\binom{n-2}{k}+\binom{n-2}{k-1}\Big]\Big\}\\
&= m\Big[\sum_{k=m}^{\bar{n}}\binom{n-1}{k} - \sum_{k=m-1}^{\bar{n}-1}\binom{n-2}{k}\Big].
\end{aligned}$$

Therefore, the optimal efficiency loss is written as

$$\lambda^*_{n,m} = \frac{(n-m)\binom{n-1}{m-1}}{(n-m)\sum_{k=0}^{m-2}\binom{n-2}{k} + m\big[\sum_{k=m}^{\bar{n}}\binom{n-1}{k} - \sum_{k=m-1}^{\bar{n}-1}\binom{n-2}{k}\big]}.$$

We rewrite the optimal efficiency loss as follows:

$$\lambda^*_{n,m} = \frac{(n-m)\binom{n-1}{m-1}}{(n-m)\sum_{k=0}^{m-2}\binom{n-2}{k} + m\big[\sum_{k=m}^{\ddot{n}}\binom{n-1}{k} - \sum_{k=m-1}^{\ddot{n}-1}\binom{n-2}{k}\big]}$$

where $\ddot{n} = \tilde{n}$ if m is odd and $\ddot{n} = \bar{n}$ if m is even. $\tilde{n} = n-1$ if n is odd and $\tilde{n} = n-2$ if n is even. $\bar{n} = n-2$ if n is odd and $\bar{n} = n-1$ if n is even.

<u>*Case 3*</u>. $m = 2$: the non-deficit and worst case constraints are characterized by

$$(n-2)c^{*2} \geq \sum_{i\in N} r(c_{-i}) \geq (n-2)c^{*2} - \lambda\big[\frac{2}{n}\sum_{i\in N} c_i - c^{*1} - c^{*2}\big].$$

We apply e^{n-k} to the system above. Again $\rho_0 = \frac{n-2}{n}$ and $\rho_{n-1} = 0$. For $k = 1$,

$$(n-2)(1-\frac{1}{n}) \geq (n-1)\rho_1 \geq (n-2)\big(1-\frac{1}{n}-\frac{\lambda}{n}\big)$$

and for $n-1 \geq k \geq 2$,

$$0 \geq k\rho_{k-1} + (n-k)\rho_k \geq -\lambda\cdot\frac{2}{n}(n-k).$$

Then, we have

$$(n-2)(n-1-\lambda) - \lambda\sum_{\substack{k=3\\ odd}}\binom{n}{k}(n-k)\frac{2}{n} \leq 0$$

and find the optimal loss.

<u>*Case 4*</u>. $m = 3$: the non-deficit and worst constraints are written as

$$(n-3)c^{*3} \geq \sum_{i\in N} r(c_{-i}) \geq (n-3)c^{*3} - \lambda\left[\frac{3}{n}\sum_{i\in N} c_i - c^{*1} - c^{*2} - c^{*3}\right].$$

Applying e^{n-k} for $0 \leq k \leq n$, we have $\rho_0 = \frac{n-3}{n}$, $\rho_{n-1} = 0$ and for $k = 1$,

$$(n-3)\left(1-\frac{1}{n}\right) \geq (n-1)\rho_1 \geq (n-3)\left(1-\frac{1}{n}\right) - \lambda\left(\frac{3}{n}(n-1)-2\right)$$

for $k = 2$,

$$(n-3) \geq 2\rho_1 + (n-2)\rho_2 \geq (n-3) - \lambda\left(\frac{3}{n}(n-2)-1\right)$$

and for $n-1 \geq k \geq 3$,

$$0 \geq k\rho_{k-1} + (n-k)\rho_k \geq -\lambda\frac{3}{n}(n-k).$$

Finding

$$\binom{n}{1}(n-3)\Big(1-\frac{1}{n}\Big) \geq \binom{n}{2}\Big[(n-3)-\lambda\Big(\frac{3}{n}(n-2)-1\Big)\Big]-\lambda\sum_{\substack{k=4\\ even}}\binom{n}{k}(n-k)\frac{3}{n}$$

gives

$$\lambda\Big\{\frac{3}{n}\sum_{\substack{k=2\\ even}}(n-k)\binom{n}{k}-\binom{n}{2}\Big\} \geq \frac{n(n-3)^2}{2}.$$

Statement (ii)

<u>*Case 1*</u>. *m is odd:*

We rewrite the optimal efficiency loss as

$$\lambda^*_{n,m} = \frac{(n-m)\binom{n-1}{m-1}}{(n-m)B_{n-2}^{\to(m-2)}+m\big[B_{n-1}^{m,\tilde{n}}-B_{n-2}^{m-1,\tilde{n}-1}\big]}.$$

If n is even, we write

$$\begin{aligned} B_{n-1}^{m,\tilde{n}}-B_{n-2}^{m-1,\tilde{n}-1} &= \sum_{k=m}^{n-2}\Big[\binom{n-1}{k}-\binom{n-2}{k-1}\Big] = \sum_{k=m}^{n-2}\binom{n-2}{k} \\ &= 2^{n-2}-B_{n-2}^{\to(m-1)} \end{aligned}$$

and if n is odd, we write

$$\begin{aligned} B_{n-1}^{m,\tilde{n}}-B_{n-2}^{m-1,\tilde{n}-1} &= \sum_{k=m-1}^{n-2}\Big[\binom{n-1}{k+1}-\binom{n-2}{k}\Big] = (n-2)+\sum_{k=m-1}^{n-3}\binom{n-2}{k+1} \\ &= (n-2)+2^{n-2}-B_{n-2}^{\to(m-1)}. \end{aligned}$$

Then, we have $\binom{n-1}{m-1} \simeq \frac{n^{m-1}}{(m-1)!}$. Note that $B_{n-2}^{\to(m-2)}$ is a polynomial of degree $m-2$ and $B_{n-2}^{\to(m-1)}$ is a polynomial of degree $m-1$. Thus, we have

$$\begin{aligned} &(n-m)B_{n-2}^{\to(m-2)}+m\big(2^{n-2}-B_{n-2}^{\to(m-1)}\big) \simeq m2^{n-2} \\ &(n-m)B_{n-2}^{\to(m-2)}+m\big((n-2)+2^{n-2}-B_{n-2}^{\to(m-1)}\big) \simeq m2^{n-2}. \end{aligned}$$

Therefore, we conclude

$$\lambda^*_{n,m} \simeq \frac{(n-m)}{m!}\cdot\frac{n^{m-1}}{2^{n-2}}.$$

<u>*Case 2*</u>. *m is even:*

We rewrite the optimal efficiency loss as

$$\lambda^*_{n,m} = \frac{(n-m)\binom{n-1}{m-1}}{(n-m)B_{n-2}^{\to(m-2)}+m\big[B_{n-1}^{m,\tilde{n}}-B_{n-2}^{m-1,\tilde{n}-1}\big]}.$$

Since $\binom{n-1}{m-1} \simeq \frac{n^{m-1}}{(m-1)!}$ and

$$B_{n-1}^{m,\bar{n}} - B_{n-2}^{m-1,\bar{n}-1} = \sum_{k=m-1}^{\bar{n}-1} \left[\binom{n-1}{k+1} - \binom{n-2}{k} \right] = \sum_{k=m-1}^{n-3} \binom{n-2}{k+1} = 2^{n-2} - B_{n-2}^{\to(m-1)},$$

we conclude $\lambda^*_{n,m} \simeq \frac{(n-m)}{m!} \; \frac{n^{m-1}}{2^{n-2}}$. ■

We provides optimal redistribution schemes for $m \geq 3$ odd, corresponding to Theorem 1.2 in the following lemma.

Corollary 1 *The optimal redistribution scheme for m odd is as follows:*

$$\begin{aligned} r^*_{n,m}(c_{-i}) =& \sum_{k=1}^{3} \alpha^*_k (c_{-i})^{*k} + \sum_{k=4}^{m-1} \beta^*_k (c_{-i})^{*k} + \sum_{k=m}^{m+3} \gamma^*_k (c_{-i})^{*k} \\ &+ \sum_{k=4}^{n-m-2} \xi^*_{m+k} (c_{-i})^{*(m+k)} + \omega^*_{n-1} (c_{-i})^{*(n-1)} \end{aligned}$$

where

$$\alpha^*_1 = 0; \;\; \alpha^*_2 = \frac{2\lambda^*_{n,m}(n-m)}{n(n-2)}; \;\; \alpha^*_3 = \frac{-2\lambda^*_{n,m}(n-m)}{(n-3)(n-2)};$$

$$\begin{aligned} \beta^*_k =& \frac{(n-m)}{k} \cdot \frac{n-k\lambda^*_{n,m}}{n} - \frac{n}{k}\left(\frac{n-m}{n-k}\right)\Bigg\{ \frac{n-k\lambda^*_{n,m}}{n} - \frac{\binom{k}{2}}{\binom{n}{2}} \frac{\sum_{j=2}^{k-1} \binom{n-1}{j}}{\binom{n-2}{k-2}} \\ &+ \frac{2\binom{k}{2}}{n\binom{n-2}{k-2}} \sum_{\substack{j=2 \\ even}}^{k-2} \frac{(n-j\lambda^*_{n,m})\binom{n-2}{j-2}}{j!} - \frac{2\binom{k}{2}}{n\binom{n-2}{k-2}} \Bigg\} \quad \text{if } k \text{ is even;} \\ \beta^*_k =& -\frac{n-m}{n-k} + \frac{n}{n-k}\left(\frac{n-m}{n-k+1}\right)\Bigg\{ \frac{n-(k-1)\lambda^*_{n,m}}{n} - \frac{\binom{k-1}{2}}{\binom{n}{2}} \sum_{j=2}^{k-2} \frac{\binom{n-1}{j}}{\binom{n-2}{k-3}} \\ &+ \frac{2\binom{k-1}{2}}{n\binom{n-2}{k-3}} \sum_{\substack{j=2 \\ even}}^{k-3} \frac{(n-j\lambda^*_{n,m})\binom{n-2}{j-2}}{j!} - \frac{2\binom{k-1}{2}}{n\binom{n-2}{k-3}} \Bigg\} \quad \text{if } k \text{ is odd;} \end{aligned}$$

Letting $L_{n,m} = \frac{n-m}{n} - \sum_{k=1}^{3} \alpha^*_k - \sum_{k=4}^{m-1} \beta^*_k$,

$$\begin{aligned} \gamma^*_m =& \frac{n}{n-m} L_{n,m}; \;\; \gamma^*_{m+1} = \frac{m}{n}\lambda^*_{n,m} - \frac{nmL_{n,m}}{2\binom{n-m}{2}}; \\ \gamma^*_{m+2} =& -\frac{m}{(n-(m+2))}\lambda^*_{n,m} + \frac{n\binom{m+1}{2}L_{n,m}}{3\binom{n-m}{3}}; \\ \gamma^*_{m+3} =& \frac{m}{n}\lambda^*_{n,m} + \frac{m(m+2)}{2\binom{n-m-2}{2}}\lambda^*_{n,m} - \frac{n\binom{m+2}{3}L_{n,m}}{4\binom{n-m}{4}}; \end{aligned}$$

$$\xi^*_{m+k} = -\frac{n}{m+k}\left\{\frac{m(m+k)}{n(n-(m+k))}\lambda^*_{n,m} + \frac{m\lambda^*_{n,m}}{n\binom{n-1}{m+k}}\sum_{j=0}^{k-3}\binom{n-2}{m+j} - \frac{m}{n-m}\frac{\binom{n-1}{m}}{\binom{n-1}{m+k}}L_{n,m}\right\} \text{ if } k \text{ is even;}$$

$$\xi^*_{m+k} = \frac{m}{n}\lambda^*_{n,m} + \frac{n}{n-(m+k)}\left\{\frac{m(m+k-1)}{n(n-(m+k-1))}\lambda^*_{n,m} + \frac{m}{n}\lambda^*_{n,m}\sum_{j=0}^{k-4}\frac{\binom{n-2}{m+j}}{\binom{n-1}{m+k-1}}\right.$$
$$\left. - \frac{m}{n-m}\frac{\binom{n-1}{m}}{\binom{n-1}{m+k-1}}L_{n,m}\right\} \text{ if } k \text{ is odd;}$$

$$\omega^*_{n-1} = -\frac{m}{n(n-1)}\lambda^*_{n,m} \quad \text{if } n \text{ is odd;}$$
$$\omega^*_{n-1} = 0 \quad \text{if } n \text{ is even.}$$

Proof To find the optimal redistribution scheme, notice that when $\lambda = \lambda^*_{n,m}$, we have $X_1 = (n-m)(n-1)/n$. When $2 \le k \le m-1$, $X_k = n-m$ if k is an odd number and $X_k = (n-m)\left(1-\frac{k}{n}\lambda^*_{n,m}\right)$ if k is an even number. When $m \le k \le n-1$, $X_k = 0$ if k is an odd number and $X_k = -\lambda^*_{n,m}\frac{m(n-k)}{n}$ if k is an even number. Recalling $X_i = (M\rho)_i$, we can find the optimal redistribution scheme has coefficients.

The last term is given as

$$\rho_{n-2} = -\frac{m}{n(n-1)}\lambda^*_{n,m} \text{ if } n \text{ is odd and } 0 \text{ if } n \text{ is even.}$$

The first three terms are given as

$$\rho_0 - \rho_1 = 0$$
$$\rho_1 - \rho_2 = \frac{2\lambda^*_{n,m}(n-m)}{n(n-2)}$$
$$\rho_2 - \rho_3 = \frac{-2\lambda^*_{n,m}(n-m)}{(n-3)(n-2)}.$$

We will find the coefficient $\rho_k - \rho_{k+1}$ for all k, $3 \le k \le m-2$. We have

$$\rho_{2h} = \left(\frac{n-m}{n-2h}\right)\left\{\frac{n-2h\lambda^*_{n,m}}{n} - \sum_{j=0}^{h-2}\frac{2h(2h-1)\binom{n-2}{2j+1}}{(2j+2)(2j+3)\binom{n-2}{2h-2}}\right.$$
$$\left. + \sum_{j=1}^{h-1}\frac{(n-2j\lambda^*_{n,m})(2h)(2h-1)\binom{n-2}{2j-2}}{n(2j)!\binom{n-2}{2h-2}} - \frac{2h(2h-1)}{n\binom{n-2}{2h-2}}\right\}$$

for $2 \le h \le \frac{m-1}{2}$ and

$$\rho_{2h+1} = \frac{n-m}{n-2h-1} - \frac{2h+1}{n-2h-1}\rho_{2h}$$

for $2 \le h \le \frac{m-3}{2}$. Since we can write

$$\rho_{2h-1} - \rho_{2h} = \frac{(n-m)}{2h}\cdot\frac{n-2h\lambda^*_{n,m}}{n} - \frac{n}{2h}\rho_{2h}$$

for $2 \leq h \leq \frac{m-1}{2}$ and

$$\rho_{2h} - \rho_{2h+1} = -\frac{n-m}{n-2h-1} + \frac{n}{n-2h-1}\rho_{2h}$$

for $2 \leq h \leq \frac{m-3}{2}$, we conclude that

$$\begin{aligned}\rho_{2h-1} - \rho_{2h} = &\frac{(n-m)}{2h} \cdot \frac{n - 2h\lambda^*_{n,m}}{n} - \frac{n}{2h}\left(\frac{n-m}{n-2h}\right)\left\{\frac{n-2h\lambda^*_{n,m}}{n} - \sum_{j=0}^{h-2} \frac{2h(2h-1)\binom{n-2}{2j+1}}{(2j+2)(2j+3)\binom{n-2}{2h-2}}\right. \\ &\left. + \sum_{j=1}^{h-1} \frac{(n-2j\lambda^*_{n,m})(2h)(2h-1)\binom{n-2}{2j-2}}{n(2j)!\binom{n-2}{2h-2}} - \frac{2h(2h-1)}{n\binom{n-2}{2h-2}}\right\}\end{aligned}$$

for $2 \leq h \leq \frac{m-1}{2}$ and

$$\begin{aligned}\rho_{2h} - \rho_{2h+1} = &-\frac{n-m}{n-2h-1} + \frac{n}{n-2h-1}\left(\frac{n-m}{n-2h}\right)\left\{\frac{n-2h\lambda^*_{n,m}}{n} - \sum_{j=0}^{h-2} \frac{2h(2h-1)\binom{n-2}{2j+1}}{(2j+2)(2j+3)\binom{n-2}{2h-2}}\right. \\ &\left. + \sum_{j=1}^{h-1} \frac{(n-2j\lambda^*_{n,m})(2h)(2h-1)\binom{n-2}{2j-2}}{n(2j)!\binom{n-2}{2h-2}} - \frac{2h(2h-1)}{n\binom{n-2}{2h-2}}\right\}\end{aligned}$$

for $2 \leq h \leq \frac{m-3}{2}$. From $\sum_{k=0}^{m-2} \rho_k - \rho_{k+1} = \rho_0 - \rho_{m-1}$ and $\rho_0 = \frac{n-m}{n}$, we can compute ρ_{m-1}.

Now we will find the remaining coefficients $\rho_k - \rho_{k+1}$ for $m-1 \leq k \leq n-3$. We have the first four terms as

$$\begin{aligned}\rho_{m-1} - \rho_m =& \frac{n}{n-m}\rho_{m-1} \\ \rho_m - \rho_{m+1} =& \frac{m}{n}\lambda^*_{n,m} - \frac{nm}{(n-m)(n-(m+1))}\rho_{m-1} \\ \rho_{m+1} - \rho_{m+2} =& -\frac{m}{(n-(m+2))}\lambda^*_{n,m} + \frac{nm(m+1)}{(n-m)(n-(m+1))(n-(m+2))}\rho_{m-1} \\ \rho_{m+2} - \rho_{m+3} =& \frac{m}{n}\lambda^*_{n,m} + \frac{m(m+2)}{(n-(m+3))(n-(m+2))}\lambda^*_{n,m} \\ &- \frac{nm(m+1)(m+2)}{(n-m)(n-(m+3))(n-(m+2))(n-(m+1))}\rho_{m-1}.\end{aligned}$$

We can write

$$\begin{aligned}\rho_{m+2h} = &\frac{m(m+2h)}{n(n-(m+2h))}\lambda^*_{n,m} + \frac{m}{n}\lambda^*_{n,m}\sum_{j=0}^{h-2}\left(\frac{m+2j+2}{m+2h+1}\right)\frac{\binom{n}{m+2j+2}}{\binom{n}{m+2h+1}} \\ &- \left(\frac{m}{n-m}\right)\left(\frac{m+1}{m+2h+1}\right)\frac{\binom{n}{m+1}}{\binom{n}{m+2h+1}}\rho_{m-1}\end{aligned}$$

for $2 \leq h \leq \frac{n-m-2}{2}$. With

$$\rho_{m+2h-1} = -\frac{n-(m+2h)}{m+2h}\rho_{m+2h},$$

we have

$$\rho_{m+2h-1} - \rho_{m+2h} = -\frac{n}{m+2h}\Bigg\{\frac{m(m+2h)}{n(n-(m+2h))}\lambda^*_{n,m} + \frac{m}{n}\lambda^*_{n,m}\sum_{j=0}^{h-2}\left(\frac{m+2j+2}{m+2h+1}\right)\frac{\binom{n}{m+2j+2}}{\binom{n}{m+2h+1}}$$
$$- \left(\frac{m}{n-m}\right)\left(\frac{m+1}{m+2h+1}\right)\frac{\binom{n}{m+1}}{\binom{n}{m+2h+1}}\rho_{m-1}\Bigg\}$$

for $2 \le h \le \frac{n-m-2}{2}$. With

$$\rho_{m+2h+1} = -\frac{m}{n}\lambda^*_{n,m} - \frac{m+2h+1}{n-(m+2h+1)}\rho_{m+2h},$$

$$\rho_{m+2h} - \rho_{m+2h+1} = \frac{m}{n}\lambda^*_{n,m} + \frac{n}{n-(m+2h+1)}\Bigg\{\frac{m(m+2h)}{n(n-(m+2h))}\lambda^*_{n,m}$$
$$+ \frac{m}{n}\lambda^*_{n,m}\sum_{j=0}^{h-2}\left(\frac{m+2j+2}{m+2h+1}\right)\frac{\binom{n}{m+2j+2}}{\binom{n}{m+2h+1}}$$
$$- \left(\frac{m}{n-m}\right)\left(\frac{m+1}{m+2h+1}\right)\frac{\binom{n}{m+1}}{\binom{n}{m+2h+1}}\rho_{m-1}\Bigg\}$$

for $2 \le h \le \frac{n-m-4}{2}$. ■

The following lemma provides optimal redistribution schemes for $m = 2$, corresponding to Theorem 1.2.

Corollary 2 *The following linear redistribution scheme defines an optimal mechanism for $m = 2$:*

$$r^*_{n,2}(c_{-i}) = \sum_{k=1}^{5}\alpha^*_k(c_{-i})^{*k} + \sum_{k=6}^{n-1}\beta^*_k(c_{-i})^{*k}$$

where

$$\alpha^*_1 = \frac{n-2}{n(n-1)}\lambda^*_{n,2};\ \alpha^*_2 = 1 - \frac{\lambda^*_{n,2}}{n-1};\ \alpha^*_3 = -\frac{2}{n-3} + \frac{2\lambda^*_{n,2}}{n}\frac{n^2-4n+6}{(n-1)(n-3)};$$
$$\alpha^*_4 = \frac{3}{\binom{n-3}{2}} - \frac{\lambda^*_{n,2}(n^2-4n+6)}{(n-1)\binom{n-3}{2}};$$
$$\alpha^*_5 = -\frac{4}{\binom{n-3}{3}} + \frac{\lambda^*_{n,2}(n^4-9n^3+43n^2-83n+60)}{6\binom{n}{2}\binom{n-3}{3}};$$

$$\beta^*_k = \frac{\binom{n}{k}\left[2\binom{n-1}{2} + \lambda^*_{n,2}[n - 2\sum_{l=0}^{k-3}\binom{n-2}{l}]\right]}{n\binom{n-1}{k-1}\binom{n-1}{k}} - \frac{2\lambda^*_{n,2}}{n-k} \quad \textit{if } k \textit{ is even;}$$
$$\beta^*_k = \frac{\binom{n}{k}\left[(n-2)(\lambda^*_{n,2}-n+1) + 2\lambda^*_{n,2}\sum_{l=2}^{k-4}\binom{n-2}{l}\right]}{n\binom{n-1}{k-1}\binom{n-1}{k}} + \frac{2\lambda^*_{n,2}}{n}\left[\frac{n}{n-k+1} + \frac{\binom{n-2}{k-2}}{\binom{n-1}{k}}\right] \quad \textit{if } k \textit{ is odd.}$$

Proof of Theorem 1.1

When $m = 1$, $ps(c) = (n-1)c^{*1}$ and $es(c) = \frac{1}{n}\sum_{i\in N} c_i - c^{*1}$. Applying e^{n-k} with $k = 0$ and $k = n$, the no deficit and worst case constraints give $\rho_0 = \frac{n-1}{n}$ and $\rho_{n-1} = 0$. With $k = 1$, the constraints are $-(1+\lambda)\big(\frac{n-1}{n}\big) \le (n-1)\rho_1 \le -\frac{n-1}{n}$. And for k, $2 \le k \le n-1$, the constraints give $-\lambda\frac{n-k}{n} \le k\rho_{k-1} + (n-k)\rho_k \le 0$. Setting M and ρ as before, we find the same hyperplane and

$$-(n-1) \ge \binom{n}{1}X_1 + \binom{n}{3}X_3 + \cdots = \binom{n}{2}X_2 + \binom{n}{4}X_4 + \cdots \ge -\lambda \sum_{\substack{k=2\\ even}}^{\hat{n}} \binom{n}{k}\Big(\frac{n-k}{n}\Big)$$

leads to the following inequality:

$$\lambda \ge \frac{n-1}{\sum_{\substack{k=2\\ even}}^{\hat{n}} \binom{n}{k}(\frac{n-k}{n})}$$

where $\hat{n}$ is $n-1$ if n is odd and $\hat{n}$ is $n-2$ if n is even. Therefore, the optimal efficiency loss to efficient surplus is given as $\lambda^*_{n,1} = \frac{n-1}{2^{n-2}-1}$.

The optimal redistribution mechanism is as follows: If $n = 3$, $\rho_0 - \rho_1 = 1$ and $\rho_1 = -\frac{1}{3}$. If $n = 4$, $\rho_0 - \rho_1 = 1$, $\rho_1 - \rho_2 = -\frac{1}{4}$, and $\rho_2 = 0$. If $n \ge 5$, the first three terms are given as

$$\begin{aligned}
\rho_0 - \rho_1 &= 1\\
\rho_1 - \rho_2 &= \frac{8 - 8n - 2^n n + 4n^2}{(-4+2^n)(-2+n)n}\\
\rho_2 - \rho_3 &= -\frac{2(8 - 2^n - 6n + 2n^2)}{(-4+2^n)(-3+n)(-2+n)}
\end{aligned}$$

and the last term is given as

$$\rho_{n-2} = -\frac{4}{(-4+2^n)\,n}$$

if n is odd and $\rho_{n-2} = 0$ if n is even. The residual terms are computed as

$$\rho_{2h+1} - \rho_{2h+2} = \frac{4(n-1)}{n(2^n-4)} + \frac{4}{2^n-4}\cdot\frac{\sum_{l=1}^{h}\binom{n-1}{2l}}{\binom{n-2}{2h+1}} - \frac{1}{\binom{n-2}{2h+1}}$$

for $1 \le h \le \frac{n-5}{2}\cdot 1_{\{n:odd\}} + \frac{n-4}{2}\cdot 1_{\{n:even\}}$, and

$$\rho_{2h+2} - \rho_{2h+3} = -\frac{4(n-1)}{(n-2h-3)(2^n-4)} - \frac{4}{2^n-4}\frac{\sum_{l=1}^{h}\binom{n-1}{2l}}{\binom{n-2}{2h+2}} + \frac{1}{\binom{n-2}{2h+2}}$$

for $1 \le h \le \frac{n-5}{2}\cdot 1_{\{n:odd\}} + \frac{n-6}{2}\cdot 1_{\{n:even\}}$. ■

Proof of Theorem 1.3

When $m = n-1$ and $m \ge 2$, $ps(c) = c^{*(n-1)}$ and $es(c) = -\frac{1}{n}\sum_{i\in N} c_i + c^{*n}$. When cost profile is e^{n-k}, $ps(c) = 1$ for $0 \le k \le n-2$ and $ps(c) = 0$ for $n-1 \le k \le n$. Likewise, $es(c) = 1 - \frac{1}{n}(n-k)$

for $0 \le k \le n-1$ and $es(c) = 0$ for $k = n$. At each profile e^{n-k}, the no deficit and worst case constraints give $1 - \frac{1}{n} - \frac{\lambda}{n} \le (n-1)\rho_1 \le 1 - \frac{1}{n}$ for $k = 1$, $1 - \lambda\frac{k}{n} \le k\rho_{k-1} + (n-k)\rho_k \le 1$ for $2 \le k \le n-2$ and $-\lambda\big(1-\frac{1}{n}\big) \le (n-1)\rho_{n-2} \le 0$ for $k = n-1$. Using the same M and ρ and finding the hyperplane, we have

$$(n-1-\lambda) + \sum_{\substack{k=3\\ odd}}^{n-2} \Big(1 - \lambda\frac{k}{n}\Big)\binom{n}{k} \le \sum_{\substack{k=1\\ odd}}^{n-2} \binom{n}{k} X_k = \sum_{\substack{k=2\\ even}}^{n-1} \binom{n}{k} X_k \le \sum_{\substack{k=2\\ even}}^{n-3} \binom{n}{k}$$

if n is odd and

$$(n-1) + \sum_{\substack{k=3\\ odd}}^{n-3} \binom{n}{k} \ge \sum_{\substack{k=1\\ odd}}^{n-1} \binom{n}{k} X_k = \sum_{\substack{k=2\\ even}}^{n-2} \binom{n}{k} X_k \ge \sum_{\substack{k=2\\ even}}^{n-2} \binom{n}{k}\Big(1 - \lambda\frac{k}{n}\Big)$$

if n is even. Then, we have the following inequality:

$$\lambda \ge \frac{\sum_{\substack{k=1\\ odd}}^{n-2} \binom{n}{k} - \sum_{\substack{k=0\\ even}}^{n-3} \binom{n}{k}}{\sum_{\substack{k=1\\ odd}}^{n-2} \frac{k}{n}\binom{n}{k}}$$

when n is odd, and

$$\lambda \ge \frac{\sum_{\substack{k=0\\ even}}^{n-2} \binom{n}{k} - \sum_{\substack{k=1\\ odd}}^{n-3} \binom{n}{k}}{\sum_{\substack{k=2\\ even}}^{n-2} \frac{k}{n}\binom{n}{k}}$$

when n is even. Then, the optimal efficiency loss to efficient surplus is written as $\lambda^*_{n,n-1} = \frac{n-1}{2^{n-2}-1}$. ■

Proof of Theorem 1.4

$$ps(c) - \lambda\; es(c) \le \sum_{i \in N} r(c_{-i}) \le ps(c) + \mu\; es(c)$$

for all $c \in \mathcal{R}^N_+$ is rewritten as

$$(n-m)c^{*m} - \lambda\big\{\frac{m}{n}c_N - \tau_m(c)\big\} \le \sum_{i \in N} r(c_{-i}) \le (n-m)c^{*m} + \mu\big\{\frac{m}{n}c_N - \tau_m(c)\big\}.$$

We apply the profiles $e^{n-k} = (0, \cdots, 0, 1, \cdots, 1)$ and set $\rho_k = \gamma(\epsilon^{n-1-k})$. Then, we have for $0 \le k \le m-1$, $es(c) = \frac{k}{n}(n-m)$ and $ps(c) = n-m$. For $m \le k \le n$, $es(c) = \frac{m}{n}(n-k)$ and $ps(c) = 0$. We have $\rho_0 = \frac{n-m}{n}$. For $k = 1$,

$$(n-m)\big(1 - \frac{1}{n}\big) - \lambda\frac{n-m}{n} \le (n-1)\rho_1 \le (n-m)\big(1 - \frac{1}{n}\big) + \mu\frac{n-m}{n}.$$

If $2 \le k \le m-1$, we have

$$(n-m) - \lambda k\big(1 - \frac{m}{n}\big) \le k\rho_{k-1} + (n-k)\rho_k \le (n-m) + \mu k\big(1 - \frac{m}{n}\big)\big\},$$

and for $m \le k \le n-1$, we have

$$-\lambda \frac{m}{n}(n-k) \le k\rho_{k-1} + (n-k)\rho_k \le \mu \frac{m}{n}(n-k)$$

and finally $\rho_{n-1} = 0$. Thus, for $m \ge 3$, we have

$$(n-m)\big(1-\frac{1}{n}\big) - \lambda\big(\frac{n-m}{n}\big) \le X_1 \le (n-m)\big(1-\frac{1}{n}\big) + \mu\big(\frac{n-m}{n}\big)$$

and for $2 \le k \le m-1$,

$$(n-m) - \lambda k \frac{n-m}{n} \le X_k \le (n-m) + \mu k \frac{n-m}{n}$$

and for $m \le k \le n-1$, we have

$$-\lambda \frac{m}{n}(n-k) \le X_k \le \mu \frac{m}{n}(n-k).$$

(i) $m \ge 3$ is odd:

Using the hyperplane argument again, we have

$$\begin{aligned}
&(n-m)(n-1) + \mu(n-m) + \sum_{\substack{k=3 \\ odd}}^{m-2} \binom{n}{k}\big((n-m) + \mu k \frac{n-m}{n}\big) + \sum_{\substack{k=m \\ odd}} \binom{n}{k} \mu \frac{m}{n}(n-k) \\
&\ge \binom{n}{1} X_1 + \binom{n}{3} X_3 + \cdots = \binom{n}{2} X_2 + \binom{n}{4} X_4 + \cdots \ge \\
&\sum_{\substack{k=2 \\ even}}^{m-1} \binom{n}{k}\big((n-m) - \lambda k \frac{n-m}{n}\big) + \sum_{\substack{k=m+1 \\ even}} \binom{n}{k}\big(-\lambda \frac{m}{n}(n-k)\big).
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\lambda \Big\{ \frac{n-m}{n} \sum_{\substack{k=2 \\ even}}^{m-1} k \binom{n}{k} + \frac{m}{n} \sum_{\substack{k=m+1 \\ even}} \binom{n}{k}(n-k) \Big\} \\
&+ \mu \Big\{ (n-m) + \frac{n-m}{n} \sum_{\substack{k=3 \\ odd}}^{m-2} k \binom{n}{k} + \sum_{\substack{k=m \\ odd}} \binom{n}{k}(n-k)\frac{m}{n} \Big\} \\
&\ge \sum_{\substack{k=2 \\ even}}^{m-1} \binom{n}{k}(n-m) - \sum_{\substack{k=3 \\ odd}}^{m-2} \binom{n}{k}(n-m) - (n-m)(n-1).
\end{aligned}$$

Rearranging, we have

$$\begin{aligned}
&\lambda \Big\{ (n-m) \sum_{k=0}^{m-2} \binom{n-2}{k} + m \sum_{k=m} \binom{n-2}{k} \Big\} \\
&+ \mu \Big\{ (n-m) \sum_{k=0}^{m-3} \binom{n-2}{k} + m \sum_{k=m-1} \binom{n-2}{k} \Big\} \ge (n-m)\binom{n-1}{m-1}
\end{aligned}$$

and thus, the optimal frontier is given as follows:

$$\lambda^*_{n,m}\left\{(n-m)\sum_{k=0}^{m-2}\binom{n-2}{k}+m\sum_{k=m}\binom{n-2}{k}\right\}$$
$$+\,\mu^*_{n,m}\left\{(n-m)\sum_{k=0}^{m-3}\binom{n-2}{k}+m\sum_{k=m-1}\binom{n-2}{k}\right\}=(n-m)\binom{n-1}{m-1}.$$

$$\frac{\lambda^*_{n,m}}{A(n,m)}+\frac{\mu^*_{n,m}}{B(n,m)}=1$$

where

$$A(n,m)=\frac{\binom{n-1}{m-1}}{\sum_{k=0}^{m-2}\binom{n-2}{k}+\frac{m}{n-m}\sum_{k=m}\binom{n-2}{k}}$$

$$B(n,m)=\frac{\binom{n-1}{m-1}}{\sum_{k=0}^{m-3}\binom{n-2}{k}+\frac{m}{n-m}\sum_{k=m-1}\binom{n-2}{k}}.$$

It is easy to check $A(n,m)\simeq B(n,m)$. The optimal frontier for the case where $m\geq 4$ is even is the same. Likewise, we can check for the case of $m\geq 4$ where m is even.

(ii) $m=1$:

For $k=1$, we have

$$-\frac{n-1}{n}(\lambda+1)\leq X_1\leq\frac{n-1}{n}(\mu-1)$$

and for $2\leq k\leq n-1$, we have

$$-\frac{\lambda}{n}(n-k)\leq X_k\leq\frac{\mu}{n}(n-k).$$

Using the hyperplane, we have

$$(n-1)(\mu-1)+\mu\sum_{\substack{k=3\\ odd}}\binom{n}{k}\left(\frac{n-k}{n}\right)\geq-\lambda\sum_{\substack{k=2\\ even}}\binom{n}{k}\left(\frac{n-k}{n}\right)$$

which is arranged as

$$\lambda\left\{\sum_{k=1}\binom{n-1}{k}-\sum_{\substack{k=1\\ odd}}\binom{n-1}{k}\right\}+\mu\left\{(n-1)+\sum_{k=2}\binom{n-1}{k}-\sum_{\substack{k=2\\ even}}\binom{n-1}{k}\right\}\geq n-1.$$

The optimal frontier is given as follows:

$$\lambda^*_{n,1}\sum_{k=1}\binom{n-2}{k}+\mu^*_{n,1}\sum_{k=0}\binom{n-2}{k}=n-1.$$

Thus, $A(n,1)\simeq B(n,1)$

(iii) $m = 2$:

For $k = 1$, we have

$$(n-2)\left(1-\frac{1}{n}\right) - \lambda\left(\frac{n-2}{n}\right) \leq X_1 \leq (n-2)\left(1-\frac{1}{n}\right) + \mu\left(\frac{n-2}{n}\right),$$

and for $k \geq 2$, we have

$$-\lambda\frac{2}{n}(n-k) \leq X_k \leq \mu\frac{2}{n}(n-k).$$

According to the hyperplane argument, we have

$$(n-2)(n-1) - \lambda(n-2) - 2\lambda \sum_{\substack{k=3 \\ odd}} \binom{n}{k}\left(\frac{n-k}{n}\right) \leq 2\mu \sum_{\substack{k=2 \\ even}} \binom{n}{k}\left(\frac{n-k}{n}\right).$$

Arranging the inequality, we have the optimal frontier as follows:

$$(n-2)(n-1) = \lambda^*_{n,2}\left\{(n-2) + 2\sum_{k=2}\binom{n-2}{k}\right\} + 2\mu^*_{n,2}\sum_{k=1}\binom{n-2}{k}.$$

It is easy to see that $A(n,2) \simeq B(n,2)$. ■

Proof of Proposition 1

We will apply cost profile e^{n-k} for k, $0 \leq k \leq n$. Recall for $0 \leq k \leq m-1$, $es(c) = \frac{k}{n}(n-m)$ with $ps(c) = n-m$ and for $m \leq k \leq n$, $es(c) = \frac{m}{n}(n-k)$ with $ps(c) = 0$.

$V_i \leq \frac{m}{n}c_i$ with anonymity requires $\frac{n-m}{n}c^{*m} \leq r(c^{*1}, \cdots, c^{*(m-1)}, c^{*(m+1)}, \cdots, c^{*n})$. This is written as

$$\begin{aligned}
&(\rho_0 - \rho_1)c^{*1} + \cdots + (\rho_{m-2} - \rho_{m-1})c^{*(m-1)} - \frac{n-m}{n}c^{*m} \\
&+ (\rho_{m-1} - \rho_m)c^{*(m+1)} + \cdots + \rho_{n-2}c^{*n} \geq 0.
\end{aligned}$$

This inequality holds if and only if $\rho_k \geq 0$ for all k, $m \leq k \leq n-2$ and $\rho_k \geq \frac{n-m}{m}$ for all k, $0 \leq k \leq m-1$. The non-deficit constraint implies

$$\begin{aligned}
(n-m)c^{*m} &\geq (n-1)(\rho_0 - \rho_1)c^{*1} + [(\rho_0 - \rho_1) + (n-2)(\rho_1 - \rho_2)]c^{*2} \\
&+ \cdots + [(m-1)(\rho_{m-2} - \rho_{m-1}) + (n-m)(\rho_{m-1} - \rho_m)]c^{*m} + \cdots \\
&+ [(n-2)(\rho_{n-3} - \rho_{n-2}) + \rho_{n-2}]c^{*(n-1)} + (n-1)\rho_{n-2}c^{*n}
\end{aligned}$$

and by Lemma 1, this holds if and only if $\rho_{n-2} \leq 0$, $(n-2)\rho_{n-3} + 2\rho_{n-2} \leq 0, \cdots, m\rho_{m-1} + (n-m)\rho_m \leq 0$, $(m-1)\rho_{m-2} + (n-m+1)\rho_{m-1} \leq n-m, \cdots, \rho_0 + (n-1)\rho_1 \leq n-m$ and $n\rho_0 \leq n-m$. With unanimity upper bound, this implies $\rho_{n-2} = \cdots = \rho_m = 0$. Since $\rho_m = 0$, the non-deficit constraint gives $m\rho_{m-1} \leq 0$ but this contradicts $\rho_{m-1} \geq \frac{n-m}{n}$ given by unanimity upper bound. Therefore, there is no anonymous linear VCG mechanism satisfying unanimity upper bound and non-deficit. ■

Proof of Theorem 2.1

We know that the pivotal mechanism is anonymous and individually rational. It generates no deficit, that is, $\mu^*_{n,1} = 0$, but its efficiency loss is $\lambda^*_{n,1} = \infty$. If λ is restricted to be finite, we have that for $k = 0$, $es(c) = 0$ and $ps(c) = n - 1$, so the worst case constraint implies $\rho_0 = \frac{n-1}{n}$. For $k = n$, the worst case constraint implies $\rho_{n-1} = 0$. For k, $1 \le k \le n-1$, the worst case constraint gives

$$-\lambda\frac{1}{n}(n-k) \le k\rho_{k-1} + (n-k)\rho_k \le \mu\frac{1}{n}(n-k)$$

and individual rationality implies $r(c_{-i}) \ge 0$, which is $\rho_k \ge 0$ for all $0 \le k \le n-1$. The worst case and individual rationality constraints are together written as

$$0 \le k\rho_{k-1} + (n-k)\rho_k \le \mu\frac{1}{n}(n-k)$$

for $1 \le k \le n-1$. For X in the range of M, we have

$$0 \le X_1 \le \frac{n-1}{n}(\mu - 1).$$

If $n = 2$, this inequality pair with $\rho_1 = 0$ gives $\mu \ge 1$, so that $\mu^*_{2,1} = 1$ and $\lambda^*_{2,1} = 0$. From $\rho_0 = \frac{1}{2}$, the optimal mechanism redistribution scheme is $r(c_{-i}) = \frac{1}{2}(c_{-i})^{*1}$.

Let $n \ge 3$. For $2 \le k \le n-1$ we have

$$0 \le X_k \le \mu\frac{m}{n}(n-k).$$

Note that from $0 \le X_1 \le \frac{n-1}{n}(\mu - 1)$, we should have $\mu \ge 1$. Let $\mu = 1$. Set $\rho_k = 0$ for all k, $1 \le k \le n-2$, then the inequality constraints are satisfied. With $\rho_0 = \frac{n-1}{n}$, we can set $r(c_{-i}) = \frac{n-1}{n}(c_{-i})^{*1}$ and compute the efficiency loss of this redistribution scheme. $\Delta(c) = \frac{n-1}{n}(c^{*1} - c^{*2})$ and $es(c) = \frac{\sum_{i\in N} c_i}{n} - c^{*1}$.

$$\mu = \sup_{c\in\mathcal{R}^N_+} \frac{|\Delta(c)|}{es(c)} = \frac{n-1}{n} \sup_{c\in\mathcal{R}^N_+} \frac{c^{*2} - c^{*1}}{\frac{\sum_{i\in N} c_i}{n} - c^{*1}} = \frac{n-1}{n} \sup_{c\in\mathcal{R}^N_+} \frac{c^{*2} - c^{*1}}{\frac{c^{*1} + (n-1)c^{*2}}{n} - c^{*1}} = 1.$$

Therefore, the optimal $\mu = 1$ with $\lambda = 0$, and the optimal redistribution scheme is $r(c_{-i}) = \frac{n-1}{n}(c_{-i})^{*1}$. ■

Proof of Theorem 2.2

We found $\rho_0 = \frac{n-2}{n}$ and $\rho_{n-1} = 0$.

$$\max\left\{0, \frac{n-2}{n}(n-1-\lambda)\right\} \le X_1 \le \frac{n-2}{n}(n-1+\mu).$$

For $2 \le k \le n-1$, $0 \le X_k \le \mu\frac{2}{n}(n-k)$. Suppose $n - 1 \ge \lambda$. Then, we have

$$(n-2)(n-1-\lambda) \le \binom{n}{1}X_1 + \sum_{\substack{k=3\\ odd}}^{\tilde{n}} \binom{n}{k} X_k = \sum_{\substack{k=2\\ even}}^{\hat{n}} \binom{n}{k} X_k \le \mu\frac{2}{n} \sum_{\substack{k=2\\ even}}^{\hat{n}} \binom{n}{k}(n-k)$$

and

$$(n-2)(n-1) = (n-2)\lambda^*_{n,2} + 2\sum_{k=0}^{n-3} \binom{n-2}{k} \mu^*_{n,2}.$$

The maximal $\lambda^*_{n,2} = n-1$, so $\lambda^*_{n,2}$ satisfies $\lambda \leq n-1$.

Recall $B(n,2) = \frac{(n-1)(n-2)}{2^{n-1}-2}$. For $n \geq 5$, we have

$$B(n,2) - B(n-1,2) = \frac{n-2}{2}\left[\frac{n-1}{2^{n-2}-1} - \frac{n-3}{2^{n-3}-1}\right] = -\frac{(n-2)[(n-5)2^{n-3}+2]}{2(2^{n-2}-1)(2^{n-3}-1)} < 0,$$

so $B(n,2)$ is strictly decreasing in n. $B(4,2) = \frac{3}{n-1}$ gives the result. ■

We provide the optimal redistribution schemes corresponding to Theorem 2.2 in the following lemma.

Corollary 3 *For any $\mu^*_{n,2} > 0$ chosen, the optimal redistribution scheme is as follows:*

$$r^*(c_{-i}) = \sum_{k=1}^{6} \alpha^*_k (c_{-i})^{*k} + \sum_{k=7}^{n-1} \beta^*_k (c_{-i})^{*k}$$

where

$$\begin{aligned}
\alpha^*_1 &= -\frac{\mu^*_{n,2}(2^{n-1}-2) - 2\binom{n-1}{2}}{n(n-1)} \\
\alpha^*_2 &= \frac{\mu^*_{n,2} n(2^{n-2}-1)^2 - 2\binom{n-1}{2}^2}{n\binom{n-1}{2}(2^{n-2}-1)} \\
\alpha^*_3 &= \frac{2\left[\binom{n-1}{2}^2 - \mu^*_{n,2}(2^{n-2}-1)^2\right]}{(2^{n-2}-1)\binom{n-1}{2}(n-3)} \\
\alpha^*_4 &= -\frac{2\binom{n-1}{2}}{n(2^{n-2}-1)} - \frac{2\left[\binom{n-1}{2}^2 - \mu^*_{n,2}(2^{n-2}-1)^2\right]}{(2^{n-2}-1)\binom{n-1}{3}(n-4)} \\
\alpha^*_5 &= \frac{2}{(n-5)(2^{n-2}-1)}\left[\binom{n-1}{2} + \frac{\binom{n-1}{2}^2 - \mu^*_{n,2}(2^{n-2}-1)^2}{\binom{n-1}{4}}\right] \\
\alpha^*_6 &= -\frac{2\binom{n-1}{2}}{(2^{n-2}-1)(n-5)} - \frac{2\binom{n-1}{2}}{n(2^{n-2}-1)}\frac{\binom{n-1}{3}}{\binom{n-1}{6}} - \frac{\binom{n-1}{2}^2 - \mu^*_{n,2}(2^{n-2}-1)^2}{3\binom{n-1}{6}(2^{n-2}-1)};
\end{aligned}$$

$$\begin{aligned}
\beta^*_k &= \frac{2\left[\binom{n-1}{2}\sum_{l=1}^{k-1}\binom{n-2}{l} - \mu^*_{n,2}(2^{n-2}-1)^2\right]}{(2^{n-2}-1)\binom{n-1}{k-1}(n-k)} \quad \textit{if } k \textit{ is odd;} \\
\beta^*_k &= -\frac{2\left\{\binom{n-1}{2}\left[n\sum_{l=1}^{k-4}\binom{n-2}{l} + k\binom{n-1}{k-2}\right] - n\mu^*_{n,2}(2^{n-2}-1)^2\right\}}{n(2^{n-2}-1)\binom{n-1}{k-1}(n-k)} \\
&\quad - \frac{2\binom{n-1}{2}}{(2^{n-2}-1)(n-k+1)} \quad \textit{if } k \textit{ is even.}
\end{aligned}$$

Proof If $\lambda^*_{n,2} = 0$, then

$$\mu^*_{n,2} = \frac{(n-2)(n-1)}{2\sum_{k=0}^{n-3}\binom{n-2}{k}} = \frac{\binom{n-1}{2}}{2^{n-2}-1}.$$

From $X_1 = \frac{n-2}{n}(n-1)$, $\rho_1 = \frac{n-2}{n}$. $X_k = 0$ for k odd, $3 \le k \le \tilde{n}$ and $X_k = \frac{2}{n}(n-k)\frac{\binom{n-1}{2}}{2^{n-2}-1}$ for k even, $2 \le k \le \hat{n}$. We have $\rho_0 = \rho_1 = \frac{n-2}{n}$ and $\rho_{n-1} = 0$. Recall that $\lambda^*_{n,2}$ and $\mu^*_{n,2}$ satisfy

$$2\binom{n-1}{2} = (n-2)\lambda^*_{n,2} + 2\sum_{k=0}^{n-3}\binom{n-2}{k}\mu^*_{n,2}.$$

Let $C(n,m) = 2\binom{n-1}{2}$, $A(n,m) = n-2$ and $B(n,m) = 2\sum_{k=0}^{n-3}\binom{n-2}{k}$. Let $L = \frac{C(n,m)}{B(n,m)} = \frac{\binom{n-1}{2}}{2^{n-2}-1}$. For k even, $2 \le k \le \hat{n}$,

$$\rho_k = \frac{2L}{n}\cdot 1_{\{k\ge 4\}} + \frac{2L}{n}\frac{\sum_{l=3}^{k-2}\binom{n-2}{l}}{\binom{n-1}{k}}\cdot 1_{\{k\ge 6\}} + \frac{2\binom{n-1}{2}}{\binom{n-1}{k}}\frac{L-1}{n}$$

and for k odd, $\tilde{n} \ge k \ge 3$,

$$\rho_k = -\frac{k}{n-k}\left\{\frac{2L}{n}\cdot 1_{\{k\ge 5\}} + \frac{2L}{n}\frac{\sum_{l=3}^{k-3}\binom{n-2}{l}}{\binom{n-1}{k-1}}\cdot 1_{\{k\ge 7\}} + \frac{2\binom{n-1}{2}}{\binom{n-1}{k-1}}\frac{L-1}{n}\right\}.$$

Therefore, $a^*_1 = \rho_0 - \rho_1 = 0$ and $a^*_2 = \rho_1 - \rho_2 = 1 - \frac{2L}{n}$. For $k \ge 3$, we have

$$\begin{aligned}
a^*_k =& \rho_{k-1} - \rho_k = \frac{n}{n-k}\left\{\frac{2L}{n}\cdot 1_{\{k\ge 5\}} + \frac{2L}{n}\frac{\sum_{l=3}^{k-3}\binom{n-2}{l}}{\binom{n-1}{k-1}}\cdot 1_{\{k\ge 7\}} + \frac{2\binom{n-1}{2}}{\binom{n-1}{k-1}}\frac{L-1}{n}\right\}\\
=& \frac{n}{n-k}\left\{\frac{2\binom{n-1}{2}}{n(2^{n-2}-1)}\cdot 1_{\{k\ge 5\}} + \frac{2\binom{n-1}{2}}{n(2^{n-2}-1)}\frac{\sum_{l=3}^{k-3}\binom{n-2}{l}}{\binom{n-1}{k-1}}\cdot 1_{\{k\ge 7\}}\right.\\
&\left. + \frac{2\binom{n-1}{2}}{\binom{n-1}{k-1}}\frac{\binom{n-1}{2}-(2^{n-2}-1)}{n(2^{n-2}-1)}\right\}
\end{aligned}$$

if k is odd and

$$\begin{aligned}
a^*_k =& -\frac{2L}{n}\left\{1 + \frac{k-1}{(n-k+1)}\cdot 1_{\{k\ge 6\}}\right\} - \frac{2L}{n}\left\{\frac{\sum_{l=3}^{k-4}\binom{n-2}{l}}{\binom{n-1}{k-1}}\cdot 1_{\{k\ge 8\}} + \frac{\sum_{l=3}^{k-2}\binom{n-2}{l}}{\binom{n-1}{k}}\cdot 1_{\{k\ge 6\}}\right\}\\
& - 2\binom{n-1}{2}\frac{L-1}{n}\frac{\binom{n}{k}}{\binom{n-1}{k-1}\binom{n-1}{k}}\\
=& -\frac{2\binom{n-1}{2}}{n(2^{n-2}-1)}\left\{1 + \frac{k-1}{n-k+1}\cdot 1_{\{k\ge 6\}}\right\} - \frac{2\binom{n-1}{2}}{n(2^{n-2}-1)}\left\{\frac{\sum_{l=3}^{k-4}\binom{n-2}{l}}{\binom{n-1}{k-1}}\cdot 1_{\{k\ge 8\}}\right.\\
& \left. + \frac{\sum_{l=3}^{k-2}\binom{n-2}{l}}{\binom{n-1}{k}}\cdot 1_{\{k\ge 6\}}\right\} - 2\binom{n-1}{2}\frac{\binom{n-1}{2}-(2^{n-2}-1)}{n(2^{n-2}-1)}\frac{\binom{n}{k}}{\binom{n-1}{k-1}\binom{n-1}{k}}
\end{aligned}$$

if k is even.

Given any $\mu^*_{n,2}$, $0 < \mu^*_{n,2} \le \frac{C(n,m)}{B(n,m)}$, we can find the corresponding redistribution scheme. Let $T = \frac{\mu^*_{n,2}B(n,m)-C(n,m)}{A(n,m)(n-1)}$. Now $X_1 = \frac{(n-1)(n-2)}{n}(1+T)$ instead of $X_1 = \frac{(n-1)(n-2)}{n}$ for the $\mu^*_{n,2} = L$ case. For k odd, $k \ge 3$, still we have $X_k = 0$ and for k even, $k \ge 2$, we have $X_k = \mu^*\frac{2}{n}(n-k)$ instead of $L\frac{2}{n}(n-k)$ for the $\mu^*_{n,2} = L$ case. Then, $\rho_1 = \frac{n-2}{n}[1+T]$ with $\rho_0 = \frac{n-2}{n}$. For k even, $2 \le k \le \hat{n}$,

$$\rho_k = \frac{2L}{n}\cdot 1_{\{k\ge 4\}} + \frac{2L}{n}\frac{\sum_{l=3}^{k-2}\binom{n-2}{l}}{\binom{n-1}{k}}\cdot 1_{\{k\ge 6\}} + \frac{2\binom{n-1}{2}}{\binom{n-1}{k}}\frac{L-1-T}{n}$$

and for k odd, $\tilde{n} \geq k \geq 3$,

$$\rho_k = -\frac{k}{n-k}\left\{\frac{2L}{n}\cdot 1_{\{k\geq 5\}} + \frac{2L}{n}\frac{\sum_{l=3}^{k-3}\binom{n-2}{l}}{\binom{n-1}{k-1}}\cdot 1_{\{k\geq 7\}} + \frac{2\binom{n-1}{2}}{\binom{n-1}{k-1}}\frac{L-1-T}{n}\right\}.$$

Therefore, $a_1 = \rho_0 - \rho_1 = -\frac{n-2}{n}T$ and $a_2 = \rho_1 - \rho_2 = 1 + T - \frac{2L}{n}$. For $k \geq 3$, we have

$$a_k = \rho_{k-1} - \rho_k = \frac{n}{n-k}\left\{\frac{2L}{n}\cdot 1_{\{k\geq 5\}} + \frac{2L}{n}\frac{\sum_{l=3}^{k-3}\binom{n-2}{l}}{\binom{n-1}{k-1}}\cdot 1_{\{k\geq 7\}} + \frac{2\binom{n-1}{2}}{\binom{n-1}{k-1}}\frac{L-1-T}{n}\right\}$$

if k is odd and

$$a_k = -\frac{2L}{n}\left\{1_{\{k\geq 4\}} + \frac{k-1}{(n-k+1)}\cdot 1_{\{k\geq 6\}}\right\} - \frac{2L}{n}\left\{\frac{\sum_{l=3}^{k-4}\binom{n-2}{l}}{\binom{n-1}{k-1}}\cdot 1_{\{k\geq 8\}} + \frac{\sum_{l=3}^{k-2}\binom{n-2}{l}}{\binom{n-1}{k}}\cdot 1_{\{k\geq 6\}}\right\}$$
$$- 2\binom{n-1}{2}\frac{L-1-T}{n}\frac{\binom{n}{k}}{\binom{n-1}{k-1}\binom{n-1}{k}}$$

if k is even. From $L = \frac{\binom{n-1}{2}}{2^{n-2}-1}$, we have

$$T = \frac{\mu^*_{n,2}}{L} - 1 = \frac{\mu^*_{n,2}(2^{n-2}-1) - \binom{n-1}{2}}{\binom{n-1}{2}}$$

and

$$L - 1 - T = \frac{\binom{n-1}{2}}{2^{n-2}-1} - \mu^*_{n,2}\cdot\frac{2^{n-2}-1}{\binom{n-1}{2}} = \frac{\binom{n-1}{2}^2 - \mu^*_{n,2}\cdot(2^{n-2}-1)^2}{(2^{n-2}-1)\binom{n-1}{2}}.$$

Plugging the functional forms of L, T and $L-1-T$ in n and m into a_k's, we have the coefficient a^*_k for $3 \leq k \leq n-1$ as follows:

$$a^*_k = \frac{n}{n-k}\left\{\frac{2\binom{n-1}{2}}{n(2^{n-2}-1)}\cdot 1_{\{k\geq 5\}} + \frac{2\binom{n-1}{2}}{n(2^{n-2}-1)}\frac{\sum_{l=3}^{k-3}\binom{n-2}{l}}{\binom{n-1}{k-1}}\cdot 1_{\{k\geq 7\}}\right.$$
$$\left. + \frac{2\left[\binom{n-1}{2}^2 - \mu^*_{n,2}\cdot(2^{n-2}-1)^2\right]}{n(2^{n-2}-1)\binom{n-1}{k-1}}\right\}$$

if k is odd and

$$a^*_k = -\frac{2\binom{n-1}{2}}{n(2^{n-2}-1)}\left\{1_{\{k\geq 4\}} + \frac{k-1}{(n-k+1)}\cdot 1_{\{k\geq 6\}}\right\} - \frac{2\binom{n-1}{2}}{n(2^{n-2}-1)}\left\{\frac{\sum_{l=3}^{k-4}\binom{n-2}{l}}{\binom{n-1}{k-1}}\cdot 1_{\{k\geq 8\}}\right.$$
$$\left. + \frac{\sum_{l=3}^{k-2}\binom{n-2}{l}}{\binom{n-1}{k}}\cdot 1_{\{k\geq 6\}}\right\} - \frac{2\binom{n}{k}}{\binom{n-1}{k-1}\binom{n-1}{k}}\frac{\binom{n-1}{2}^2 - \mu^*_{n,2}\cdot(2^{n-2}-1)^2}{n(2^{n-2}-1)}$$

if k is even where $0 < \mu^*_{n,2} \leq \frac{\binom{n-1}{2}}{2^{n-2}-1}$. ■

Proof of Theorem 2.3

Let $\tilde{n} = n-1$ and $\hat{n} = n-2$ if n is even and $\tilde{n} = n-2$ and $\hat{n} = n-1$ if n is odd.

Firstly, we will show for any m, $3 \le m \le n-2$ *in Case 1 and 2*:

Case 1. m is odd:

Again $\rho_0 = \frac{n-m}{n}$ and $\rho_{n-1} = 0$. For $1 \le k \le n-1$, individual rationality and the worst case constraint require

$$\max\left\{0, (n-m)\left(1 - \frac{\lambda}{n} - \frac{1}{n}\right)\right\} \le X_1 \le (n-m)\left(\frac{\mu}{n} + \frac{n-1}{n}\right)$$

and for $m-1 \ge k \ge 2$

$$\max\left\{0, (n-m)\left(1 - \frac{\lambda k}{n}\right)\right\} \le X_k \le (n-m)\left(1 + \mu\frac{k}{n}\right)$$

and for $n-1 \ge k \ge m$,

$$0 \le X_k \le \mu\frac{m}{n}(n-k).$$

If $\lambda \le \frac{n}{m-1}$, then, $\max\left\{0, (n-m)\left(1-\frac{\lambda k}{n}\right)\right\} = (n-m)\left(1-\frac{\lambda k}{n}\right)$ and $\max\left\{0, (n-m)\left(1-\frac{\lambda}{n}-\frac{1}{n}\right)\right\} = (n-m)\left(1-\frac{\lambda}{n}-\frac{1}{n}\right)$.

We have

$$\begin{aligned}
&(n-m)(n+\mu-1) + (n-m)\sum_{\substack{k=3\\ odd}}^{m-2}\binom{n}{k}\left(1+\mu\frac{k}{n}\right) + \sum_{\substack{k=m\\ odd}}^{\tilde{n}}\binom{n}{k}\mu\frac{m}{n}(n-k)\\
&\ge \sum_{\substack{k=1\\ odd}}^{m-2}\binom{n}{k}X_k + \sum_{\substack{k=m\\ odd}}^{\tilde{n}}\binom{n}{k}X_k = \sum_{\substack{k=2\\ even}}^{m-1}\binom{n}{k}X_k + \sum_{\substack{k=m+1\\ even}}^{\hat{n}}\binom{n}{k}X_k\\
&\ge (n-m)\sum_{\substack{k=2\\ even}}^{m-1}\binom{n}{k}\max\left\{0, 1-\frac{\lambda k}{n}\right\}.
\end{aligned}$$

Assuming that $\lambda \le \frac{n}{m-1}$, we have

$$\begin{aligned}
&\lambda\left[(n-m)\sum_{\substack{k=2\\ even}}^{m-1}\binom{n}{k}\frac{k}{n}\right] + \mu\left[(n-m)\sum_{\substack{k=1\\ odd}}^{m-2}\binom{n}{k}\frac{k}{n} + \sum_{\substack{k=m\\ odd}}^{\tilde{n}}\binom{n}{k}(n-k)\frac{m}{n}\right]\\
&\ge (n-m)\left\{\sum_{\substack{k=0\\ even}}^{m-1}\binom{n}{k} - \sum_{\substack{k=1\\ odd}}^{m-2}\binom{n}{k}\right\}
\end{aligned}$$

and thus,

$$\begin{aligned}
&\lambda^*_{n,m}\left[(n-m)\sum_{\substack{k=2\\ even}}^{m-1}\binom{n}{k}\frac{k}{n}\right] + \mu^*_{n,m}\left[(n-m)\sum_{\substack{k=1\\ odd}}^{m-2}\binom{n}{k}\frac{k}{n} + \sum_{\substack{k=m\\ odd}}^{\tilde{n}}\binom{n}{k}(n-k)\frac{m}{n}\right]\\
&= (n-m)\left\{\sum_{\substack{k=0\\ even}}^{m-1}\binom{n}{k} - \sum_{\substack{k=1\\ odd}}^{m-2}\binom{n}{k}\right\}.
\end{aligned}$$

This is rewritten as

$$\mu^*_{n,m}\Bigg[(n-m)\sum_{k=0}^{m-3}\binom{n-2}{k}+m\Bigg[\sum_{k=m-1}^{\tilde{n}}\binom{n-1}{k}-\sum_{k=m-2}^{\tilde{n}-1}\binom{n-2}{k}\Bigg]\Bigg]$$
$$+\lambda^*_{n,m}\Bigg[(n-m)\sum_{k=0}^{m-2}\binom{n-2}{k}\Bigg]=(n-m)\binom{n-1}{m-1}.$$

Now we will check if $\lambda^*_{n,m}$ satisfies the assumption $\lambda \leq \frac{n}{m-1}$. Let $\mu^*_{n,m}=\delta\lambda^*_{n,m}$ for some $\delta \geq 0$. Then,

$$\lambda^*_{n,m}=\frac{(n-m)\Big\{\sum_{\substack{k=0\\ even}}^{m-1}\binom{n}{k}-\sum_{\substack{k=1\\ odd}}^{m-2}\binom{n}{k}\Big\}}{(n-m)\Big\{\sum_{\substack{k=2\\ even}}^{m-1}\binom{n}{k}\frac{k}{n}+\sum_{\substack{k=1\\ odd}}^{m-2}\delta\binom{n}{k}\frac{k}{n}\Big\}+\delta\sum_{\substack{k=m\\ odd}}^{\tilde{n}}\binom{n}{k}(n-k)\frac{m}{n}}.$$

$\lambda^*_{n,m} \leq \frac{n}{m-1}$ holds if and only if

$$(n-m)(m-1)\Bigg\{\sum_{\substack{k=0\\ even}}^{m-1}\binom{n}{k}-\sum_{\substack{k=1\\ odd}}^{m-2}\binom{n}{k}\Bigg\}$$
$$\leq n\Bigg\{(n-m)\Bigg\{\sum_{\substack{k=2\\ even}}^{m-1}\binom{n}{k}\frac{k}{n}+\sum_{\substack{k=1\\ odd}}^{m-2}\delta\binom{n}{k}\frac{k}{n}\Bigg\}+\delta\sum_{\substack{k=m\\ odd}}^{\tilde{n}}\binom{n}{k}(n-k)\frac{m}{n}\Bigg\}.$$

Since $\binom{n}{k}k=n\binom{n-1}{k-1}$ and $\binom{n}{k}(n-k)=n\binom{n-1}{k}$, the right hand side of the previous inequality is written as

$$n\Bigg\{(n-m)\Bigg\{\sum_{\substack{k=2\\ even}}^{m-1}\binom{n}{k}\frac{k}{n}+\sum_{\substack{k=1\\ odd}}^{m-2}\delta\binom{n}{k}\frac{k}{n}\Bigg\}+\delta\sum_{\substack{k=m\\ odd}}^{\tilde{n}}\binom{n}{k}(n-k)\frac{m}{n}\Bigg\}$$
$$=(n-m)\Bigg\{n\sum_{\substack{k=2\\ even}}^{m-1}\binom{n-1}{k-1}+n\delta\sum_{\substack{k=1\\ odd}}^{m-2}\binom{n-1}{k-1}\Bigg\}+nm\delta\sum_{\substack{k=m\\ odd}}^{\tilde{n}}\binom{n-1}{k}.$$

Thus, $\lambda^*_{n,m} \leq \frac{n}{m-1}$ if and only if

$$(n-m)\Bigg\{(m-1)\Bigg(\sum_{\substack{k=0\\ even}}^{m-1}\binom{n}{k}-\sum_{\substack{k=1\\ odd}}^{m-2}\binom{n}{k}\Bigg)-n\sum_{\substack{k=2\\ even}}^{m-1}\binom{n-1}{k-1}\Bigg\}$$
$$\leq \delta\Bigg\{n(n-m)\sum_{\substack{k=1\\ odd}}^{m-2}\binom{n-1}{k-1}+mn\sum_{\substack{k=m\\ odd}}^{\tilde{n}}\binom{n-1}{k}\Bigg\}.$$

The right hand side of the inequality is nonnegative. We will show that the left hand side is always negative and thus, the inequality holds. Let

$$A(n)=(m-1)\Bigg(\sum_{\substack{k=0\\ even}}^{m-1}\binom{n}{k}-\sum_{\substack{k=1\\ odd}}^{m-2}\binom{n}{k}\Bigg)-n\sum_{\substack{k=2\\ even}}^{m-1}\binom{n-1}{k-1}.$$

Observe that $A(n) < 0$ if $m = 3$. For $m \geq 5$, we check first

$$\begin{aligned}A(m+1) &= (m-1)\Bigg(\sum_{\substack{k=0\\ even}}^{m-1}\binom{m+1}{k} - \sum_{\substack{k=1\\ odd}}^{m-2}\binom{m+1}{k}\Bigg) - (m+1)\sum_{\substack{k=2\\ even}}^{m-1}\binom{m}{k-1} \\ &= (m-1)m - (m+1)\sum_{\substack{k=2\\ even}}^{m-1}\binom{m}{k-1} \leq (m-1)m - m(m+1) < 0\end{aligned}$$

and show that $A(n)$ is decreasing in n.

$$\begin{aligned}A(n) - A(n+1) =&(m-1)\Bigg(\sum_{\substack{k=0\\ even}}^{m-1}\binom{n}{k} - \sum_{\substack{k=1\\ odd}}^{m-2}\binom{n}{k}\Bigg) - n\sum_{\substack{k=2\\ even}}^{m-1}\binom{n-1}{k-1} \\ &- (m-1)\Bigg(\sum_{\substack{k=0\\ even}}^{m-1}\binom{n+1}{k} - \sum_{\substack{k=1\\ odd}}^{m-2}\binom{n+1}{k}\Bigg) + (n+1)\sum_{\substack{k=2\\ even}}^{m-1}\binom{n}{k-1}.\end{aligned}$$

Since $\binom{n-1}{k-1} - \binom{n}{k-1} = -\binom{n-1}{k-2}$, $\binom{n}{k} - \binom{n+1}{k} = -\binom{n}{k-1}$ and $n\binom{n-1}{k-2} + \binom{n}{k-1} = k\binom{n}{k-1}$, we have

$$A(n) - A(n+1) =(m-1)\Bigg[\sum_{\substack{k=1\\ odd}}^{m-2}\binom{n}{k-1} - \sum_{\substack{k=2\\ even}}^{m-1}\binom{n}{k-1}\Bigg] + \sum_{\substack{k=2\\ even}}^{m-1}k\binom{n}{k-1}.$$

Finally, we write

$$\begin{aligned}A(n) - A(n+1) =&(m-1)\Bigg[\sum_{\substack{k=1\\ odd}}^{m-2}\binom{n}{k-1} - \sum_{\substack{k=2\\ even}}^{m-3}\binom{n}{k-1}\Bigg] + \sum_{\substack{k=2\\ even}}^{m-3}k\binom{n}{k-1} \\ =&(m-1)\Bigg[\sum_{\substack{k=0\\ even}}^{m-3}\binom{n}{k} - \sum_{\substack{k=1\\ odd}}^{m-4}\binom{n}{k}\Bigg] + \sum_{\substack{k=2\\ even}}^{m-3}k\binom{n}{k-1} > 0\end{aligned}$$

and thus, $A(n)$ is decreasing in n. We conclude that $A(n) < 0$, that is, the desired inequality holds for $\lambda^*_{n,m}$.

<u>*Case 2*</u>. *m is even:*

We have

$$\begin{aligned}&\max\Bigg\{0, n(n-m)\Bigg(1 - \frac{\lambda}{n} - \frac{1}{n}\Bigg)\Bigg\} + \sum_{\substack{k=3\\ odd}}^{m-1}\binom{n}{k}\max\Bigg\{0, (n-m)\Bigg(1 - \frac{\lambda k}{n}\Bigg)\Bigg\} \\ &\leq \sum_{\substack{k=1\\ odd}}^{m-1}\binom{n}{k}X_k + \sum_{\substack{k=m+1\\ odd}}^{\tilde{n}}\binom{n}{k}X_k = \sum_{\substack{k=2\\ even}}^{m-2}\binom{n}{k}X_k + \sum_{\substack{k=m\\ even}}^{\hat{n}}\binom{n}{k}X_k \\ &\leq (n-m)\sum_{\substack{k=2\\ even}}^{m-2}\binom{n}{k}\Bigg(1 + \mu\frac{k}{n}\Bigg) + \mu\sum_{\substack{k=m\\ even}}^{\hat{n}}\binom{n}{k}\frac{m}{n}(n-k)\end{aligned}$$

Given m, the optimal surplus loss λ^* can be considered as a function of n. We should find $\hat{k}$ such that $\hat{k} = \max\{k,\ 3 \le k \le m-1, \text{odd} |\ \frac{n}{k} \ge \lambda^*(n) \text{ for all } n\}$. Here $\lambda^*(n)$ is the optimal surplus loss computed with assuming $n/\hat{k} \ge \lambda^*$.

Assuming $n/\hat{k} \ge \lambda$, we have

$$\max\left\{0, (n-m)\left(1 - \frac{\lambda k}{n}\right)\right\} = (n-m)\left(1 - \frac{\lambda k}{n}\right)$$

only for $3 \le k \le \hat{k}$. Then,

$$\begin{aligned}
&(n-m)(n-\lambda-1) + (n-m)\sum_{\substack{k=3\\ odd}}^{\hat{k}} \binom{n}{k}\left(1 - \frac{\lambda k}{n}\right) \\
\le &\sum_{\substack{k=1\\ odd}}^{m-1} \binom{n}{k} X_k + \sum_{\substack{k=m+1\\ odd}}^{\tilde{n}} \binom{n}{k} X_k = \sum_{\substack{k=2\\ even}}^{m-2} \binom{n}{k} X_k + \sum_{\substack{k=m\\ even}}^{\hat{n}} \binom{n}{k} X_k \\
\le &(n-m)\sum_{\substack{k=2\\ even}}^{m-2} \binom{n}{k}\left(1 + \mu\frac{k}{n}\right) + \frac{m}{n}\mu \sum_{\substack{k=m\\ even}}^{\hat{n}} \binom{n}{k}(n-k)
\end{aligned}$$

gives

$$\begin{aligned}
&\lambda\left[(n-m)\sum_{\substack{k=1\\ odd}}^{\hat{k}} \binom{n}{k}\frac{k}{n}\right] + \mu\left[(n-m)\sum_{\substack{k=2\\ even}}^{m-2} \binom{n}{k}\frac{k}{n} + \frac{m}{n}\sum_{\substack{k=m\\ even}}^{\hat{n}} \binom{n}{k}(n-k)\right] \\
&\ge (n-m)\left[\sum_{\substack{k=1\\ odd}}^{\hat{k}} \binom{n}{k} - \sum_{\substack{k=0\\ even}}^{m-2} \binom{n}{k}\right]
\end{aligned}$$

and thus,

$$\begin{aligned}
&\lambda^*_{n,m}\left[(n-m)\sum_{\substack{k=1\\ odd}}^{\hat{k}} \binom{n}{k}\frac{k}{n}\right] + \mu^*_{n,m}\left[(n-m)\sum_{\substack{k=2\\ even}}^{m-2} \binom{n}{k}\frac{k}{n} + \frac{m}{n}\sum_{\substack{k=m\\ even}}^{\hat{n}} \binom{n}{k}(n-k)\right] \\
&= (n-m)\left[\sum_{\substack{k=1\\ odd}}^{\hat{k}} \binom{n}{k} - \sum_{\substack{k=0\\ even}}^{m-2} \binom{n}{k}\right].
\end{aligned}$$

Let $\mu^*_{n,m} = \delta\lambda^*_{n,m}$ for some $\delta \ge 0$. Then, the optimal surplus loss computed with assuming $n/\hat{k} \ge \lambda^*_{n,m}$ is

$$\lambda^*_{n,m} = \frac{(n-m)\left[\sum_{\substack{k=1\\ odd}}^{\hat{k}} \binom{n}{k} - \sum_{\substack{k=0\\ even}}^{m-2} \binom{n}{k}\right]}{(n-m)\left[\sum_{\substack{k=1\\ odd}}^{\hat{k}} \binom{n}{k}\frac{k}{n} + \delta\sum_{\substack{k=2\\ even}}^{m-2} \binom{n}{k}\frac{k}{n}\right] + \frac{m}{n}\delta\sum_{\substack{k=m\\ even}}^{\hat{n}} \binom{n}{k}(n-k)}$$

and this $\lambda^*_{n,m}$ should not contradict the assumption, that is,

$$\frac{n}{\hat{k}} \geq \frac{(n-m)\Big[\sum_{\substack{k=1\\odd}}^{\hat{k}}\binom{n}{k} - \sum_{\substack{k=0\\even}}^{m-2}\binom{n}{k}\Big]}{(n-m)\Big[\sum_{\substack{k=1\\odd}}^{\hat{k}}\binom{n}{k}\frac{k}{n} + \delta\sum_{\substack{k=2\\even}}^{m-2}\binom{n}{k}\frac{k}{n}\Big] + \frac{m}{n}\delta\sum_{\substack{k=m\\even}}^{\hat{n}}\binom{n}{k}(n-k)}$$

for $3 \leq \hat{k} \leq m-1$. This inequality is equivalent to

$$\delta\Big[\sum_{\substack{k=2\\even}}^{m-2}\binom{n}{k}k + \frac{m}{n-m}\sum_{\substack{k=m\\even}}^{\hat{n}}\binom{n}{k}(n-k)\Big] \geq \hat{k}\Big[\sum_{\substack{k=1\\odd}}^{\hat{k}}\binom{n}{k} - \sum_{\substack{k=0\\even}}^{m-2}\binom{n}{k}\Big] - \sum_{\substack{k=1\\odd}}^{\hat{k}}\binom{n}{k}k.$$

Since the left hand side of the inequality is always nonnegative for any n, $n \geq m+1$, we like to have the right hand side negative for any n, $n \geq m+1$. Our objective is to find maximal $\hat{k}$, $3 \leq \hat{k} \leq m-1$ satisfying

$$\hat{k}\Big[\sum_{\substack{k=1\\odd}}^{\hat{k}}\binom{n}{k} - \sum_{\substack{k=0\\even}}^{m-2}\binom{n}{k}\Big] - \sum_{\substack{k=1\\odd}}^{\hat{k}}\binom{n}{k}k \leq 0. \tag{4}$$

Let $\hat{k} = m-1$. Then, inequality (4) is

$$\sum_{\substack{k=1\\odd}}^{m-1}\binom{n}{k}k \geq (m-1)\Big[\sum_{\substack{k=1\\odd}}^{m-1}\binom{n}{k} - \sum_{\substack{k=0\\even}}^{m-2}\binom{n}{k}\Big].$$

This is rewritten as

$$\sum_{\substack{k=1\\odd}}^{m-3}\binom{n}{k}(m-1-k) - (m-1)\sum_{\substack{k=0\\even}}^{m-2}\binom{n}{k} \leq 0.$$

Let $S(n) = \sum_{\substack{k=1\\odd}}^{m-3}\binom{n}{k}(m-1-k) - (m-1)\sum_{\substack{k=0\\even}}^{m-2}\binom{n}{k}$. Firstly observe that

$$S(m+1) = 1 + m^2 - 2^{m-1}(m+1) < 0$$

for m, $m \geq 4$. We can show that $S(n)$ is decreasing in n for $n \geq m+1$. Using $\binom{n}{k} - \binom{n+1}{k} = -\binom{n}{k-1}$, we write

$$\begin{aligned} S(n) - S(n+1) &= -(m-1)\sum_{k=1}^{m-2}(-1)^{k-1}\binom{n}{k-1} + k\sum_{\substack{k=1\\odd}}^{m-3}\binom{n}{k-1} \\ &= -(m-1)\sum_{k=0}^{m-3}(-1)^{k}\binom{n}{k} + k\sum_{\substack{k=1\\odd}}^{m-3}\binom{n}{k-1}. \end{aligned}$$

From $\sum_{j=0}^{k}(-1)^j\binom{n}{j} = (-1)^k\binom{n-1}{k}$ for $0 \leq k \leq n-1$,

$$S(n) - S(n+1) = (m-1)(-1)^{m-2}\binom{n-1}{m-3} + k\sum_{\substack{k=1\\odd}}^{m-3}\binom{n}{k-1}.$$

Since m is even, we have $S(n) - S(n+1) > 0$ implying that $S(n)$ is decreasing in n given m. Therefore, we conclude $S(n) < 0$ for all n, $n \geq m+1$ as desired. That is, $\hat{k} = m-1$ works without contradiction and the optimal efficiency loss must be computed with $\hat{k} = m-1$.

Plugging $\hat{k} = m-1$, we have

$$\mu^*_{n,m}\Bigg[(n-m)\sum_{k=0}^{m-3}\binom{n-2}{k} + m\Bigg[\sum_{k=m-1}^{\hat{n}}\binom{n-1}{k} - \sum_{k=m-2}^{\hat{n}-1}\binom{n-2}{k}\Bigg]\Bigg]$$
$$+\lambda^*_{n,m}\Bigg[(n-m)\sum_{k=0}^{m-2}\binom{n-2}{k}\Bigg] = (n-m)\binom{n-1}{m-1}.$$

Now we will show the statement for the case of $m = n-1$ for $m \geq 3$:

The two way worst case constraints are written as

$$c^{*(n-1)} - \lambda\Bigg[\frac{n-1}{n}c^{*n} - \frac{1}{n}\sum_{i=1}^{n-1}c^{*i}\Bigg] \leq \sum_{i\in N} r(c_{-i}) \leq c^{*(n-1)} + \mu\Bigg[\frac{n-1}{n}c^{*n} - \frac{1}{n}\sum_{i=1}^{n-1}c^{*i}\Bigg].$$

Applying e^{n-k} with individual rationality, we have $\rho_0 = \frac{1}{n}$ and $\rho_{n-1} = 0$. For $k=1$,

$$\max\{0, \frac{n-1-\lambda}{n}\} \leq (n-1)\rho_1 \leq \frac{n-1+\mu}{n}$$

and for $n-2 \geq k \geq 2$,

$$\max\{0, 1 - \frac{k}{n}\lambda\} \leq k\rho_{k-1} + (n-k)\rho_k \leq 1 + \frac{k}{n}\mu.$$

For $k = n-1$, we have

$$0 \leq (n-1)\rho_{n-2} \leq \mu\frac{n-1}{n}.$$

If n is odd and $\frac{n}{n-2} \geq \lambda$,

$$(n-1-\lambda) + \sum_{\substack{k=3\\ odd}}^{n-2}\binom{n}{k}\left(1 - \frac{k}{n}\lambda\right) \leq \sum_{\substack{k=2\\ even}}^{n-3}\binom{n}{k}\left(1 + \frac{k}{n}\mu\right) + \mu(n-1)$$

which is equivalent to

$$(n-1) \leq \lambda(2^{n-2} - 1) + \mu \cdot 2^{n-2}.$$

Thus, the optimal frontier is

$$(n-1) = \lambda^*(2^{n-2} - 1) + \mu^* \cdot 2^{n-2}.$$

We can easily check that the maximal λ^* on the optimal frontier satisfies $\lambda^* \leq \frac{n}{n-2}$. Likewise, if n is even and $\lambda \leq \frac{n}{n-2}$,

$$(n-1+\mu) + \sum_{\substack{k=3\\ odd}}^{n-3}\binom{n}{k}\left(1 + \frac{k}{n}\mu\right) + \binom{n}{n-1}\mu\frac{n-1}{n} \geq \sum_{\substack{k=2\\ even}}^{n-2}\binom{n}{k}\left(1 - \frac{k}{n}\lambda\right).$$

From this, we have

$$\lambda(2^{n-2} - 1) + \mu 2^{n-2} \geq n - 1$$

and the optimal frontier is

$$\lambda^*(2^{n-2} - 1) + \mu^* 2^{n-2} = n - 1.$$

The maximal λ^* on the optimal frontier doesn't contradict $\lambda \leq \frac{n}{n-2}$. ■

References

[Aggarwal et al., 2005] Aggarwal, G., Fiat, A., Goldberg, A., Hartline, J., Immorlica, N., Sudan, M. 2005. "Derandomization of auctions." *Proceedings of the Annual Symposium on Theory of Computing (STOC)*, pp. 619-625.

[Apt et al., 2008] Apt, Krzysztof, Vincent Conitzer, Mingyu Guo, and Evangelos Markakis. 2008. "Welfare Undominated Groves Mechanisms." *Internet and network economics: 4th international workshop, WINE 2008: Proceedings*, pp.426-437.

[Bailey, 1997] Bailey, M. J. 1997. "The demand revealing process: to distribute the surplus." *Public Choice*, Vol. 91, pp. 107-126.

[Cavallo, 2006] Cavallo, Ruggiero. 2006. "Optimal decision making with minimal waste: Strategyproof redistribution of VCG payments." *International Conference on Autonomous Agents and Multi-agent Systems (AAMAS)*, pp. 882-889, Hakodate, Japan.

[Conitzer and Sandholm, 2006] Conitzer V., Sandholm, T. 2006. "Failures of the VCG mechanism in combinatorial auctions and exchanges." *International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, Hakodate, Japan, pp. 521-528.

[Deb, Gosh and Seo, 2002] R. Deb, I. Gosh, T.K. Seo. 2002. "Welfare asymptotics of the pivotal mechanism for excludable public goods." *Mathematical Social Science*, Vol. 43, pp. 209-224.

[Deb and Seo, 1998] R. Deb and T.K. Seo. 1998. "Maximal surplus from the pivotal mechanism: A closed form solution." *Review of Economic Design*, Vol. 3, pp. 347-357.

[Failtings, 2005] Failtings, B. 2005. "A budget-balanced, incentive-compatible scheme for social choices" *Agent-Mediated Electronic Commerce (AMEC), LNAI.*, 3435, pp. 30-43

[Goldberg et al., 2001] Goldberg, A., Hartline, J., Wright, A. 2001. "Competitive auctions and digital goods." *Proceedings of the Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, Washington, DC, pp. 735-744.

[Goldberg et al., 2006] Goldberg, A., Hartline, J., Karlin, A., Saks, M., Wright, A., 2006. "Competitive auctions." *Games and Economic Behavior*

[Green and Laffont, 1977] Green, J., Laffont, J.-J. 1977. "Characterization of satisfactory mechanisms for the revelation of preferences for public goods." *Econometrica*, Vol. 45, pp. 427-438.

[Green and Laffont, 1979] J. Green, J.J. Laffont. 1979. *Incentives in Public Decision Making*, North-Holland, Amsterdam.

[Green et al., 1976] Green, J., Kohlberg, E., Laffont, J.J., 1976. "Partial equilibrium approach to the free rider problem." *Journal of Public Economics*, Vol. 6, 375-394.

[Guo and Conitzer, 2008a] Guo, M. and V. Conitzer. 2008a. "Undominated VCG Redistribution Mechanisms." *Proceedings of 7th International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, Estoril, Portugal.

[Guo and Conitzer, 2008b] Guo, M. and V. Conitzer. 2008b. "Optimal-in-expectation redistribution mechanisms." *International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, Estoril, Portugal.

[Guo and Conitzer, 2009] Guo, M. and V. Conitzer. 2009. "Worst-case optimal redistribution of VCG payments in multi-unit auctions." *Games and Economic Behavior*, Vol. 67, pp. 69-98.

[Hartline and McGrew, 2005] Hartline, J., McGrew, R., 2005. "From optimal limited to unlimited supply auctions." *Proceedings of the ACM Conference on Electronic Commerce (EC)*, Vancouver, Canada, pp. 175-182.

[Holmstrom, 1979] Holmstrom, B. 1979. "Groves' schemes on restricted domains." *Econometrica*, Vol. 47, pp. 1137-1144.

[Kunreuther and Easterling, 1996] Howard Kunreuther and Doug Easterling. 1996. "The role of compensation in siting hazardous facilities." *Journal of Policy Analysis and Management*, Vol. 15(3), pp. 601-622

[Laffont and Maskin, 1979] J.J. Laffont and E. Maskin. 1979. "A differential approach to expected utility maximizing mechanisms", in: J.J. Laffont (Ed.). Aggregation and Revelation of Preferences, North-Holland, New York, 1979.

[Moulin, 1986] Moulin, Hervé. 1986. "Characterizations of the pivotal mechanism." *Journal of Public Economics*, Vol. 31, pp. 53-78.

[Moulin, 2009] Moulin, Hervé. 2009. "Almost budget-balanced VCG mechanisms to assign multiple objects." *Journal of Economic Theory*, Vol. 144, pp. 96-119

[Moulin, 2010] Moulin, Hervé. 2010. "Auctioning and assigning an object: some remarkable VCG mechanisms." *Social Choice and Welfare*, Vol. 34, pp. 193-216

[Moulin and Shenker, 2001] Hervé Moulin and Scott Shenker. 2001. "Strategy-proof sharing of submodular costs: Budget balance versus efficiency." *Economic Theory*, Vol. 18 (3), pp. 511-533.

[Myerson and Satterthwaite, 1983] Myerson, R., Satterthwaite, M., 1983. "Efficient mechanisms for bilateral trading." *Journal of Economic Theory*, Vol. 28, 265-281.

[Parkes, Kalagnanam and Eso, 2001] Parkes, D., J. Kalagnanam, and M. Eso. 2001. "Achieving budget-balance with Vickrey-based payment schemes in exchanges." *Proceedings of the Seventeenth International Joint Conference on Artificial Intelligence (IJCAI)*, pp. 1161-1168, Seattle, WA

[Porter, Shoham and Tennenholtz, 2004] Porter, R., Y. Shoham and M. Tennenholtz. 2004. "Fair imposition." *Journal of Economic Theory*, Vol. 118, pp. 209-228

[Suijs, 1996] Suijs, Jeroen. 1996. "On incentive compatibility and budget balancedness in public decision making." *Economic Design*, Vol. 2, pp. 193-209.

[Zhou, 2007] L. Zhou. 2007. "The failure of groves mechanisms in canonical allocation models." Mimeo, Arizona State University