The Daycare Assignment Problem

John Kennes* Daniel Monte† Norovsambuu Tumennasan‡

April 25, 2011

Abstract

In this paper we introduce and study the daycare assignment problem. We take the mechanism design approach to the problem of assigning children of different ages to daycares, motivated by the mechanism currently in place in Denmark. The dynamic features of the daycare assignment problem distinguishes it from the *school choice problem*. For example, the children's preference relations must include the possibility of waiting and also the different combinations of daycares in different points in time. Moreover, schools' priorities are history-dependent: a school gives priority to children currently enrolled to it, as is the case with the Danish system.

First, we study the concept of stability, and to account for the dynamic nature of the problem, we propose a novel solution concept, which we call strong stability. With a suitable restriction on the priority orderings of schools, we show that strong stability and the weaker concept of static stability will coincide. We then extend the well known Gale-Shapley deferred acceptance algorithm for dynamic problems and we prove that it yields a matching that satisfies strong stability. We show that it is not Pareto dominated by any other matching, and that, if there is an efficient stable matching, it must be the Gale-Shapley one. However, contrary to static problems, the Gale-Shapley algorithm does not necessarily Pareto dominate all other strongly stable mechanisms. Most importantly, the Gale-Shapley algorithm is not strategy-proof. In fact, one of our main results is a much stronger impossibility result: For the class of dynamic matching problems that we study, there are no algorithms that satisfy strategy-proofness and strong stability.

Second, we show that, due to the overlapping generations structure of the problem, the also well known Top Trading Cycles algorithm is neither Pareto efficient nor strategy-proof.

We conclude by showing that a variation of the serial dictatorship is strategy-proof and efficient.

JEL classification : C78, D61, D78, I20.

Keywords: daycare assignment, market design, matching.

*School of Economics and Management, Aarhus University, Denmark. *E-mail:* jkennes@econ.au.dk
†Department of Economics, Simon Fraser University, Canada. *E-mail:* daniel_monte@sfu.ca
‡School of Economics and Management, Aarhus University, Denmark. *E-mail:* ntumennasan@econ.au.dk

1 Introduction

In this paper we study the problem of assigning children to daycares using the mechanism design approach. This problem is motivated by the current Danish system of allocating children to daycare slots. The system currently in place in Denmark has many shortcomings, such as instability, Pareto inefficiency and lack of a clear optimal strategy for parents in how to report their preferences.

The defining feature of the daycare assignment problem is that it is dynamic: children of different ages may be allocated to the same daycare, and each child may be allocated to different daycares at different periods. Parents must balance the objectives of getting back to work while at the same time getting their children into a quality daycare facility. Moreover, schools' priorities may be history dependent: in Denmark, a school gives priority to children previously allocated to that same school and to children not allocated to any school in the previous period. Thus, a parent can influence the priorities of the daycares simply by waiting. This is illustrated by the following piece of advice from the Roskilde municipality website.[1]

> Roskilde Municipality has a child-care guarantee. It means that your child could be looked after when it is 26 weeks, if the Roskilde Municipality has received your application for a place within 8 weeks after the baby is born. In other situations, the application must be sent within 3 months before the needed placement date. The guarantee is for the care of the entire territory of the municipality, so if you want a special institution for your child, you have to wait longer.

The mechanism design approach has been intensively studied in the context of the *school choice problem*, which consists in assigning children of a specific age to schools.[2] One of the main objectives in this literature has been to study mechanisms that satisfy one or more well defined positive properties, such as Pareto efficiency, strategy-proofness, or stability (which has been referred to as "justified envy" in the context of the school choice problem).

The daycare assignment problem differs significantly from the school choice problem due to its dynamic nature. Children of different ages are allocated to the same daycare, and parents may prefer to wait rather than place their children in certain daycares. This time dimension of the daycare assignment problem makes it computationally and conceptually more complicated than the school choice problem. First, the concept of stability, or justified envy, must be strengthened when used in a dynamic environment to be meaningful. The main intuition here is that justified

[1] See http://www.musicon.dk/webtop/site.aspx?p=14906.
[2] See [5] for an important paper in the area, and also [13] for a recent survey.

envy is harder to define: the priorities of each school in a future year will depend on the allocation in the current year. For example, if a lower-priority child stays in a daycare in period t, then in $t+1$ she might have a higher priority in that same daycare (in particular, this is true under the current assignment mechanism in place in Denmark). Thus, in the discussion of the concept of justified envy for period $t+1$, it is not clear whether the allocation to which it should be analyzed is the one in t or the one in $t+1$.

Second, strategy-proofness may be more difficult to achieve in a dynamic environment. There are two reasons for why a player may misreport its own true preferences: first, it may be afraid of losing a spot at a higher ranked school–this motive is also present in static problems; second, and most importantly, each child may misreport its own preferences so as to affect the priority rankings of schools in the following period. We will show in this paper that this second motive is indeed very strong and is the driving force of some of our results: specially that neither the Gale-Shapley deferred acceptance algorithm nor the Top Trading Cycles will be strategy-proof. In addition, note that if an assignment algorithm in place is not strategy-proof, then computing the optimal strategy for the parents is substantially more complicated in a dynamic problem than it is in a static one.

The problem introduced in this paper contributes to the theory of matching, by introducing the dynamic version of the school choice problem. Moreover, we believe that it is also a very important problem for policy discussions, for two main reasons. First, because a central tenet of the Danish welfare state is a very high tax rate combined with subsidy schemes for high-quality welfare services (including child care). This high tax relies heavily on two incomes per family. A central part of the welfare infrastructure is, thus, the functioning of child care for pre-school children subsidized and governed by Danish municipalities. The daycare assignment mechanism currently in place in Denmark has many shortcomings, and we will present some of them.

Most importantly, the study of better ways to allocate children to daycares is particularly relevant since this is a crucial age for a child's development. There is an emerging literature that reveals the high return to investments in early childhood development. This research contends that high-quality programs focused on birth to age 5 produces a higher per-dollar return than K-12 schooling and later job training in the United States (Cunha, Heckman and Schennach (2010) and Cunha, Heckman, Lochner, and Masterov (2006)). The many benefits of quality early childhood education are to reduce the need for special education and remediation, and to cut juvenile delinquency, teenage pregnancy and dropout rates.

In this paper, we will most often consider the case in which the priorities of schools are only history dependent in a rather weak sense: the priority ranking of each school will only change for children that were previously allocated to it. For all other children, the priorities will remain the same. We denote this condition by *independence of previous assignment*. Moreover, we will often consider a restriction on preferences, which we call *independence*. This restriction implies that preferences over schools are somehow stable – there are no complementarities, for example. Even with only this weak link between periods, the problem becomes substantially different to the static case, leading to the negative results mentioned in the previous paragraph.

Whenever possible, we extend the concepts of the static problem of school assignment to the dynamic and new problem of the daycare assignment. Most importantly, we develop the concept of stability in the dynamic context, which we call *strong stability*. We show that there does not exist an algorithm that satisfies strong stability for all priority orderings and all preference profiles. However, if we impose the above mentioned restriction on the priority orderings of schools, namely that priority is independent of the assignment of previous periods in other schools, then we show that the well known Gale-Shapley deferred acceptance algorithm satisfies strong stability. It is not Pareto dominated by any other mechanism that satisfy strong stability, and, if there exists an efficient and strongly stable matching, it must be the Gale-Shapley one.

However, contrary to the results in static two-sided matching problems, we show that the Gale-Shapley deferred acceptance algorithm is not strategy-proof for the class of problems that we look at. We then prove our first impossibility result for this class of problems: there does not exists a mechanism that is both strategy-proof and strongly stable.

Given this impossibility result, we then look for mechanisms that are strategy-proof and efficient. We find that the top trading cycles, also commonly used in the school choice problem, is neither efficient nor strategy-proof. A variation of this algorithm, which we call top trading cycles by cohort is also not strategy-proof. We then conclude by showing a version of the serial dictator assignment rule. This mechanism is efficient and strategy-proof.

Since the work of Abdulkadiroglu and Sonmez [5], mechanism design has been used by many researchers to design new algorithms for the assignment of children to schools. This literature has shown that some of the systems currently in place have many shortcomings, and new systems that overcome some of these problems have been proposed. These new mechanisms have been adopted recently in Boston and New York school systems and the early evidence suggests that these mechanisms are an improvement over the previous systems. This form of market design and intervention, by proposing algorithms that improve on the current system by overcoming shortcomings of the algorithms currently in place, has been quite successful in terms of outcomes of reassigned children. See Abdulkadiroglu, Pathak, and Roth [1] and Abdulkadiroglu, Pathak, Roth, and Sönmez [2] for a discussion of the practical considerations in the student assignment mechanisms in New York City and Boston.

The structure of this paper is as follows. In section 2, we present a short description of the daycare system currently in place in Denmark. In section 3 we describe the model in detail. We study stability and show the existence of stable algorithms in section 4, with a discussion of the Gale-Shapley algorithm and its properties in subsection 4.2.1. In section 5 we prove an impossibility result regarding the concepts of strategy-proofness and strong stability. We then show in section 7 that efficiency and strategy-proofness are not incompatible. We conclude in section 8. Some proofs are in the appendix.

2 The Danish Daycare System

Denmark is divided into 5 regions and 98 municipalities. The municipalities are responsible for the cost and operation of daycare institutions: they select their assignment mechanism and then oversee the execution of the mechanism. Parents pay a user fee per child which accounts for 20-30% of the total cost. Municipalities can decide on the level of the parents' share, but the maximum parental share is set by law to be around 33%. Day-care institutions are directed at preschool children from the ages of 6 months to 6 years. The day-care institutions consists of "Vuggestuer" day nursery (child-minding with children ages 6 months to 3 years), "Børnehaver" (pre-schools with children 3 years to 6 years) and "Integrerede" institutioner" age-integrated institutions (daycare for children ages 6 months to 6 years combined in one institution). The daycares are generally of high quality and most parents use these services. In 2004, 94% of all 3 to 6-year-old children were enrolled in a centre-based early childhood care or education centre. Vuggestues are also used by the majority of parents.

The local governments use slightly different mechanisms. In the appendix we include an English translation of the assignment algorithm currently in place in the Aarhus Municipality. An example demonstrates various weaknesses of the Aarhus allocation mechanism, but given that it requires some notations introduced in the following section, we include the example in the appendix.

Below we highlight the main features of the Aarhus mechanism, which are common across most municipalities, including Copenhagen.

1. Children of varying ages from 6 months to 3 years can go to same daycare;

2. The assignment algorithm runs once a month;

3. Even if a child has a spot in some daycare she can participate in the assignment algorithm;

4. Children currently allocated to a daycare, will not be displaced from the daycare involuntarily;

5. Each daycare gives higher priority to children who do not have a spot at any daycare over the children who have one in any daycare except the original one — this is called a "guaranteed spot".

In the next section, we construct a model that captures the above mentioned features of the Danish system.

3 Model

Time is discrete and $t = -1, 0, \cdots, \infty$. There are a finite number of infinitely lived daycares/schools. Let $S = \{s_1, \cdots, s_m\}$ be the set of schools. Each school $s \in S$ has a maximal capacity r_s which is assumed to be constant always, i.e., the capacity does not depend on time. There is an age limit for children to attend school. We assume that children can start schooling at age 1 and move to the next level of schooling at age 3. Consequently, children can attend school when they are 1 and 2 years old. School attendance is not mandatory. Let h stand for the option of staying home. Let $\bar{S} = S \cup \{h\}$. For technical convenience, we treat h as a school with unbounded capacity. Each period t, there is a new set of children $I_t = \{1, \cdots, n_t\}$ who are 1 year old. Consequently, at any period t the set of school age children is $I_{t-1} \cup I_t$. As time passes the set of school age children evolves in the "overlapping generations" fashion. The set of all children is $I = \cup_t I_t$.

First, we extend the definition of matching to a dynamic context. For the static problem, matching maps the set of children to the set of schools. Here, matching is a collection of functions that map the school age children to the set of schools.

Definition 1 (Matching)**.** *A matching μ is a collection of functions $\mu = (\mu^{-1}, \cdots, \mu^{t-1}, \mu^t, \mu^{t+1}, \cdots)$ where $\mu^t : I_t \cup I_{t-1} \times \bar{S} \to \{0, 1\}$ such that*

1. *For all $i \in I_{t-1} \cup I_t$, $\sum_{s \in \bar{S}} \mu^t(i, s) = 1$,*

2. *For all $s \in S$, $\sum_{i \in I_{t-1} \cup I_t} \mu^t(i, s) \leq r_s$.*

We refer to μ^t as the period t matching.

If child i is assigned a seat at school s in period t, then $\mu^t(i,s) = 1$. Requirement (1) above says that each child is assigned a spot at one school at most, while requirement (2) says that each school cannot house more children than its capacity. We assume that at time $t = -1$ the matching is exogenoulsy given (for example, it may be that all of these initial children stay at home in their first year). In other words, for each matching we consider the matching at period -1 is constant.

With slight abuse of notation, $\mu^t(i)$ denotes the school which is matched with child i under the period t matching μ^t, i.e., $\mu^t(i) = s$ whenever $\mu^t(i,s) = 1$, for each $i \in I_{t-1} \cup I_t$. Similarly, $\mu^t(s)$ is used to denote the set of children who are assigned a seat at school s under period t matching μ^t, i.e., $\mu^t(s) = \{i \in I_{t-1} \cup I_t : \mu^t(i,s) = 1\}$.

Each child is characterized by a strict preference ordering $\succ_i$ over $\bar{S}^2$. The notation (s, s') corresponds to the allocation in which a child attends school s at age 1 and school s' at age 2. Throughout the paper, we maintain the following assumptions on preferences:

Assumption 1 (Preferences)**.**

1. *(No complementarities)* If $(s,s) \succ_i (s',s')$ for some $s, s' \in \bar{S}$ and $i \in I$, then $(s,s) \succ_i (s,s')$ and $(s,s) \succ_i (s',s)$.

2. *(Weak Independence)* If $(s,s) \succ_i (s',s')$ for some $s, s' \in \bar{S}$ and $i \in I$, then $(s,s'') \succ_i (s',s'')$ and $(s'',s) \succ_i (s'',s')$ for any $s'' \neq s'$. On the other hand, $(s,s'') \succ_i (s',s'')$ or $(s'',s) \succ_i (s'',s')$ for some $s \neq s'' \in \bar{S}$ and $s' \in \bar{S}$ implies that $(s,s) \succ_i (s',s')$.

There are two direct implications of the assumptions above. From the first condition and strict preferences we obtain that for any $s, s' \in \bar{S}$ and $i \in I$, at least one of the following conditions is satisfied

$$(i)\ (s,s) \ \succ_i \ (s,s') \text{ and } (s,s) \ \succ_i \ (s',s); \text{ or}$$
$$(ii)\ (s',s') \ \succ_i \ (s,s') \text{ and } (s',s') \ \succ_i \ (s',s).$$

Moreover, the two conditions above may be satisfied at the same time. This would be the case, for example, in which a child incurs a large enough cost (not necessarily monetary) from changing schools.

A second implication is the following. Suppose that for some s and $s_1 \neq s' \neq s_2$, $(s,s) \succ_i (s_1,s_1)$, $(s',s') \succ_i (s_2,s_2)$. Then we must have that $(s,s') \succ_i (s_1,s_2)$. To see this, note that from assumption 1, we have that $(s,s') \succ_i (s_1,s')$ and $(s_1,s') \succ_i (s_1,s_2)$ as $s_1 \neq s' \neq s_2$. Consequently, $(s,s') \succ_i (s_1,s') \succ_i (s_1,s_2)$.

In this paper, we often consider a stronger version of the weak independence assumption which we call *independence*. Recall that if child's preferences satisfy weak independence, then whenever attending school s in both periods is preferred to attending school s' in both periods, attending s and a third school s'' must be better than attending s' and s''. However, weak independence does not rule out the possibility that the child prefers attending school s' in both periods to attending s in one period and s' in the other. Independence, however, rules out this possibility.

Definition 2 (Independence)**.** *We say that a child i's preferences satisfy* Independence *if, for any* $s \neq s' \in \bar{S}$

$$(s, s) \succ_i (s', s') \iff (s, s'') \succ_i (s', s'') \text{ and } (s'', s) \succ_i (s'', s') \text{ for all } s'' \in \bar{S}.$$

When defining the preferences, we are following a more general axiomatic approach. Before we proceed further let us give a specific example that illustrates a more parametric approach.

Example 1. *Suppose that by attending school s for one period, child i benefits $b_i(s)$ which does not depend on the child's age. Each child has a time discount of δ. Moreover, child i incurs a cost of c_i only from the school to school change at age 2, i.e., the cost of any home to school change is* 0*. Finally, the utility of child i attending schools s and s' at her respective ages of 1 and 2 is*

$$U_i(s, s') = \begin{cases} b_i(s) + \delta b_i(s') - c_i & \textit{if } s \neq s' \textit{ and } s \neq h \\ b_i(s) + \delta b_i(s') & \textit{otherwise} \end{cases}$$

Clearly, the underlying preferences for this children satisfy assumption 1 and furthermore, they satisfy property Independence if the cost c_i of school to school change is 0 or sufficiently small. $\diamond$

At any time $t \geq 0$, each school ranks all the school age children by priority. Priorities do not represent school preferences but rather, they are imposed by local municipality. For example, in the existing assignment mechanism in Denmark, all schools give priority to their currently enrolled children. Similarly, the children with special needs are given higher priority by the schools tailored to meet those needs. In practice, age usually factors into where a child stands in the priority ranking of a school. Specifically, older children are given priority.

Henceforth, we assume that each institution gives the highest priority to its currently enrolled children, which is a feature of the assignment mechanism currently in place in Denmark. A rationale behind this priority is that no school forces its current enrollee out in order to free a spot for some other child. Because of this assumption, the priority ranking of each school is dependent on its attendees of the previous period.

The priority of the schools over the children must be carefully defined. Note that, contrary to the school choice problem, this priority ordering is a function of the previous period's matching. In particular, as we noted previously, children currently enrolled at a school have priority over outsiders at that same school. We will denote the strict, binary relation which generates the priority ranking of school s at period t by $\rhd_s^t(\mu^{t-1})$. That is, if at period t children i has a higher priority than children j at school s given that in period $t-1$ the matching was μ^{t-1}, then we denote $i \rhd_s^t \left(\mu^{t-1}\right) j$.

We impose the following assumptions on the priority rankings of the schools, which implies that they are Markovian with previous period's matching as the state variable.

Assumption 2 (Priority Orderings of Schools)**.** *Each school's priority ranking satisfies the following conditions:*

1. *(Priority for currently enrolled children) If $i \in I_{t-1}$ and $i \in \mu^{t-1}(s)$ for some $s \in S$, then $i \rhd_s^t (\mu^{t-1}) j$ for all $j \notin \mu^{t-1}(s)$.*

2. *(Weak consistency of different period rankings) If $i \rhd_s^{t-1} (\mu^{t-2}) j$ for some $i, j \in I_{t-1}$, $s \in S$ and μ, then $i \rhd_s^t \left(\mu^{t-1}\right) j$ in any of the following cases:*

   - $\mu^{t-1}(i) = \mu^{t-1}(j) = s$
   - $\mu^{t-1}(i) = s, h$ *and* $\mu^{t-1}(j) = h$
   - $\mu^{t-1}(j) \neq s, h$

3. *(Weak irrelevance of previous assignment) If $i \rhd_s^t (\mu^{t-1}) j$ for some $i, j \in I_{t-1}$, $s \in S$, and μ with $\mu^{t-1}(i) \neq s, h$ and $\mu^{t-1}(j) \neq s, h$, then $i \rhd_s^t \left(\bar{\mu}^{t-1}\right) j$ for any $\bar{\mu}$ satisfying one of the following conditions.*

   - $\bar{\mu}^{t-1}(i) = \bar{\mu}^{t-1}(j) = s$
   - $\bar{\mu}^{t-1}(i) = s, h$ *and* $\bar{\mu}^{t-1}(j) = h$
   - $\bar{\mu}^{t-1}(j) \neq s, h$

4. *(Weak irrelevance of difference in age) If $i \rhd_s^t (\mu^{t-1}) j$ for some $i \in I_{t-1}$, $j \in I_t$, $s \in S$, and μ with $\mu^{t-1}(i) \neq s, h$, then $i \rhd_s^t \left(\bar{\mu}^{t-1}\right) j$ for all $\bar{\mu}$. In addition, if $j \rhd_s^t (\mu^{t-1}) i$ for some $i \in I_{t-1}$, $j \in I_t$, $s \in S$, and μ with $\mu^{t-1}(i) \neq s, h$, then $j \rhd_s^t \left(\bar{\mu}^{t-1}\right) i$ for all $\bar{\mu}$ with $\bar{\mu}^{t-1}(i) \neq s, h$.*

Loosely speaking, the last three assumptions mean that the priority ranking of any school does not depend on the attendees of other schools (excluding staying home). Specifically, the second one says that if child i has higher priority than child j at school s in period $t-1$, then child i keeps her advantage over child j in the following period unless child j attends school s (h) while child i does not attend s (s or h). The third one says that at any period, school s's relative ranking of any two children is not affected by the fact that one child has attended school $s' \neq s$ and the other $s'' \neq s$. The fourth assumption says that at any period school s's relative ranking of any two children is not affected by the fact that one child has attended school $s' \neq s$ at period $t-1$ while the other is one year old at period t. Here we remark that assumption 2 does not rule out the possibility that a school s gives priorities to the children who have not attended any school over the ones who have attended some school other than s in the previous period. This possibility is ruled out if the priority rankings of the schools satisfy the *Independence of Past Attendance* (IPA) property. We sometimes will concentrate exclusively on the cases in which *IPA* is satisfied. Now let us present the formal definition below.

Definition 3 (*Independence of Past Attendance*)**.** *The priority ranking of a school satisfies the* Independence of Past Attendance *(IPA) property if*

1. *(Consistency of different period rankings) If* $i \rhd_s^{t-1} (\mu^{t-2}) j$ *for some* $i, j \in I_{t-1}$, $s \in S$ *and* μ, *then* $i \rhd_s^t \left(\mu^{t-1}\right) j$ *in any of the following cases:*
   - $\mu^{t-1}(i) = \mu^{t-1}(j) = s$
   - $\mu^{t-1}(j) \neq s$
2. *(Irrelevance of previous assignment) If* $i \rhd_s^t (\mu^{t-1}) j$ *for some* $i, j \in I_{t-1}$, $s \in S$, *and* μ *with* $\mu^{t-1}(i) \neq s$ *and* $\mu^{t-1}(j) \neq s$, *then* $i \rhd_s^t \left(\bar{\mu}^{t-1}\right) j$ *for any* $\bar{\mu}$ *satisfying one of the following conditions.*
   - $\bar{\mu}^{t-1}(i) = \bar{\mu}^{t-1}(j) = s$
   - $\bar{\mu}^{t-1}(j) \neq s$
3. *(Irrelevance of difference in age) If* $i \rhd_s^t (\mu^{t-1}) j$ *for some* $i \in I_{t-1}$, $j \in I_t$, $s \in S$, *and* μ *with* $\mu^{t-1}(i) \neq s$, *then* $i \rhd_s^t \left(\bar{\mu}^{t-1}\right) j$ *for all* $\bar{\mu}$. *In addition, if* $j \rhd_s^t (\mu^{t-1}) i$ *for some* $i \in I_{t-1}$, $j \in I_t$, $s \in S$, *and* μ *with* $\mu^{t-1}(i) \neq s, h$, *then* $j \rhd_s^t \left(\bar{\mu}^{t-1}\right) i$ *for all* $\bar{\mu}$ *with* $\bar{\mu}^{t-1}(i) \neq s$.

In practice, *IPA* is often not satisfied: many schools give priority to two year old children who have not attended any school in the previous period over one year old children and the two year old children who have attended school in the previous period. In particular, given a concept called "guaranteed spots" , *IPA* is also not satisfied in the current Danish daycare assignment mechanism.

Remark 1. *The school choice problem is a very special case of the daycare assignment problem. To see this, suppose that the set of children consists of only children who are one year old at period* -1 *and let every child stay home when they are one. The schools' priorities are well defined at period 0. In addition, the children rank the schools at period 0 fixing that their period* -1 *matches are h. Now one can see that this special case of our daycare assignment problem is a school choice problem.*

Remark 2. *The daycare assignment problem differs from the school choice problem in many aspects. However, the most crucial aspect of the daycare assignment problem that distinguishes it from the school choice problem is that the history dependence of the schools' priorities and children's preferences. To see this, suppose that the children's preferences satisfy* independence *and somehow the schools' priorities at any period are independent of the previous period's matching. Now because independence is satisfied, at any period any child's rankings of the schools do not depend on the future or past matchings. Hence, the children's rankings in each period is uniquely defined without depending on history. Now as the schools' priorities are not history dependent by supposition, one can treat the daycare assignment problem as separate, independent school choice problems in different periods. Consequently, all the results from the school choice problem would be valid in our setting. However, in the daycare assignment problem, we firmly believe that history dependence must be considered very seriously.*

Given remark 2, we often restrict our attention to the cases in which both *independence* and *IPA* are satisfied. Observe that in these cases, history dependence is as minimal as possible but we obtain many results different from the ones found in the school choice problem. Hence, only a minimal history dependence is needed to distinguish the daycare assignment problem from the school choice problem.

3.1 Properties of a Matching

The matching literature has identified Pareto efficiency and stability as the two main desirable properties. The main goal of this subsection is to adapt these concepts to our daycare assignment problem.

Extending the concept of Pareto efficiency to our setting is straightforward. The main reason is the following: In the definition of Pareto efficiency, one considers only the well-beings of one side of the market, namely the children. In addition, children's preferences are exogenously defined and not history dependent. Hence, the definition of Pareto efficiency in our setting coincides with the one in the school assignment problem: a matching μ is *Pareto efficient* if no other matching strictly improves at least one child without hurting the others. We give the formal definition below.

Definition 4 (Pareto Efficiency)**.** *A matching μ is* Pareto efficient *if there is no other $\bar{\mu} \neq \mu$, such that for some $t \geq 0$ and some $i \in I$, $\left(\bar{\mu}^t(i), \bar{\mu}^{t+1}(i)\right) \succ_i \left(\mu^t(i), \mu^{t+1}(i)\right)$ and for $\forall j \in I$, either $\left(\bar{\mu}^t(j), \bar{\mu}^{t+1}(j)\right) = \left(\mu^t(j), \mu^{t+1}(j)\right)$ or $\left(\bar{\mu}^t(j), \bar{\mu}^{t+1}(j)\right) \succ_j \left(\mu^t(j), \mu^{t+1}(j)\right)$.*

Adopting the definition of stable matching in our setting is not straightforward. As [5] points out, already in static settings, one has to be careful in interpreting stable matchings for the school choice problem. To be specific, in the context of college admissions, under a stable matching no college-student pair should be able to improve themselves. However, in the context of school choice, the schools have priorities but not preferences, thus, it is unclear how a school can improve itself. Thus, [5] suggests to interpret stable matchings as the ones free of *justified envy.* That is, under a stable matching, if a child likes a school better than her current match, then this school should not assign a seat to any child who has a lower priority than the child. In this case, no child can justify her desire to change her current match with some other school.

The stability concepts we construct in this paper are based upon the idea of *justified envy* freeness. The dynamic nature of our setting presents some challenges that are absent in the school choice problem. However, before spelling them out, let us first define the weak stability concept that we perceive as the analog of the stability concept in the school choice problem.

Whether a matching is weakly stable depends on whether some child can *justify* her envy of another at some period while keeping her past/future match the same. In other words, at some period t, child i justifies her envy of child j if child j attends a school s in which she has a lower priority than child i and at the same time child i would improve if she changed only her period t match with s. If a matching is free of this type of justified envy, then this matching is weakly stable. In a way, for weak stability, we are analyzing the problem at some time t, assuming that the matching of every other period $t' \neq t$ is fixed. In this sense, the weak stability concept is analogous to the stability concept in the school choice problem.

Definition 5 (Weak Stability)**.** *A matching μ is* weakly stable *if at any period t, there does not exist a school-child pair (s, i) such that (1) and (2) below hold at the same time*

1. *(a) $(s, \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^{t+1}(i))$, or*

(b) $(\mu^{t-1}(i), s) \succ_i (\mu^{t-1}(i), \mu^t(i))$,

2. $\left|\mu^t(s)\right| < r_s$ *or/and* $i \rhd_s^t (\mu^{t-1}) j$ *for some* $j \in \mu^t(s)$.

Condition (1) above refers to the fact that child i would be strictly better off by switching to some school s rather than the school specified by the matching μ. On top of that, condition (2) implies that either there are spots left at the preferred school s of child i, or the school is in full capacity but some child j allocated to this school under the matching μ has lower priority than child i at that school.

In the definition of weak stability, one considers only the one period deviations which has two shortcomings: (1) because the children can attend school for two periods, a child can imagine situations in which she changes her match in both periods and (2) the schools' priorities, which have to be considered for stability, evolve depending on the past matchings. These shortcomings are magnified if *independence* or *IPA* is not satisfied. To illustrate this point, we consider the following two examples.

Example 2 (Justified Envy under Failure of Independence)**.** *Consider a matching that assigns child i a seat at school s' when she is both 1 and 2 years old. However, there is another school s such that child i improves only if she switches to school s when she is both 1 and 2 years old. Observe that child i's preferences do not satisfy* independence*. Moreover, suppose that when child i is 1 year old, she is placed in school s's priority ranking higher than another child i' who is assigned a seat at school s at that time. With this information, we cannot rule out the possibility that the matching is* weakly stable*. The reason is that child i prefers attending s' for 2 periods to attending school s when she is 1 and s' when she is 2.*

However, one can reasonably argue that child i's envy of child i' is justified because she has a right to attend school s ahead of child i' at age 1. Then, in the following period, she will be in the highest priority group at school s. This gives her a right to attend school s when she is 2. ◇

Example 3 (Justified Envy under Failure of IPA)**.** *Suppose there are 2 schools: s and s'. School s has a capacity of only 1 child while school s' has a capacity of 2 children. Child i and child i' are born at the same period. Both children's preferences satisfy the following property: $(s, s) \succ (s', s) \succ (h, s) \succ (s', s')$. Suppose that school s gives higher priority to child i than i' at period t when the children are 1 year old. However, i' is given higher priority over child i by school s at period $t + 1$ if at period t, i' does not attend any school while i attends s'. Clearly, school s's priority ranking does not satisfy IPA.*

Consider a matching which assigns both children a seat in school s' at period t but assigns child i a spot at school s and child i' a spot at school s' at period $t+1$. Implicitly, period t spot of school s is assigned to some other child who has higher priority at school s over both children. With this information only, we cannot prove that the matching is not weakly stable.

However, one can argue that child i' envies child i in a justified manner: if she is stays home at period t and attends school s at period $t+1$, then she would definitely improve. In addition, she would have been ranked ahead of child i in the priority ranking of school s at period $t+1$. ⋄

To account for the issues raised by the examples above, we will define a stronger concept of stability. First, we need the following notation: for any $i, j \in I_t$, $s \in \bar{S}$ and μ such that $\mu(i) \neq \mu(j)$ and $\mu(j) \in S$, let

$$\bar{M}^t(i,j,\mu) = \left\{\bar{\mu}^t : \bar{\mu}^t(i) = \mu^t(j) \,\&\, \bar{\mu}^t(j) \neq \mu^t(j), \,\&\, \bar{\mu}^t(i') = \mu^t(i') \,\forall\, i' \neq i, j \in I_{t-1} \cup I_t\right\}.$$

That is, the set $\bar{M}^t(i,j,\mu)$ is a set of matchings at period t such that i replaces j over the allocation specified by the matching μ^t, j is placed at a different school and all other children's assignments are left unchanged. One may think of this as the set of all hypothetical matchings at time t such that i replaces j who then finds a school somewhere else — perhaps home, or some other school — and all other children remain in the same school. Implicit in the solution concept of strong stability and the construction of the set $\bar{M}^t(i,j,\mu)$ is the assumption that children are not "farsighted." Under this view, an allocation of a particular period is considered "unfair" (or subject to justified envy) if the child takes the matching of all other children at all other periods as given. In particular, when the child "feels" that she has justified envy over some child in a particular school, for the following period, she imagines that this child over whom she had priority will either stay at home, or be placed in some other school that will not affect the next period's matching and all other children remain matched as originally. When evaluating that the matching μ is subject to justified envy, the child does not evaluate the entire general equilibrium effect of a new allocation that would take into consideration her justified envy and possibly everyone else's.

Definition 6 (Strong Stability)**.** *Matching μ is* strongly stable *if it is weakly stable and the following conditions are satisfied. At any period t, there does not exist a triplet (s, s', i) such that $(s, s') \succ_i (\mu^t(i), \mu^{t+1}(i))$ and one of the following conditions hold:*

1. $\left|\mu^t(s)\right| < r_s$ *and* $\left|\mu^{t+1}(s')\right| < r_{s'}$,

2. $|\mu^t(s)| < r_s$, $|\mu^{t+1}(s')| = r_{s'}$, *and, for some* $j' \in \mu^{t+1}(s')$, $i \rhd_{s'}^{t+1} (\bar{\mu}^t) j'$ *where* $\bar{\mu}^t$ *is the period* t *matching with* $\bar{\mu}^t(i) = s$ *and* $\bar{\mu}^t(i') = \mu^t(i')$ *for all* $i' \neq i \in I_{t-1} \cup I_t$,

3. $|\mu^t(s)| = r_s$, $|\mu^{t+1}(s')| < r_{s'}$, *and, for some* $j \in \mu^t(s)$, $i \rhd_s^t (\mu^{t-1}) j$,

4. $|\mu^t(s)| = r_s$, $|\mu^{t+1}(s')| \geq r_{s'}$, *for some* $j \in \mu^t(s)$, $j' \in \mu^{t+1}(s')$ *and for any* $\bar{\mu}^t \in \bar{M}(i, j, \mu)$, $i \rhd_s^t (\mu^{t-1}) j$ *and* $i \rhd_{s'}^{t+1} (\bar{\mu}^t) j'$.[3]

We interpret justified envy in the dynamic context as the existence of a profile of schools for which a child prefers to its current match and such that in some "reasonable" way it would be "fair" for her to go to the preferred schools. Specifically, a reasonable way may mean one the four cases: (1) both of these schools have unassigned spots; (2) in the first period a preferred school has an unassigned spot and in the second, the child has a higher priority over another child allocated at a preferred school; (3) a preferred school in the second period is operating with less than full capacity and in the first period the child is placed on a higher priority than some other child already allocated there, and finally (4) in the first year the child has a higher priority than some other child in a particular school and in the second year, the child has a higher priority than some other child even if there had been a reallocation in the first period, in which she replaced some child in year 1, as long as in this new allocation, all other children remained in the same school.

Remark 3. *Strong stability is a refinement of weak stability and we believe that it is a natural concept that captures the meaning of justified envy in our setting. Yet we must remark that the definition of strong stability is stronger than what examples 2 and 3 call for. In other words, one can slightly weaken definition 6 so that a matching is strongly stable if it is weakly stable and free of justified envy discussed in examples 2 and 3. However, doing so does not change any of the results in the next section. Given this, weakening the definition of strong stability is not beneficial from the technical perspective.*

3.2 Properties of Mechanism

Let P_i denote the reported preference ordering of child $i \in I$ and P be the product space of the reported preference ordering of every child i. A mechanism φ is an algorithm that constructs, sequentially, a matching for the daycare assignment problem, given the reported preferences and the priority orderings. That is, mechanism φ maps the reported preferences P and the function $\rhd^t(\cdot)$ to a matching μ. Recall that μ^{-1} is fixed and exogenously given. Let $\varphi_i\left(P, \rhd^t(\cdot)\right)$ denote

[3] Observe that $\mu(j) \neq h$ as h has an unlimited capacity. Hence, $M^t(i, j, \mu)$ is well defined.

the assignment pair of child i when she is 1 and 2. Strategy-proofness is defined as an incentive for reporting the true preferences. Formally, reporting the true preferences is a weakly dominant strategy for the children.

Definition 7 (Strategy-Proofness). *A mechanism φ is* strategy-proof *if for all $i \in I$, all $\rhd^t(\cdot)$, all P_i, all $t \geq 0$, all $\hat{P}_i$, and all $\hat{P}_{-i}$,*

$$\varphi_i\left(P_i, \hat{P}_{-i}, \rhd^t(\cdot)\right) \succ_i \varphi_i\left(\hat{P}_i, \hat{P}_{-i}, \rhd^t(\cdot)\right) \, OR \, \varphi_i\left(P_i, \hat{P}_{-i}, \rhd^t(\cdot)\right) = \varphi_i\left(\hat{P}_i, \hat{P}_{-i}, \rhd^t(\cdot)\right),$$

where P_i is i's true preferences while $\hat{P}_i$ and $\hat{P}_{-i}$ are the reported preferences of i and the others.

Definition 8 (Stability and Efficiency).

1. *A mechanism φ is* efficient*, if for all P and $\rhd^t(\cdot)$, it yields an efficient matching.*
2. *A mechanism φ is* strongly (weakly) stable*, if for all P and all $\rhd^t(\cdot)$, it yields a strongly (weakly) stable matching.*

4 Stable Matchings and Their Properties

In this section, we explore different questions regarding weakly and strongly stable matchings under the assumption that the planner knows the children's preferences as well as the schools' priorities. Even in this case, given that our problem differs from the school assignment problem significantly, we need to explore the fundamental questions such as the relation between the stability concepts and the existence of stable matchings.

4.1 The Relation between Strong and Weak Stability

Now we will explore under what conditions, the concepts of weakly and strongly stable matchings will coincide. From examples 2 and 3, one can conjecture that weakly and strongly stable matchings may be equivalent if the children's preferences satisfy *Independence* and the schools' priority rankings satisfy *IPA*. Indeed this is the case, as we will show in the next two lemmas.

Lemma 1. *Suppose that all schools' preference rankings satisfy* IPA*. If μ is weakly but not strongly stable, then for some period t and some school-child pair (s, i),*

1. $\mu^t(i) = \mu^{t+1}(i)$,
2. $(s, s) \succ_i (\mu^t(i), \mu^{t+1}(i))$,

3. $\left|\mu^t(s)\right| < r_s$ *or/and* $i \rhd_s^t (\mu^{t-1}) j$ *for some* $j \in \mu^t(s)$*.*

The proof is in the appendix.

Next we show that the solution concept for the daycare assignment problem, the strong stability, is in fact equivalent to the static concept of weak stability for a large class of problems. Precisely, if the children's preferences satisfy *Independence* and the school's priority rankings satisfy *IPA*, the two concepts are equivalent.

Theorem 1 (Equivalence of Weak and Strong Stability)**.** *Suppose every child's preferences satisfy* Independence *and every school's priority ranking satisfies* IPA*. Then matching* μ *is strongly stable if and only if it is weakly stable.*

Proof. By definition, any strongly stable matching is weakly stable. Hence, we need to show that any weakly stable matching is strongly stable. Suppose otherwise, i.e., there exists a weakly stable matching μ which is not strongly stable. By lemma 1, if μ is weakly but not strongly stable, then for some period t and some school-child pair (s, i),

1. $\mu^t(i) = \mu^{t+1}(i)$,

2. $(s, s) \succ_i (\mu^t(i), \mu^{t+1}(i))$,

3. $\left|\mu^t(s)\right| < r_s$ or/and $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$.

Clearly, $(s, s) \succ_i (\mu^t(i), \mu^t(i))$. In addition, each child's preferences satisfy *Independence*, hence, $(s, \mu^t(i)) \succ_i (\mu^t(i), \mu^t(i))$. By combining this with the 3rd condition above, one obtains that μ is not weakly stable. □

4.2 The Existence of Stable Matchings

Now we turn our attention to the question of whether strongly stable matchings exist. The answer to this question is negative if the schools' priority rankings do not satisfy IPA.

Theorem 2. *If the priority listings of the schools do not satisfy* IPA*, then the existence of strongly stable matchings is not guaranteed.*

Proof. We construct an example with no strongly stable matching in which *IPA* is violated.
Suppose there are 2 schools, $\{s, s'\}$. Schools s and s' have capacities of 1 and 3, respectively. In each period, there are two one-year old children and they are identical in all aspects. Their preferences

satisfy the following property: $(s,s) \succ (h,s) \succ (s',s') \succ (h,h)$. Moreover, the children's preferences satisfy *independence*.

At any period, the schools use the following priority ranking: (1) the previous period's attendees (2) two year old children who have not attended any school in the previous period. (Note that condition (2) violates *IPA*).

1. Consider any matching with $\mu^t(i) = h$ for some i and t. There must be a unassigned spot at one of the schools at period t. By assigning this spot to child i at t, one can improve her. Thus, no such matching would satisfy strong stability.

2. Consider any matching with $(\mu^t(i), \mu^{t+1}(i)) = (s, s')$ for some i and t. Clearly, child i has the highest priority at schools s in period $t+1$ and in addition, $(s,s) \succ (s,s')$ by independence. Hence, child i can be improved in a justified manner.

3. Consider any matching such that for $i \in I_t$, $\mu^{t+1}(i) = s$. Then one of the following happens: (1) one of the one-year old children at $t+1$ attends school s at $t+2$ or (2) none of the one-year old children at $t+1$ attends school s at time $t+2$. In the former case, either we are back to case 1 or one of the one-year old children in $t+1$ matches with (s', s'). This child prefers (h, s) to (s', s'). In addition, at $t+2$ she has priority over any one-year old or any two year old who attended s' at $t+1$ (recall that the other one year old at $t+1$ matches with (s', s)). Hence, this child can be improved in a justified manner. In case (2), either we are back to case 1 or both children attend s' at periods $t+1$ and $t+2$. Then each child prefers (h, s) to (s', s'). In addition, at $t+2$, each child has priority over any one year old at school s or school s has an unassigned seat. Hence, either children can be improved in a justified manner.

□

In the counter example used for the proof of theorem 2, the children's preferences satisfy *independence*. However, independence does not play any role for theorem 2, i.e., one construct an example needed for theorem 2 in which the children's preferences satisfy *independence*. Hence, we conclude that the existence of strongly stable matchings is not guaranteed without *IPA* regardless of *independence* is satisfied or not. But with *IPA*, is the existence guaranteed? The answer to this question is positive but before we present the formal result, let us introduce the algorithm used for the existence result.

4.2.1 Gale-Shapley Deferred Acceptance Algorithm and Its Properties

The Gale and Shapley deferred acceptance algorithm was originally designed to deal with static two-sided matching problems. To run this algorithm at certain period t, one needs to know the schools' priority rankings over all school age children as well as the children's preferences over schools. In the class of problems studied in this paper, the schools' priority rankings are well defined given the previous period's matching. However, the children's preferences are defined over the pairs of schools, since each child can attend different schools for two consecutive periods. Hence, to run the original Gale-Shapley mechanism, one needs to derive one period preferences for each child at a given period, based on the past matchings and the original preferences of the children over the pairs of schools; we do not want to derive one period preferences based on the future matchings as the current matchings affect next period's priority rankings of the schools.

For now, let us assume that at period t, we have derived the one period preference relation $\mathcal{P}_i(\mu^{t-1})$ for each $i \in I_{t-1} \cup I_t$ depending on μ^{t-1} matchings. Let $\mathcal{P}(\mu^{t-1}) = \left\{\mathcal{P}_i(\mu^{t-1})\right\}_{i \in I_{t-1} \cup I_t}$. Thus, $s\mathcal{P}_i(\mu^{t-1})s'$ means that at time t, player i prefers school s to s' given the period $t-1$ matching μ^{t-1}. Note that this definition relies critically on the previous period's matching (for example, there could be high switching costs for the children). With this concept of one-period preferences, we will define stability in a static context that will be used in some of our proofs.

Definition 9 (Static Stability). *Period t matching μ^t is statically stable under $\mathcal{P}(\mu^{t-1})$ and μ^{t-1}, if there exists no school-child pair (s, i) such that*

1. $s\mathcal{P}_i(\mu^{t-1})\mu^t(i)$,
2. $|\mu^t(s)| < r_s$ *or/and* $i \rhd_s^t (\mu^{t-1})j$ *for some* $j \in \mu^t(s)$.

Now we will define the one-period preferences that we will use for the Gale and Shapley deferred acceptance algorithm.

Definition 10 (Isolated Preference Relation). *For given μ^{t-1},*

1. *the* isolated preference relation *for $i \in I_t$ is the preference relation $\succ_i^1$ such that $s' \succ_i^1 s''$ if and only if $(s', s') \succ_i (s'', s'')$ for any $s' \neq s'' \in \bar{S}$,*
2. *the* isolated preference relation *for $i \in I_{t-1}$ is the preference relation $\succ_i^2 (\mu^{t-1})$ depending on previous period's matching and such that $s' \succ_i^2 (\mu^{t-1})s''$ if and only if $(\mu^{t-1}(i), s') \succ_i (\mu^{t-1}(i), s'')$ for any $s' \neq s'' \in \bar{S}$.*

The Gale and Shapley deferred acceptance algorithm:

The algorithm is the same in each period, and it only uses the matching results of the previous period. In period t, assume that the previous period's matching is obtained by using the Gale and Shapley algorithm. At period t, the schools assign their spot to the all school age children in finite rounds as follows:

Round 1: Each child proposes to her first choice according to her isolated preferences. Each school tentatively assigns its spots to the proposers according to its priority ranking. If the number of proposers to school s is greater than the number of available spots r_s, then the remaining proposers are rejected.

In general, at:

Round k: Each child who was rejected in the previous round proposes to her next choice according to her isolated preferences. Each school considers the pool of children who it had been holding plus the current proposers. Then it tentatively assigns its spots to this pool of children according to its priority ranking. The remaining proposers are rejected.

The algorithm terminates when no child proposal is rejected and each child is assigned her final tentative assignment.

Given that the children's preferences as well as schools' priority rankings are strict, it is easy to see that the Gale and Shapley deferred acceptance algorithm yields a unique matching. We refer to this matching as the Gale and Shapley matching and use the notation μ_{GS} for it.

With the next result we show that when assuming *IPA*, strong stability is equivalent to static stability under isolated preferences.

Lemma 2. *Matching μ is weakly stable if and only if for all t, μ^t is statically stable under isolated preferences and μ^{t-1}. Furthermore, if each school's preference rankings satisfy* IPA *μ is strongly stable if for all t, μ^t is statically stable under isolated preferences and μ^{t-1}.*

The proof for this lemma is in the Appendix.

Corollary 1. *The Gale and Shapley matching is weakly stable. Furthermore, if the priority ranking of each school satisfies* IPA*, then the Gale and Shapley matching is strongly stable.*

Proof. It is well known that μ_{GS}^t is statically stable under isolated preferences and μ_{GS}^{t-1}. Then lemma 2 yields the result. $\square$

As we already mentioned, examples 2 and 3 illustrate the need of strengthening the weak stability concept into the strong stability one if *independence* or *IPA* is not satisfied. However, corollary 1 demonstrates that *IPA* is a sufficient condition for the existence of strongly stable matchings even if *independence* is not satisfied. In addition, theorem 2 shows that with or without *independence*, the existence of strongly stable matchings is not guaranteed without *IPA*. In this sense, *IPA* is a more critical condition than *independence* for the existence of strongly stable matchings. Perhaps, this is a good news from the policy maker's perspective in the sense that the policy maker can change the schools' priorities but not the children's preferences.

In static settings, one of the most significant results is that the Gale and Shapley matching Pareto dominates all other stable matchings.[4] This result is no longer valid in our daycare assignment problem. In fact, there could be multiple weakly/strongly stable matchings that do not Pareto dominate one another. The following example illustrates this point.

Example 4. *There are 3 schools $\{s, s_1, s_2\}$. All schools have a capacity of one child. There is no school age child until period $t-1$. At period $t-1$, only one child i is 1 year old. At period t, there are 2 one-year old children $\{i_1, i_2\}$. At period $t+1$, child i' is 1 year old. Each school gives the highest priority to the child who has attended the school if the child is still of the school age. If children $\bar{\imath} \neq \bar{\imath}' \in \{i, i_1, i_2, i'\}$ have not attended school $\bar{s} = s, s_1, s_2$ in the previous period, then school $\bar{s}$ ranks child $\bar{\imath}$ and child $\bar{\imath}'$ according to the following rankings.*

$$\begin{array}{ccccccc} i & \rhd_s & i_1 & \rhd_s & i_2 & \rhd_s & i' \\ i & \rhd_{s_1} & i' & \rhd_{s_1} & i_2 & \rhd_{s_1} & i_1 \\ i & \rhd_{s_2} & i_1 & \rhd_{s_2} & i_2 & \rhd_{s_2} & i' \end{array}$$

Each child's preferences satisfy independence. *Child i's top choice is (s, s). The preferences of children i_1, i_2 and i' satisfy the following conditions:*

$$\begin{array}{ccccc} (s_1, s_1) & \succ_{i_1} & (s_2, s_2) & \succ_{i_1} & (s, s), \\ (s, s) & \succ_{i_2} & (s_2, s_2) & \succ_{i_2} & (s_1, s_1), \\ (s_1, s_1) & \succ_{i'} & (s_2, s_2) & \succ_{i'} & (s, s). \end{array}$$

The Gale and Shapley matching μ is as follows: $\mu^{t-1}(i) = \mu^t(i) = s$, $\mu^t(i_1) = \mu^{t+1}(i_1) = s_1$, $\mu^t(i_2) = s_2$, $\mu^{t+1}(i_2) = s$, $\mu^{t+1}(i') = s_2$ and $\mu^{t+2}(i') = s_1$. Because the schools' priority rankings

[4] See [12].

satisfy IPA*, thanks to corollary 1, we obtain that* μ *is strongly stable.*

Now let us consider the following matching $\bar{\mu}$*:* $\bar{\mu}^{t-1}(i) = \bar{\mu}^{t}(i) = s$, $\bar{\mu}^{t}(i_1) = \bar{\mu}^{t+1}(i_1) = s_2$, $\bar{\mu}^{t}(i_2) = s_1$, $\bar{\mu}^{t+1}(i_2) = s$, $\bar{\mu}^{t+1}(i') = s_1$ *and* $\bar{\mu}^{t+2}(i') = s_1$*. It easy to check* $\bar{\mu}$ *is strongly stable.*

Now observe that matching μ *does not Pareto dominate matching* $\bar{\mu}$ *because child* i' *prefers* $\bar{\mu}$ *to* μ*. In fact,* $\bar{\mu}$ *is not Pareto dominated by any strongly stable matching. To see this, observe that the only matching that Pareto dominates* $\bar{\mu}$ *is the one in which children 1 and 2 switch their matches in period* t*. But this is not strongly stable because child* i_1 *has a justified envy of child* i' *at* $t+1$*.* $\diamond$

First observe that in the above example both *IPA* and *independence* are satisfied. Hence, the weakly and strongly stable matchings coincide. Hence the example above shows that there may exist other mechanisms that produce matchings that are weakly/strongly stable and are not Pareto dominated by the Gale-Shapley matching. This is the first main distinction between the matching produced by the Gale-Shapley algorithm in the static school choice problem versus the dynamic problem of the daycare assignment.

Given the importance of this result when compared to the static case, we state the result below.

Theorem 3 (Gale-Shapley matching does not necessarily Pareto dominate all stable matchings)**.** *The Gale and Shapley matching does not necessarily Pareto dominate all weakly/strongly stable matchings.*

In the light of example 4, one must explore whether any strongly stable matching Pareto dominates the matching from the Gale and Shapley deferred acceptance algorithm. This, indeed, is impossible which we show in the following proposition.

Proposition 1 (Gale-Shapley matching is not Pareto dominated by any other strongly stable mechanism)**.** *Suppose each school's priority rankings satisfy* IPA*. Then the Gale-Shapley matching is not Pareto dominated by any other strongly stable matchings.*

Sketch of the Proof. Here, we will only sketch the proof. The formal proof is in the appendix.

The proof is by contradiction: suppose that there exists a strongly stable matching μ that Pareto dominates the matching resulting from the Gale-Shapley, μ_{GS}. We proceed in 3 steps.

First, we show that in the initial period it must be true that for all 2-year old children the allocation in the two matchings must coincide. The main intuition is that the matching produced by the Gale-Shapley algorithm must be statically stable and must Pareto dominate any matching μ^0 that is statically stable, following a well known property of the Gale-Shapley mechanism. Therefore, there does not exist a statically stable mechanism that Pareto dominates μ_{GS} and improves the allocation of a 2-year old child in the first period.

For step 2, which is less straightforward, we show that the 1-year old children also cannot be improved in their allocation. First, note that if the new Pareto dominant matching is different than the Gale-Shapley matching in period 0 for children $i \in I_0$, then these children must be "worse off" in period zero, only to be improved next period. Formally, $\mu^0_{GS}(i) \succ^1_i \mu^0(i)$, but $(\mu^1(i), \mu^1(i)) \succ_i (\mu^1_{GS}(i), \mu^1_{GS}(i))$. The intuition is that child i must always be at least as good in period 1 than she is at period 0, due to strong stability and the assumption that currently allocated children have priorities on the second period. By lemma 2, we know that μ^1 is statically stable under isolated preferences and μ^0. Now suppose we ran the Gale and Shapley algorithm at period 1 under isolated preferences and μ^0. Let us denote the resulting matching $\bar{\mu}^1$. If $\bar{\mu}^1$ is statically stable under isolated preferences and μ^0_{GS}, then from lemma 2, we know that μ^1_{GS} is a stable matching under isolated preferences and μ^0_{GS}. In addition, it must Pareto dominate $\bar{\mu}^1$ in terms of the isolated preferences, since $\bar{\mu}^1$ is statically stable and μ^1_{GS} must Pareto dominate all stable matchings (see [12]). From [12], we know that if $\bar{\mu}^1(i) \neq \mu^1(i)$, then $\bar{\mu}^1(i) \succ^2_i (\mu^0)\mu^1(i)$. Iterating assumption 1, we show in the formal proof that: $(\bar{\mu}^1(i), \bar{\mu}^1(i)) \succ_i (\mu^1(i), \mu^1(i))$ and $(\mu^0_{GS}(i), \mu^1_{GS}(i)) \succ_i (\mu^0(i), \mu^1(i))$. However, recall that μ Pareto dominates μ_{GS}. This is a contradiction. Thus, after showing that $\bar{\mu}^1$ is statically stable under isolated preferences and μ^0_{GS}, which we show in the appendix, this step of the proof is complete.

The final step of the proof is by induction: in period 1, use the same argument for children $i \in I_1$, that we have used for children $i \in I_0$ in period 0, and similarly for any time period t. □

Corollary 1 shows that if the planner wants to eliminate the *justified envy*, then she should use the Gale and Shapley algorithm. In addition, as shown in proposition 1, the Gale and Shapley matching is not Pareto dominated by any other strongly stable matchings. Hence, the Gale and Shapley deferred acceptance algorithm is indeed one of the most important algorithms in the daycare assignment problem.

Now we study if any strongly stable matching is efficient. The next proposition yields that unless one follows the Gale and Shapley algorithm, then any strongly stable matching is not efficient.

Proposition 2. *Suppose that the priority rankings of all schools satisfy* IPA*. Then any strongly stable matching different from the Gale and Shapley matching is not efficient.*

Proof. Consider any strongly stable matching μ with some period t matching that is different from the one that the Gale and Shapley deferred acceptance algorithm under isolated preferences and

μ^{t-1} yields. Consider any $i \in I_t$. Then $\mu^t(i) = \mu^{t+1}(i)$ or $(\mu^{t+1}(i), \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^t(i))$; otherwise, μ is not strongly stable because, in this case, child i would have the higher priority at school $\mu^t(i)$ and $(\mu^t(i), \mu^t(i)) \succ_i (\mu^t(i), \mu^{t+1}(i))$ by assumption 1.

For each child $i \in I_{t-1} \cup I_t$, define her preference relation to be $\mathcal{P}_i^t$ such that $s\mathcal{P}_i^t s'$ if and only if

$$(\mu^{t-1}(i), s) \succ_i (\mu^{t-1}(i), s') \text{ whenever } i \in I_{t-1}$$

$$(s, \mu^{t+1}(i)) \succ_i (s', \mu^{t+1}(i)) \text{ whenever } i \in I_t$$

Because μ is strongly stable, there cannot exist any school-child pair (s, i) such that

1. $(\mu^{t-1}(i), s) \succ_i (\mu^{t-1}(i), \mu^t(i))$ or $(s, \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^{t+1}(i))$,
2. $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1})j$ for some $j \in \mu^t(s)$.

In terms of $\mathcal{P}$, these conditions mean that there is no school-child pair (s, i) such that

1. $s\mathcal{P}_i^t \mu^t(i)$,
2. $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1})j$ for some $j \in \mu^t(s)$.

In other words, μ^t is a statically stable matching under $\mathcal{P}$ and μ^{t-1}.

Consider matching $\bar{\mu}$ such that $\bar{\mu}^\tau = \mu^\tau$ for all $\tau \neq t$ but $\bar{\mu}^t$ is the resulting matching from the Gale and Shapley deferred acceptance algorithm under $\mathcal{P}$ and μ^{t-1}.

From [12], we know that $\bar{\mu}^t$ must Pareto dominate every other stable matching under $\mathcal{P}$ and μ^{t-1}. This and that μ^t is a statically stable matching under $\mathcal{P}$ and μ^{t-1} imply that $\bar{\mu}^t(i)\mathcal{P}_i\mu^t(i)$ for all $i \in I_{t-1} \cup I_t$ if $\bar{\mu}^t(i) \neq \mu^t(i)$. Consequently, if $\bar{\mu}^t(i) \neq \mu^t(i)$ for some $i \in I_{t-1}$, then $(\mu^{t-1}(i), \bar{\mu}^t(i)) \succ_i (\mu^{t-1}(i), \mu^t(i))$. Similarly, if $\bar{\mu}^t(i) \neq \mu^t(i)$ for some $i \in I_t$ then $(\bar{\mu}^t(i), \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^{t+1}(i))$. Now consider $\bar{\mu}$ and μ. Clearly, $\bar{\mu}$ Pareto dominates μ if $\bar{\mu}^t(i) \neq \mu^t(i)$ for some $i \in I_{t-1} \cup I_t$. Hence, it must be that $\bar{\mu}^t(i) = \mu^t(i)$ for all $i \in I_{t-1} \cup I_t$.

Consider $\hat{\mu}$ such that $\hat{\mu}^\tau = \mu^\tau$ for all $\tau \neq t$ but $\hat{\mu}^t$ is the resulting matching from the Gale and Shapley deferred acceptance algorithm under isolated preferences and $\hat{\mu}^{t-1}$. Clearly, $\bar{\mu}^{t-1} = \hat{\mu}^{t-1}$, hence, the priority rankings of the schools are the same under both $\bar{\mu}$ and $\hat{\mu}$. In addition, for each $i \in I_{t-1}$, the induced preference relation $\succ_i^2 (\mu^{t-1})$ is equivalent to $\mathcal{P}$. Now consider any child $i \in I_t$. Then under $\mathcal{P}$, the relative ranking of $\mu^{t+1}(i)$ weakly improves from the one under $\succ_i^1$. In all other aspects, $\mathcal{P}_i$ and $\succ_i^2 (\mu^{t-1})$ are the same. Now recall that $\bar{\mu}^t(i) = \mu^t(i)$ for all $i \in I_{t-1} \cup I_t$. In addition, recall that $\mu^t(i) = \mu^{t+1}(i)$ or $(\mu^{t+1}(i), \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^t(i))$. Therefore, under both $\mathcal{P}_i$ and $\succ_i^2 (\mu^{t-1})$, the set of schools that are strictly preferred to $\mu^t(i)$ is the same. Consequently, we obtain that under $\mathcal{P}$ and isolated preferences, for each $i \in I^{t-1} \cup I^t$, the set of schools that are strictly preferred to $\mu^t(i)$ is the same. In addition, because the Gale and Shapley algorithm is used for both cases and $\bar{\mu}^t(i) = \mu^t(i)$ for all $i \in I_{t-1} \cup I_t$, it must be $\bar{\mu}^t = \hat{\mu}^t$ thanks to theorem 9 in [11]. Consequently, $\mu^t = \hat{\mu}^t$, which contradicts that μ^t differs from the matching that the Gale and Shapley algorithm yields. □

Proposition 2 means that if any strongly stable matching is efficient, then it must be the one from the Gale and Shapley deferred acceptance algorithm. However, from [15], it is well known that the Gale and Shapley deferred acceptance algorithm (in static settings) does not necessarily yield a Pareto efficient matching. This is still the case in our setting because the school choice problem is a special case of our problem as we already pointed out in Remark 1.

5 Strategy Proofness and Stability: Impossibility Results

It is well known that in static settings, when the Gale and Shapley deferred acceptance algorithm is used, reporting one's true preference ordering is a weakly dominant strategy. Hence, the algorithm is strategy proof. In this section, we explore if any mechanism is strategy-proof and yields a strongly stable matching.

Even when *independence* and *IPA* are satisfied, strategy-proofness is more difficult to achieve in the daycare assignment problem. In static problems, a child misreports her preferences if she can obtain a better placement. This motive is present in the daycare assignment problem. That is, a child could obtain a spot at a higher ranked school without hurting her placement in the other period. But we know from the school choice literature that there are important strategy-proof mechanisms — e.g., the Gale and Shapley algorithm or Top Trading Cycles algorithm. However, in the daycare assignment problem, there is another motive which is not present in the school choice problem: a child misrepresents her preferences to affect the priority rankings of schools when she is two. This way she obtains a better placement when she is two, but she sacrifices her placement when she is one. The second motive is indeed very strong that derives the following impossibility result.

Theorem 4 (Impossibility Result)**.** *The existence of a strategy-proof and weakly stable mechanism is not guaranteed.*

Proof. Consider the following example: there are 4 schools $\{s, s_1, s_2, \bar{s}\}$. All schools have a capacity of one child. There is no school age child until period $t-1$. Suppose $I_{t-1} = \{i, \bar{\imath}\}$, $I_t = \{i_1, i_2\}$

and $I_{t+1} = \{i'\}$. Then starting period $t+3$, there are no school age children. Each school gives the highest priority to the child who has attended the school if the child is still of the school age. If children $j \neq j' \in \{i, \bar{\imath}, i_1, i_2, i'\}$ have not attended school $s' = s, s_1, s_2$ in the previous period, then school s' ranks child j and child j' according to the following rankings.

$$\begin{array}{ccccccc} i & \rhd_s & i' & \rhd_s & i_1 & \rhd_s & i_2 \\ i & \rhd_{s_1} & i_1 & \rhd_{s_1} & i_2 & \rhd_{s_1} & i' \\ i & \rhd_{s_2} & i_1 & \rhd_{s_2} & i' & \rhd_{s_2} & i_2 \\ \bar{\imath} & \rhd_{\bar{s}} & i_1 & \rhd_{\bar{s}} & i' & \rhd_{\bar{s}} & i_2 \end{array}$$

We consider 2 preference profiles which differs only in child i_1's preferences. Each child's preferences satisfy *independence*. Child i's top choice is (s, s) while child $\bar{\imath}$'s is $(\bar{s}, \bar{s})$. The preferences of children i_2 and i' satisfy the following conditions:

$$\begin{array}{ccccccc} (s_2, s_2) & \succ_{i_2} & (s_1, s_1) & \succ_{i_2} & (s, s) & \succ_{i_2} & (\bar{s}, \bar{s}) \\ (s_2, s_2) & \succ_{i'} & (s, s) & \succ_{i'} & (s_1, s_1) & \succ_{i_2} & (\bar{s}, \bar{s}) \end{array}$$

Child i_1 has 2 types of preferences, $\succ^1_{i_1}$ and $\succ^2_{i_1}$ which are:

$$\begin{array}{ccccccc} (s, s) & \succ^1_{i_1} & (s_1, s_1) & \succ^1_{i_1} & (s_2, s_2) & \succ^1_{i_1} & (\bar{s}, \bar{s}) \\ (s, s) & \succ^2_{i_1} & (\bar{s}, \bar{s}) & \succ^1_{i_2} & (s_2, s_2) & \succ^2_{i_1} & (s_1, s_1) \end{array}$$

In addition, suppose $(s_2, s) \succ^1_{i_1} (s_1, s_1)$.

For this example, we prove that there is no mechanism that is strategy proof and yields strongly stable matchings in 3 steps.

Step 1. Under profile 1, the only weakly stable matching is as follows: $\mu^{t-1}(i) = \mu^t(i) = s$, $\mu^{t-1}(\bar{\imath}) = \mu^t(\bar{\imath}) = \bar{s}$, $\mu^t(i_1) = \mu^{t+1}(i_1) = s_1$, $\mu^t(i_2) = \mu^{t+1}(i_2) = s_2$, $\mu^{t+1}(i') = s$ and $\mu^{t+2}(i') = s_2$.

Proof of Step 1. It is easy to see that the Gale and Shapley deferred acceptance algorithm yields the matching given in step 1, hence is weakly stable. Now only thing left to show is that there is no other weakly stable matching under profile 1.

Let $\hat{\mu}$ be weakly stable. It is clear that $\hat{\mu}^{t-1}(i) = \hat{\mu}^t(i) = s$, $\hat{\mu}^{t-1}(\bar{\imath}) = \hat{\mu}^t(\bar{\imath}) = \bar{s}$ and $\hat{\mu}^{t+2}(i') = s_2$. Consequently, we obtain that $\hat{\mu}^t(i_1) = s_1$ because child i_1 has higher priority in school s_1 at period t than anyone but i. However, i must match with s at period t. Hence, $\hat{\mu}^t(i_1) = s_1$. This implies that $\hat{\mu}^t(i_2) = s_2$. Then i_2 has the highest priority at school s_2 at period $t+1$. Since s_2 is the top choice for i_2, $\hat{\mu}^{t+1}(i_2) = s_2$. Consequently, $\hat{\mu}_2(i') = s$ which means $\hat{\mu}^{t+1}(i_1) = s_1$. This means that $\hat{\mu} = \mu$. This proves the step 1.

Step 2. Under profile 2, the only weakly stable matching is $\bar{\mu}$ as follows: $\bar{\mu}^{t-1}(i) = \bar{\mu}^t(i) = s$,

$\bar{\mu}^{t-1}(\bar{\imath}) = \bar{\mu}^t(\bar{\imath}) = \bar{s}$, $\bar{\mu}^t(i_1) = s_2$, $\bar{\mu}^t(i_2) = s_1$, $\bar{\mu}^{t+1}(i_1) = s$, $\bar{\mu}^{t+1}(i_2) = s_1$, $\bar{\mu}^{t+1}(i') = s_2$ and $\bar{\mu}^{t+2}(i') = s_2$.

Proof of Step 2. It is easy to see that the Gale and Shapley deferred acceptance algorithm yields the matching given in claim 2, hence is weakly stable. Now only thing left to show is that there is no other weakly stable matching under profile 2.

Let $\hat{\mu}$ be a weakly stable matching. It is clear that $\hat{\mu}^{t-1}(i) = \hat{\mu}^t(i) = s$, $\hat{\mu}^{t-1}(\bar{\imath}) = \hat{\mu}^t(\bar{\imath}) = \bar{s}$ and $\hat{\mu}^{t+2}(i') = s_2$. Consequently, we obtain that $\hat{\mu}^t(i_1) = s_2$ because child i_1 has higher priority in school s_2 at period t than i_2. This means that $\hat{\mu}^t(i_2) = s_1$.

Now let us argue that $\hat{\mu}^{t+1}(i') = s_2$. If not, $\hat{\mu}^{t+1}(i_1) = s_2$; otherwise, child i' has higher priority than child i_2 at school s_2 and s_2 is the top choice of child i'. Hence, this contradicts with $\hat{\mu}$ being weakly stable. Thus, $\hat{\mu}^{t+1}(i_1) = s_2$. But because $(s_2, \bar{s}) \succ^2_{i_1} (s_2, s_2)$ and child i_1 has higher priority at school $\bar{s}$ than anyone but $\bar{\imath}$, $\hat{\mu}$ is weakly stable. This is a contradiction. Hence, $\hat{\mu}^{t+1}(i') = s_2$.

Because $\hat{\mu}^{t+1}(i') = s_2$, $\hat{\mu}^{t+1}(i_1) = s$ because i_1 has higher priority at school s than i_2. Then $\hat{\mu}^{t+1}(i_2) = s_1$.

Step 3. For this example, there exists no strategy proof mechanism that yields weakly stable matchings.

Proof of Step 3. On contrary, suppose that, for this example, there exists strategy proof mechanism that yields weakly stable matchings. Suppose that the children's preferences are under profile 1. By truthfully reporting her type, child i_1 attends school s_1 when she is both 1 and 2 years old. However, by misreporting her preferences as if under profile 2, she attends s_2 in period t but s in period $t+1$. By assumption, $(s_2, s) \succ^1_{i_1} (s_1, s_1)$. Hence, child i_1 misrepresents her preferences under profile 1, hence, the mechanism is not strategy proof. □

In the example used for the proof of theorem 4, type 1 child i_1 likes school s better than any other school. Clearly, there is no chance that she can attend s in period t. In addition, she cannot attend s at $t+1$ because child i' attends s. But observe that child i' wants to attend school s_2 but cannot do so because child i_2 attends s_2. The most important aspect is that child i_2 has higher priority over child i' at school s_2 in period $t+1$ only because she attends school s_2 in period t. Child i_1 can eliminate child i_2's advantage over i' if she attends school s_2 in period t. By doing this, i_1 enables i' to attend s_2 at $t+1$. Ultimately, she frees a spot at school s for herself at $t+1$. This is the reason why type 1 child i_1 has an incentive to misreport her preferences.

Remark 4. *Observe that in the example used for the proof of theorem 4, both* IPA *and* independence *are satisfied. Hence, a very minimal history dependence leads to a very negative result. This result* *a major difference between the school choice problem and the daycare assignment problem.*

Theorem 4 has two important corollaries which we present next.

Corollary 2. *The existence of a mechanism that is strategy proof and yields a strongly stable matching is not guaranteed.*

Proof. Recall that each strongly stable matching is weakly stable. This and theorem 4 prove this corollary. □

Corollary 3. *The Gale and Shapley deferred acceptance algorithm is not necessarily strategy-proof.*

6 Top Trading Cycles

We have shown that the well known Gale-Shapley deferred acceptance algorithm, which is widely used in the school choice problem, is not a particularly appealing algorithm for the dynamic problem, since it is not strategy-proof.

Most importantly, we showed that stability and strategy-proofness are incompatible for dynamic problems. This may suggest that eliminating justified envy in a dynamic problem may not be the most appropriate objective when designing an assignment mechanism, at least not if strategy-proofness is to be preserved. In the remaining sections of this paper, we investigate whether strategy-proofness is compatible with efficiency. First, we consider another well known mechanism, the top trading cycles, in our setting.

Consider an algorithm that uses the top trading cycles mechanism each period, according to the isolated preferences of children. We refer to the mechanism described by [5], which was inspired by [14] and [18]. It can be described as follows:

Step 1: Each child points to her preferred school. Each school points to its highest ranked child. The process goes on, until it reaches a cycle, which it eventually will. A cycle can be written as $\{i_1, s_1, i_2, s_2, ..., i_k, s_k\}$, where here, s_j is child $i_j's$ preferred school, whereas child i_l is the highest ranked child in school s_{l-1}, for $l = 2, ..., k$; and child i_1 is the highest ranked child at school s_k. All children in the cycle are allocated to their preferred school.

In general, at:

Step k: All children allocated in steps 1,...,$k-1$ do not participate in step k. Each remaining child points to its preferred school, among the set of schools with remaining spots. Each pointed school points to the highest priority child among the remaining children. The process goes on until it reaches a cycle, which it eventually will. All children in the cycle are allocated to the schools that they have pointed to.

The process continues until all children are allocated.

The top trading cycle mechanism is shown to be both efficient and strategy-proof for static problems. Here, we investigate its properties in our daycare assignment problem. In our next proposition, we study whether the top trading cycles mechanism yield an efficient matching. The answer is negative.

Proposition 3 (Top Trading Cycles is not necessarily Pareto Efficient)**.** *If the top trading cycles mechanism is applied at every period using the isolated preferences of the children, then the resulting matching, denoted by μ_{TTC}, is not necessarily Pareto efficient.*

Proof. For the proof of this proposition, it suffices to construct an counter example. Suppose in period 0, two children i_1 and i_2 are two years old and two children j_1 and j_2 are one year old. There are 4 schools s_1, s_2, s_3 and s_4 and each school has a capacity of 1 child. The schools' priorities satisfy *IPA* and the children's preferences satisfy *independence*. The schools' priorities are given as follows under the assumption that the children have not attended any school in the previous period:

$$\begin{array}{ccccccc} i_1 & \rhd_{s_1} & i_2 & \rhd_{s_1} & j_1 & \rhd_{s_1} & j_2 \\ i_2 & \rhd_{s_2} & i_1 & \rhd_{s_2} & j_2 & \rhd_{s_2} & j_1 \\ i_1 & \rhd_{s_3} & i_2 & \rhd_{s_3} & j_1 & \rhd_{s_3} & j_2 \\ i_1 & \rhd_{s_3} & i_2 & \rhd_{s_3} & j_2 & \rhd_{s_3} & j_1 \end{array}$$

Child i_1's top choice is s_1 while child i_2's is s_2. The other two children's preferences satisfy the following conditions:

$$\begin{array}{ccccccccccc} (s_2, s_2) & \succ_{j_1} & (s_1, s_1) & \succ_{j_1} & (s_4, s_2) & \succ_{j_1} & (s_3, s_1) & \succ_{j_1} & (s_3, s_3) & \succ_{j_1} & (s_4, s_4) \\ (s_2, s_2) & \succ_{j_2} & (s_1, s_1) & \succ_{j_2} & (s_3, s_1) & \succ_{j_2} & (s_4, s_2) & \succ_{j_2} & (s_3, s_3) & \succ_{j_2} & (s_4, s_4) \end{array}$$

The top trading cycle algorithm using isolated preferences yields the following matching μ_{TTC}: $\mu^0_{TTC}(i_1) = s_1$, $\mu^0_{TTC}(i_2) = s_2$, $\mu^0_{TTC}(j_1) = s_3$, $\mu^0_{TTC}(j_2) = s_4$, $\mu^1_{TTC}(j_1) = s_1$, $\mu^1_{TTC}(j_2) = s_2$. It is not hard to see that μ_{TTC} is Pareto dominated by the matching μ: $\mu^0(i_1) = s_1$, $\mu^0(i_2) = s_2$, $\mu^0(j_1) = s_4$, $\mu^0(j_2) = s_3$, $\mu^1(j_1) = s_2$, $\mu^1(j_2) = s_1$.

□

Note that the example used in the proof of proposition 3 that, not only the top trading cycles is not necessarily efficient, but also a variation of it, done by cohorts. Precisely, consider the following mechanism. At any period t, The children born in period $t-1$ are allocated according to the top trading cycles mechanism (see [5]). Once every children $i \in I_{t-1}$ is allocated, most schools will have less, if any, spots available. Consider only the schools with open spots and use the top trading cycles for the generation born in period t, where from the initial number of spots for each school, we have subtracted the number of 2-year-old children already allocated. For this round, consider only the priority of schools over the children of generation t. i.e., a young child cannot replace an already allocated 2-year-old child. This variation of the top trading cycles is also is not Pareto efficient.

In the example below, we show that the top trading cycles (using isolated preferences) may not be a strategy proof mechanism.

Example 5 (Top Trading Cycles may not be Strategy-Proof)**.** *Assume that there are 4 schools* $\{s'', s', s, \bar{s}\}$, *and 4 children:* $\{\bar{\imath}, i, i', i''\}$, *with* $\bar{\imath} \in I_{-1}$ *and* $\{i, i', i''\} \in I_0$. *Assume also that there are no children born in period 1,* $I_1 = \varnothing$. *The schools' priorities satisfy* IPA *and the children's preferences satisfy* independence. *The priorities of the schools satisfy the following conditions:*

$$\begin{array}{lllllll} s'': & i & \rhd_{s''} & i'' & \rhd_{s''} & j, & \forall j \neq i, i' \\ s': & i' & \rhd_{s'} & j, & \forall j \neq i' & & \\ s: & i & \rhd_{s} & j, & \forall j \neq i & & \\ \bar{s}: & \bar{\imath} & \rhd_{\bar{s}} & i' & \rhd_{\bar{s}} & i & \end{array}$$

The children's preferences are:

$$\begin{array}{lllllllll} i: & \bar{s} & \succ_i & s & \succ_i & s' & \succ_i & s'' \\ i': & s'' & \succ_{i'} & \bar{s} & \succ_{i'} & s' & \succ_{i'} & s \\ i'': & s'' & \succ_{i''} & s & \succ_{i''} & s' & \succ_{i''} & \bar{s} \\ \bar{\imath}: & \bar{s} & \succ_{\bar{\imath}} & s & \succ_{\bar{\imath}} & s' & \succ_{\bar{\imath}} & s'' \end{array}$$

In addition, child i prefers $(s', \bar{s})$ to (s, s). The matching resulting from the top trading cycles is:

$$(\bar{\imath}, \bar{s}) ; (i, s) ; (i'', s'') ; (i', s'),$$

in period $t = 0$ and

$$(i'', s'') ; (i', \bar{s}) ; (i, s),$$

in period $t = 1$.

Suppose that i misreports its preferences to be: $i: \quad \bar{s} \quad \succ_i \quad s' \quad \succ_i \quad s \quad \succ_i \quad s''$ *, while all others report truthfully. The resulting matching for* $t = 0$ *is:*

$$(\bar{\imath}, \bar{s}) \, ; \left(i, s'\right), \left(i', s''\right) ; (i", s) \, ,$$

while for $t = 1$ *it is:*

$$\left(i', s''\right) ; (i, \bar{s}) \, ; \left(i'', s\right) .$$

Note that under truth-telling, i's allocation was: (s, s)*, while after misreporting it is* $(s', \bar{s})$*. Thus, i has improved herself overall by taking* s' *in the first period and altering the priority of* s'' *for the following period.* ◇

Note that the example above shows that a variation of the top trading cycles which is done by cohorts is not strategy-proof.

7 Strategy-Proofness and Efficiency

In this section, we show that strategy-proofness and efficiency are not necessarily incompatible. Consider a serial dictator algorithm adapted to our setting. We will argue that this algorithm is strategy-proof. Moreover, it is efficient. Before constructing the algorithm, recall that at period t, n_t number of children are one and they are indexed 1 through n_t. The algorithm runs as follows:

At period 0:

All 2-year old children choose the school that they want to attend in an increasing order according to their indices. All children obtain their top spot as long as the chosen school has available seats. When a school has fulfilled its slots, the child moves on to her next best choice.

When all 2-year old children have been allocated, then all 1-year old children choose their preferred school with open slots following an increasing order according to their indices.

At period 1:

In the following period, the children who are now 2-year old choose their schools according to their indices. Then the one year old children choose their schools in an increasing order according to their indices. The process repeats itself. Given that at any given period there is a finite number of school age children, this is a well-defined mechanism. Moreover, it is easy to verify that the proposed algorithm is both strategy-proof and efficient.[5]

[5] One can use the random serial dictatorship algorithm which is a slight variation of the serial dictatorship algo-

8 Conclusion

In this paper we have introduced the daycare assignment problem. This problem differs from the school choice problem due to its dynamic nature. We have proved some negative results concerning well-known mechanisms, even when preferences are assumed to satisfy some consistency across periods, and schools' priorities are linked only in a very weak sense (priorities are history dependent only through currently allocated children, and are otherwise the same). In particular, we have shown that the Gale-Shapley deferred acceptance algorithm and the Top Trading Cycles, both commonly used in the school choice problem, are not strategy-proof in the daycare assignment problem. We have extended these insights to show that there are no strongly stable mechanisms that are strategy-proof.

We conclude by presenting a version of the serial dictator, adapted to our setting, and arguing that it is strategy-proof and efficient.

9 Appendix

9.1 Aarhus Assignment Mechanism[6]

PLACE ASSIGNMENT RULES

In brief, places are assigned in this order:

1. Children with special needs, e.g., children with disabilities

2. Children with siblings in the same institution

3. Bilingual children who, after expert evaluation, are deemed in need of special assistance in day care

4. The oldest child in an assignment district (anvisningsdistrikt) who is written up for a guaranteed place. That is, a place corresponding to the rules of the place guarantee. An assignment district is the area the child lives in. It consists of 1 to 3 school districts

5. The oldest child in an assignment district who is written up for a guaranteed place. Aarhus municipality is divided into 8 major guarantee districts (garantidistrikter) along the approach roads. A guarantee district consists of one or several assignment districts

6. The oldest child in an assignment district who is written up for a guaranteed place

rithm. The random serial dictatorship algorithm is strategy-proof and ex-post efficient but not necessarily ex-ante efficient– see [8].

[6] For the original document see: https://www.borger.dk/selvbetjening/sider/fakta.aspx?sbid=8632

7. The oldest child on the waiting list for a particular institution, even if the child has another place already

Guaranteed place and desired place

You can choose a guaranteed place, but at the same time request one or more specific institutions. These wishes will be taken into account when we find a place for you. However, we cannot guarantee that you get one of these desired places. If none of the institutions you are interested in have openings, you will be offered a guaranteed place.

A guaranteed place is a place within the district you live in, or at a distance from your home which involves no more than half an hour of extra transport each way to and from work. The municipal placement guarantee is satisfied when you have been offered a place. To be assigned a guaranteed seat at a desired time, the application must be received by the placement guarantee office (Pladsanvisningen) no later than 3 months before the place is desired.

Moves outside Aarhus Municipality

If you move from Aarhus municipality and want to keep your place after the move, you must immediately inform the placement guarantee office that you are moving. Aarhus municipality will ask your new municipality to submit a subsidy certificate (tilskudsbevis). Monthly payments will then depend on the size of the subsidy from the new municipality. Any difference will be charged / adjusted per. move date.

Domiciled outside Aarhus Municipality

If you live in another municipality and want a place at an institution in Aarhus municipality, you must register on the waiting list. This can also be done at https://digitalpladsanvisning.borgerservice.dk.

When you get offered a place in Aarhus Municipality and accept it, we will ask your municipality to provide a subsidy certificate stating the starting date. We will also make an agreement about which of the two municipalities are required to collect parental fees from you. The monthly fee will depend on the size of the subsidy from your municipality.

Privacy Act (persondataloven) - the rights of citizens

"Act concerning the processing of personal data" gives you as a citizen various rights when the municipality processes information about you. The purpose of the act is to enhance transparency and thereby strengthen your legal position.

The municipality is, for example, obliged to inform the citizen of the municipality's treatment of the collected information – The so-called information obligation (oplysningspligt).

Duty to assist inquiry

Lack of response may have implications for the claimed benefit or harm the proceedings. Procedural law states that you have an obligation to assist inquiry. You are also obliged to immediately give the municipality notice of any change in your personal and financial circumstances that may cause change in benefits. Missing or incorrect information may result in claims for reimbursement.

The municipality's control of information

For control purposes, the municipality may obtain information. These may be in electronic form. They may regard economic conditions, etc. from, for example, employers, unemployment insurance, tax authorities and other public authorities, including municipalities.

Disclosure of information

Aarhus Municipality routinely transfers data to other municipalities, government institutions, counties and others with legal right to the information.

Discovery, and correction

According to the Privacy Act you are entitled to access information stored about you. You can gain access by querying "Borgercervice", City Hall, 8100 Aarhus C. If the municipality has entered incorrect information about you, you can demand this corrected.

Example 6 (Aarhus Mechanism)**.** *Suppose there are 2 schools,* $\{s_1, s_2\}$ *and each school has a capacity of one child. In each period 1 child is born, but children are identical in all other aspects. Their preferences are satisfy the following property:* $(s_1, s_1) \succ (s_2, s_1) \succ (h, s_1) \succ (s_2, s_2)$*. Both schools are in a close enough distance to all children that by offering a spot in either school, the municipality meets its obligation to find a guaranteed spot to any applicant. However, school* s_1 *is closer to all children than* s_2 *is. Consequently, any opening in* s_1 *is offered to the oldest applicant before any opening in* s_2 *is offered.*

Consider the following strategy profile: at age 1, each child applies for a spot if and only if then the two year old child has attended s_2 *in the previous period or has applied and yet does not have a spot in any school. At age 2, each child who have not applied for a spot in the previous period applies for a spot. In case of applying for a spot, one always requests a guaranteed spot. The result is that at age 1, no child attends a school while at age 2 all children attend school 1. At this equilibrium school 2 never fills its spot.*

Now let us show the above proposed strategy profile is an equilibrium. Clearly, if the two year old child has attended s_2 *in the previous period or has applied and yet does not have a spot in any school, then one year old attends school* s_1 *by requesting a guaranteed spot. Given that attending school* s_1 *at both age 1 and 2 is the most preferred option, one has no incentive to deviate. When* *then two year old child requests a guaranteed spot, by applying for a spot, one year old child only can attend school* s_2. *Then this child also attends school* s_2 *at age 2 as the spot in school* s_1 *is assigned to then 1 year old child. However,* $(h, s_1) \succ (s_2, s_2)$. *Therefore, one year old certainly worsens. This completes the proof that the proposed strategy profile is an equilibrium.*

Now the resulting matching from the above described equilibrium is that (h, s_1) *for each child. Clearly, each child matching with* (s_2, s_1) *Pareto dominates* (h, s_1). *Hence, the Aarhus allocation mechanism is not efficient. Furthermore, in each period, then the one 1 year old child can attend school* s_2 *as school* s_2 *has an unfilled spot. Consequently, the Aarhus allocation mechanism is not weakly stable.*

$\diamond$

Proof of Lemma 1. Since μ is not strongly but weakly stable, by definition 6, there must exist s, s' such that $(s, s') \succ_i (\mu^t(i), \mu^{t+1}(i))$ and one of the following conditions are satisfied:

1. $|\mu^t(s)| < r_s$ and $|\mu^{t+1}(s')| < r_{s'}$,
2. $|\mu^t(s)| < r_s$, $|\mu^{t+1}(s')| = r_{s'}$, and, for some $j' \in \mu^{t+1}(s')$, $i \rhd_{s'}^{t+1} (\bar{\mu}^t) j'$ where $\bar{\mu}^t$ is the period t matching with $\bar{\mu}^t(i) = s$ and $\bar{\mu}^t(i') = \mu^t(i')$ for all $i' \neq i \in I^{t-1} \cup I^t$,
3. $|\mu^t(s)| = r_s$, $|\mu^{t+1}(s')| < r_{s'}$, and, for some $j \in \mu^t(s)$, $i \rhd_s^t (\mu^{t-1}) j$,
4. $|\mu^t(s)| = r_s$, $|\mu^{t+1}(s')| = r_{s'}$, for some $j \in \mu^t(s)$, $j' \in \mu^{t+1}(s')$ and for any $\bar{\mu}^t \in M(i, j, \mu)$, $i \rhd_s^t (\mu^{t-1}) j$ and $i \rhd_{s'}^{t+1} (\bar{\mu}^t) j'$.

First, note that $s \neq \mu^t(i)$ and $s' \neq \mu^{t+1}(i)$; otherwise, μ is not weakly stable, which can be verified using the fact that *IPA* is satisfied.

Case 1. $s = s'$. Consequently, $(s, s) \succ_i (\mu^t(i), \mu^{t+1}(i))$. In addition, $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$. Combining this with μ being weakly stable, one obtains that $(\mu^t(i), \mu^{t+1}(i)) \succ_i (s, \mu^{t+1}(i))$. Given independence, this, in turn, implies that if $\mu^t(i) \neq \mu^{t+1}(i)$ then $(\mu^t(i), \mu^t(i)) \succ_i (s, s)$. Then, by transitivity of preferences, $(\mu^t(i), \mu^t(i)) \succ_i (\mu^t(i), \mu^{t+1}(i))$. This implies that μ is not weakly stable because child i has the highest priority at school s at period $t+1$, hence, at $t+1$, she has a right to attend school s ahead of any other child. Therefore, $\mu^t(i) = \mu^{t+1}(i)$. This is the condition we seek.

Case 2. $s \neq s'$ and $\mu^t(i) = \mu^{t+1}(i)$. Consequently, $(s, s') \succ_i (\mu^t(i), \mu^t(i))$. In addition, $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$. Combining this with μ being weakly stable, one obtains

$(\mu^t(i), \mu^t(i)) \succ_i (s, \mu^t(i))$. Recall that $(s, s') \succ_i (\mu^t(i), \mu^t(i))$. Hence, by transitivity, $(s, s') \succ_i (s, \mu^t(i))$. Then, by assumption 1 (2), $(s', s') \succ_i (\mu^t(i), \mu^t(i))$. Suppose $(s, s) \succ_i (s', s')$. Then $(s, s) \succ_i (\mu^t(i), \mu^t(i))$ and, by assumption, $\left|\mu^t(s)\right| < r_s$ or/and $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$. Hence, we have identified a pair (s, i) asked in the lemma.

Now suppose $(s', s') \succ_i (s, s)$. Since μ is weakly stable, either the allocation given by μ is preferred to this alternative allocation, or s' does not lead to justified envy. Formally, at least one of the two conditions must hold: (a) $(\mu^t(i), \mu^t(i)) \succ_i (\mu^t(i), s')$ or/and (b) $|\mu^{t+1}(s')| = r_{s'}$ and there exists no $j' \in \mu^{t+1}(s')$ such that $i \rhd_{s'}^{t+1} (\mu^t) j'$.

Suppose (a) occurs. Recall $(s, s') \succ_i (\mu^t(i), \mu^t(i))$, hence, $(s, s') \succ_i (\mu^t(i), s')$. Then assumption 1 (2) implies that $(s, s) \succ_i (\mu^t(i), \mu^t(i))$ because $s \neq s'$. Observe that the pair (s, i) is the pair asked in the lemma as we already pointed out that $(s, s) \succ_i (\mu^t(i), \mu^t(i))$, $\left|\mu^t(s)\right| < r_s$ or/and $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$.

Suppose now (b) occurs but not (a). Recall that one of the 4 conditions listed in the beginning of the proof must be satisfied. Since $|\mu^{t+1}(s')| = r_{s'}$, 1 and 3 are ruled out. If condition 2 is satisfied, then $i \rhd_{s'}^{t+1} (\bar{\mu}^t) j'$ for some $j' \in \mu^{t+1}(s')$. Furthermore, $\bar{\mu}^t$ differs from μ^t only in that $\bar{\mu}^t(i) = s$. Then, by *IPA*, $i \rhd_{s'}^{t+1} (\mu^t) j'$. This a contradiction with b occurring. If condition 4 is satisfied, then there must exist j, j' such that, for any $\bar{\mu}^t \in M(i, j, \mu)$, $i \rhd_s^t (\mu^{t-1}) j$ and $i \rhd_{s'}^{t+1} (\bar{\mu}^t) j'$. In particular, it must be true for $\bar{\mu}^t$ such that $\bar{\mu}^t(j) = h$. Observe that $\bar{\mu}^t$ differs from μ^t only in that $\bar{\mu}^t(i) = s$ and $\bar{\mu}^t(j) = h$. By *IPA*, $i \rhd_{s'}^{t+1} (\mu^t) j'$. This a contradiction with b occurring.

Case 3. $s \neq s'$ and $\mu^t(i) \neq \mu^{t+1}(i)$. Consequently, $(s, s') \succ_i (\mu^t(i), \mu^{t+1}(i))$. Since μ is weakly stable, one of the two conditions must hold: (a) $(\mu^t(i), \mu^{t+1}(i)) \succ_i (\mu^t(i), s')$ or/and (b) $|\mu^{t+1}(s')| = r_{s'}$ and no $j' \in \mu^{t+1}(s')$ with $i \rhd_{s'}^{t+1} (\mu^t) j'$ exists.

Suppose (a) occurs. Recall that by assumption, in this case 3, $(s, s') \succ_i (\mu^t(i), \mu^{t+1}(i))$, hence, $(s, s') \succ_i (\mu^t(i), s')$. Using 1 (2), this implies that $(s, s) \succ_i (\mu^t(i), \mu^t(i))$. Then, $(s, \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^{t+1}(i))$. Consider the pair (s, i). As pointed out earlier, $\left|\mu^t(s)\right| < r_s$ (conditions 1 or 2) or/and $i \rhd_s^t (\mu^{t-1}) j$ (conditions 3 or 4) for some $j \in \mu^t(s)$. This means that μ is not weakly stable which is a contradiction.

Suppose now (b) occurs but not (a), therefore $\left(\mu^t(i), s'\right) \succ_i (\mu^t(i), \mu^{t+1}(i))$. Recall that $(s, s') \succ_i (\mu^t(i), \mu^{t+1}(i))$, since μ is not strongly stable. In addition, one of the 4 conditions listed in the beginning of the proof must be satisfied. Since $|\mu^{t+1}(s')| = r_{s'}$, 1 and 3 are ruled out. If condition 2 is satisfied, then $i \rhd_{s'}^{t+1} (\bar{\mu}^t) j'$ for some $j' \in \mu^{t+1}(s')$. Furthermore, $\bar{\mu}^t$ differs from μ^t only in that $\bar{\mu}^t(i) = s$. By *IPA*, $i \rhd_{s'}^{t+1} (\mu^t) j'$. This is a contradiction with (b) occurring. If condition 4 is satisfied, then there must exist j, j' such that, for any $\bar{\mu}^t \in M(i,j,\mu)$, $i \rhd_s^t (\mu^{t-1}) j$ and $i \rhd_{s'}^{t+1} (\bar{\mu}^t) j'$. Fix $\bar{\mu}^t$ such that $\bar{\mu}^t(j) = h$. Observe that $\bar{\mu}^t$ differs from μ^t only in that $\bar{\mu}^t(i) = s$ and $\bar{\mu}^t(j) = h$. By *IPA*, $i \rhd_{s'}^{t+1} (\mu^t) j'$. This is a contradiction with (b) occurring. □

Proof of Lemma 2. Necessity. Assume μ is weakly stable. We need to show that for all t, μ^t is statically stable under isolated preferences and μ^{t-1}. Suppose otherwise. Then there must exist, t, and a school-child pair (s, i) such that

1. if $i \in I_t$, then $s \succ_i^1 \mu^t(i)$ and at least one of the following is satisfied: $|\mu^t(s)| < r_s$ or $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$,

2. if $i \in I_{t-1}$, then $s \succ_i^2 \left(\mu^{t-1}\right) \mu^t(i)$ and at least one of the following is satisfied: $|\mu^t(s)| < r_s$ or $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$.

Suppose $i \in I_t$. Then we are in case 1. Since μ is is weakly stable, the following 2 conditions cannot be satisfied at the same time: (a) $(s, \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^{t+1}(i))$ and (b) $|\mu^t(s)| < r_s$ and/or $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$. If (b) is not true, then this is a contradiction because (s, i) must satisfy the conditions given in case 1. Hence, assume that (b) is satisfied but (a) is not, i.e., $(\mu^t(i), \mu^{t+1}(i)) \succ_i (s, \mu^{t+1}(i))$. If $\mu^t(i) \neq \mu^{t+1}(i)$, assumption 1 implies that $(\mu^t(i), \mu^t(i)) \succ_i (s, s)$. By the definition of $\succ^1$, $\mu^t(i) \succ_i^1 s$ which contradicts with the assumption that $s \succ_i^1 \mu^t(i)$. Suppose $\mu^t(i) = \mu^{t+1}(i)$. Recall that $s \succ_i^1 \mu^t(i)$, hence, $(s, s) \succ_i (\mu^t(i), \mu^{t+1}(i))$. Recall that (b) is satisfied. Thus, by moving to school s in period t, child i would have the highest priority at school s at time $t+1$. Hence, μ is not strongly stable. Hence, $i \notin I_t$.

Suppose $i \in I_{t-1}$. Then we are in case 2. Because μ is weakly stable, the following 2 conditions cannot be satisfied at the same time: (a) $(\mu^{t-1}(i), s) \succ_i (\mu^{t-1}(i), \mu^t(i))$ and (b) $|\mu^t(s)| < r_s$ and/or $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$. If (b) is not true, then this is a contradiction because (s, i) must satisfy the conditions given in case 2. Hence, (b) must be satisfied but (a) is not, i.e., $(\mu^{t-1}(i), \mu^t(i)) \succ_i (\mu^{t-1}(i), s)$. By the definition of $\succ_i^2 (\mu^{t-1})$, we have that $\mu^t(i) \succ_i^2 (\mu^{t-1}) s$ which contradicts with the assumption that $s \succ_i^2 \left(\mu^{t-1}\right) \mu^t(i)$. Hence, $i \notin I_{t-1}$. Therefore, for all t, μ^t is statically stable under isolated preferences and μ^{t-1}.

Sufficiency. For any t, μ^t is statically stable under isolated preferences and μ^{t-1}. First let us show that μ is weakly stable. Suppose otherwise. Then, at some period t, there must exist a pair (s, i) such that one of the two conditions below is satisfied:

1. (a) $(s, \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^{t+1}(i))$, and

   (b) $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$.

   or

   (a) $(\mu^{t-1}(i), s) \succ_i (\mu^{t-1}(i), \mu^t(i))$, and

   (b) $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$.

Suppose case 1 occurs. If $s \neq \mu^{t+1}(i)$, then by assumption 1, and recall $(s, \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^{t+1}(i))$, we would have that:

$$(s, s) \succ_i (\mu^t(i), \mu^t(i)).$$

By definition of $\succ_i^1$, we have that $s \succ_i^1 \mu^t(i)$. This and 1b mean that μ^t is not statically stable under isolated preferences and μ^{t-1}. This is a contradiction. Suppose, on the other hand, that $s = \mu^{t+1}(i)$. If $(\mu^{t+1}(i), \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^t(i))$, then the definition of $\succ_i^1$ yields $\mu^{t+1}(i) \succ_i^1 \mu^t(i)$. This and 1b mean that μ^t is not statically stable under isolated preferences and μ^{t-1}.

Suppose $(\mu^t(i), \mu^t(i)) \succ_i (\mu^{t+1}(i), \mu^{t+1}(i))$. This and assumption 1 yield $(\mu^t(i), \mu^t(i)) \succ_i (\mu^t(i), \mu^{t+1}(i))$. Now consider period $t+1$. Then by the definition of $\succ_i^2 (\mu^t)$, $\mu^t(i) \succ_i^2 (\mu^t) \mu^{t+1}(i)$. In addition, observe that child i has the highest priority at school $\mu^t(i)$. The last 2 conditions contradict with μ^{t+1} being statically stable under isolated preferences and μ^t.

Suppose case 2 occurs. By the definition of $\succ_i^2 (\mu^{t-1})$, we have that $s \succ_i^2 (\mu^{t-1}) \mu^t(i)$ since $(\mu^{t-1}(i), s) \succ_i (\mu^{t-1}(i), \mu^t(i))$. But this and 2$b$ directly imply that μ^t is not statically stable under isolated preferences and μ^{t-1}. This is a contradiction.

We have shown that μ is weakly stable. Now we are left to show that μ is strongly stable if *IPA* is satisfied.[7] Suppose otherwise. Then by lemma 1, for some period t and some school-child pair (s, i),

1. $\mu^t(i) = \mu^{t+1}(i)$
2. $(s, s) \succ_i (\mu^t(i), \mu^{t+1}(i))$
3. $\left|\mu^t(s)\right| < r_s$ or/and $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$

The first 2 conditions and the definition of $\succ_i^1$ yield $s \succ_i^1 \mu^t(i)$. This and the third condition imply that μ^t is not statically stable under isolated preferences and μ^{t-1}. □

[7] Note that if the children's preferences satisfy independence, then theorem 1 implies the result directly

Proof of Proposition 1. Recall that time -1 matching μ^{-1} is fixed for all matchings we consider.

On contrary to the proposition, suppose that some strongly stable matching μ Pareto dominates matching μ_{GS}.

Step 1. If $i \in I_{-1}$, then $\mu^0_{GS}(i) = \mu^0(i)$.

Proof of Step 1. For any 2 year old child, her isolated preference is $\succ_i^2 (\mu^{-1})$. From lemma 2, we have that μ^0_{GS} and μ^0 are stable period 0 matchings under isolated preferences and μ^{-1}. Gale and Shapley [12] show that μ^0_{GS} Pareto dominates every other statically stable period 0 matchings under isolated preferences and μ^{-1} in terms of isolated preferences. This means $\mu^0_{GS}(i) \succ_i^2 (\mu^{-1})\mu^0(i)$ if $\mu^0_{GS}(i) \neq \mu^0(i)$. By definition of $\succ_i^2 (\mu^{-1})$, $(\mu^{-1}(i), \mu^0_{GS}(i)) \succ_i (\mu^{-1}(i), \mu^0(i))$ if $\mu^0_{GS}(i) \neq \mu^0(i)$. Hence, if μ Pareto dominates μ_{GS}, then it must be $\mu^0_{GS}(i) = \mu^0(i)$.

Step 2. If $i \in I_0$, then $\mu^0_{GS}(i) = \mu^0(i)$.

Proof of Step 2. Suppose $\mu^0_{GS}(i) \neq \mu^0(i)$ for some $i \in I_0$. Then, as in the proof of step 1, we obtain that $\mu^0_{GS}(i) \succ_i^1 \mu^0(i)$. By the definition of the isolated preferences $\succ_i^1$, we have that $(\mu^0_{GS}(i), \mu^0_{GS}(i)) \succ_i (\mu^0(i), \mu^0(i))$. In addition, strong stability implies that if $\mu^0_{GS}(i) \neq \mu^1_{GS}(i)$ then $(\mu^0_{GS}(i), \mu^1_{GS}(i)) \succ_i (\mu^0_{GS}(i), \mu^0_{GS}(i))$; otherwise, μ_{GS} is not strongly stable. If $\mu^0(i) = \mu^1(i)$, then combining the previous 2 relations, one obtains $(\mu^0_{GS}(i), \mu^1_{GS}(i)) \succ_i (\mu^0(i), \mu^0(i))$. This contradicts with μ Pareto dominating μ_{GS}. Hence, $\mu^0(i) \neq \mu^1(i)$. This and strong stability of μ imply that $(\mu^0(i), \mu^1(i)) \succ_i (\mu^0(i), \mu^0(i))$. Since μ Pareto Dominates μ_{GS}, it must be that $(\mu^0(i), \mu^1(i)) \succ_i (\mu^0_{GS}(i), \mu^1_{GS}(i))$. Recall that $(\mu^0_{GS}(i), \mu^0_{GS}(i)) \succ_i (\mu^0(i), \mu^0(i))$ and $(\mu^0_{GS}(i), \mu^1_{GS}(i)) \succ_i (\mu^0_{GS}(i), \mu^0_{GS}(i))$. These relations and assumption 1, indeed (**??**) imply that $(\mu^1(i), \mu^1(i)) \succ_i (\mu^1_{GS}(i), \mu^1_{GS}(i))$.

By lemma 2, we know that μ^1 is statically stable under isolated preferences and μ^0. Now suppose we ran the Gale and Shapley algorithm at period 1 under isolated preferences and μ^0. Let us denote the resulting matching $\bar{\mu}^1$. From [12], we know that if $\bar{\mu}^1(i) \neq \mu^1(i)$, then $\bar{\mu}^1(i) \succ_i^2 (\mu^0)\mu^1(i)$. By the definition of $\succ_i^2 (\mu^0)$, $(\mu^0(i), \bar{\mu}^1(i)) \succ_i (\mu^0(i), \mu^1(i))$. Recall that $(\mu^0(i), \mu^1(i)) \succ_i (\mu^0(i), \mu^0(i))$ and $\mu^0(i) \neq \mu^1(i)$. These imply that $\bar{\mu}^1(i) \neq \mu^0(i)$. Then by assumption 1, $(\mu^0(i), \bar{\mu}^1(i)) \succ_i (\mu^0(i), \mu^1(i))$ implies $(\bar{\mu}^1(i), \bar{\mu}^1(i)) \succ_i (\mu^1(i), \mu^1(i))$.

Before proceeding any further let us sum up the preference relations for any $i \in I_0$ if μ Pareto dominates μ:

$$(\bar{\mu}^1(i), \bar{\mu}^1(i)) \succ_i (\mu^1(i), \mu^1(i)) \succ_i (\mu^1_{GS}(i), \mu^1_{GS}(i)) \succ_i (\mu^0_{GS}(i), \mu^0_{GS}(i)) \succ_i (\mu^0(i)), \mu^0(i)) \tag{1}$$

Next we will proceed to show that $\bar{\mu}^1$ is statically stable under isolated preferences and μ^0_{GS}. Let us postpone the proof momentarily to discuss its implications. From lemma 2, we know that μ^1_{GS} is a stable matching under isolated preferences and μ^0_{GS}. In addition, it must Pareto dominate $\bar{\mu}^1$ in terms of the isolated preferences, since $\bar{\mu}^1$ is statically stable and the μ^1_{GS} must Pareto dominate all stable matchings (see [12]). Hence, if $\mu^1_{GS}(i) \neq \bar{\mu}^1(i)$, then $\mu^1_{GS}(i) \succ^2_i (\mu^0_{GS})\bar{\mu}^1(i)$. By the definition of $\succ^2_i (\mu^0)$, $(\mu^0_{GS}(i), \mu^1_{GS}(i)) \succ_i (\mu^0_{GS}(i), \bar{\mu}^1(i))$. Recalling that $(\mu^0_{GS}(i), \mu^0_{GS}(i)) \succ_i (\mu^0(i), \mu^0(i))$, we find that $(\mu^0_{GS}(i), \bar{\mu}^1(i)) \succ_i (\mu^0(i), \bar{\mu}^1(i))$. Assumption 1 and $(\bar{\mu}^1(i), \bar{\mu}^1(i)) \succ_i (\mu^1(i), \mu^1(i))$ yield $(\mu^0(i), \bar{\mu}^1(i)) \succ_i (\mu^0(i), \mu^1(i))$. Accordingly, $(\mu^0_{GS}(i), \mu^1_{GS}(i)) \succ_i (\mu^0(i), \mu^1(i))$. However, recall that μ Pareto dominates μ_{GS}. This is the contradiction we are looking for. Thus, after showing that $\bar{\mu}^1$ is statically stable under isolated preferences and μ^0_{GS} the proof is complete.

We now proceed to show that $\bar{\mu}^1$ is indeed a stable matching under isolated preferences and μ^0_{GS}. We already know from Assumption 1 and (1) that, for all $i \in I_0$, $\bar{\mu}^1(i) \succ^2_i (\mu^0)\mu^1(i)$ if $\bar{\mu}^1(i) \neq \mu^1(i)$. Also, from [12], we know that, for all $i \in I_1$, $\bar{\mu}^1(i) \succ^1_i \mu^1(i)$ if $\bar{\mu}^1(i) \neq \mu^1(i)$. Recall that $\bar{\mu}^1$ is statically stable matching under isolated preferences and μ^0. Now consider the isolated preferences in period 1 from μ^0_{GS} and suppose, under these isolated preferences, $\bar{\mu}^1$ is not stable. Therefore, there must exist a school-child pair (s, i) such that both conditions are satisfied:

I. – if $i \in I_0$, then $s \succ^2_i (\mu^0_{GS})\bar{\mu}^1(i)$, or
   – if $i \in I_1$, then $s \succ^1_i \bar{\mu}^1(i)$;

II. $|\bar{\mu}^1(s)| < |r_s|$ or/and $i \rhd^1_s (\mu^0_{GS})j$ for some $j \in \bar{\mu}^1(s)$.

Because $\bar{\mu}^1$ statically stable under the isolated preferences and μ^0, the conditions 1 and 2 below cannot be satisfied at the same time.

1. (a) if $i \in I_0$, then $s \succ^2_i (\mu^0)\bar{\mu}^1(i)$, or
   (b) if $i \in I_1$, then $s \succ^1_i \bar{\mu}^1(i)$.

2. $|\bar{\mu}^1(s)| < r_s$ or/and $i \rhd^1_s (\mu^0)j$ for some $j \in \bar{\mu}^1(s)$.

Suppose $i \in I_0$. Then $s \succ^2_i (\mu^0_{GS})\bar{\mu}^1(i)$. We show that in this case condition $1\,(a)$ is satisfied. By the definition of $\succ^2_i (\mu^0_{GS})$, $(\mu^0_{GS}(i), s) \succ_i (\mu^0_{GS}(i), \bar{\mu}^1(i))$. If $\mu^0(i) = \mu^0_{GS}$, then $(\mu^0(i), s) \succ_i (\mu^0(i), \bar{\mu}^1(i))$. This means that condition 1a is satisfied. Let $\mu^0(i) \neq \mu^0_{GS}$. Then preference relations given in (1), assumption 1, $(\mu^0_{GS}(i), s) \succ_i (\mu^0_{GS}(i), \bar{\mu}^1(i))$ and the fact that $(s, s) \succ_i \left(\bar{\mu}^1(i), \bar{\mu}^1(i)\right)$ imply that $(\mu^0(i), s) \succ_i (\mu^0(i), \bar{\mu}^1(i))$. Hence, condition $1\,(a)$ is satisfied.

Suppose $i \in I_1$. Then $s \succ^1_i \bar{\mu}^1(i)$. Since $\succ^1$ does not depend on the last period's matching, condition $1\,(b)$ is satisfied. Therefore, we find that either $1\,(a)$ or $1\,(b)$ is satisfied. This means that

2 cannot be satisfied. Clearly, it must be that $|\bar{\mu}^1(s)| = r_s$. This implies that school s's priority ranking must satisfy $i \rhd_s^1 (\mu_{GS}^0) j$ and $j \rhd_s^1 (\mu^0) i$, for at least some $j \in \bar{\mu}^1(s)$. There are 2 cases consider:

1. $i \notin \mu_{GS}^0(s)$, or

2. $i \in \mu_{GS}^0(s)$ and $i \in I_0$.

If case (1.) happens, this implies that $j \notin \mu_{GS}^0(s)$; otherwise, j would have the highest priority at school s, hence, we reach a contradiction with $i \rhd_s^1 (\mu_{GS}^0) j$. Therefore, $j \notin \mu_{GS}^0(s)$. Since school s's priority ranking satisfies *IPA*, given that $i \rhd_s^1 (\mu_{GS}^0) j$ it must be that $j \in \mu^0(s)$ and $j \in I_0$ to have the required reversal of school s's priority ranking. Then $\mu_{GS}^0(j) \neq \mu^0(j)$. This, as argued earlier in step 1, implies that $(\mu_{GS}^0(j), \mu_{GS}^0(j)) \succ_j (\mu^0(j), \mu^0(j)) = (s, s)$, where the last equality comes from the fact above, that if $j \notin \mu_{GS}^0(s)$, it must be that $j \in \mu^0(s)$. Now recall that $j \in \bar{\mu}^1(s)$.

1. Therefore, $(\mu_{GS}^0(j), \mu_{GS}^0(j)) \succ_j (\mu^0(j), \bar{\mu}^1(j))$ which is a contradiction (see preference relation 1).

Suppose (2.) happens, $i \in \mu_{GS}^0(s)$, i.e., $s = \mu_{GS}^0(i)$. We know $s \succ_i^2 (\mu_{GS}^0) \bar{\mu}^1(i)$. These conditions yield $(\mu_{GS}^0(i), \mu_{GS}^0(i)) \succ_i (\mu_{GS}^0(i), \bar{\mu}^1(i))$. This is a contradiction which we are looking for.

This completes the proof of step 2.

Step 3. The Gale and Shapley algorithm yields a strongly stable matching that is not Pareto dominated by any other strongly stable matchings.

Proof of Step 3. Proving step 3 is just a matter of reiterating the arguments of steps 1 and 2 assuming previous periods' matchings are identical with the ones resulted from the Gale and Shapley deferred acceptance algorithm. □

References

[1] Abdulkadiroglu, Atila, Parag A. Pathak, and Alvin E. Roth, (2005a) "The New York City High School Match." *American Economic Review* 95, no. 2: 364–376.

[2] Abdulkadiroglu, Atila, Parag A. Pathak, Alvin E. Roth, and Tayfun Sönmez, (2005b) "The Boston Public School Match." *American Economic Review* 95, no. 2:368–371.

[3] Abdulkadiroglu, Atila, Parag A. Pathak, and Alvin E. Roth, (2009) "Strategy-proofness versus Efficiency in Matching with Indifferences: Redesigning the NYC High School Match," *American Economic Review*, 99, 5, pp1954-1978.

[4] Abdulkadiroglu, Atila and Tayfun Sönmez (1999) "House Allocation with Existing Tenants," *Journal of Economic Theory*, vol. 88, p 233-260.

[5] Abdulkadiroglu, Atila and Tayfun Sönmez, (2003) "School Choice: A Mechanism Design Approach," *American Economic Review* 93, no. 3, 729–747.

[6] Abdulkadiroglu, Atila, Joshua D. Angrist, Susan M. Dynarski, Thomas J. Kane, and Parag A. Pathak (2010) "Accountability and Flexibility in Public Schools: Evidence from Boston Charters and Pilots," mimeo, Harvard University. (September 2010 draft)

[7] Bauchmüller, Robert, Mette Gørtz, and Astrid Würtz (2011) "Long-run Benefits from Universal High-quality Pre-schooling," AKF working paper.

[8] Che, Y. and F. Kojima (2010) "Asymptotic Equivalence of Probabilistic Serial and Random Priority Mechanisms," *Econometrica*, Vol. 78, No. 5, 1625–1672.

[9] Cunha, F, Heckman, J. J, Lochner, L. J, and Masterov, D. V. (2006) "Interpreting the Evidence on Life Cycle Skill Formation," in *Handbook of the Economics of Education*, eds. Hanushek, E. A and Welch, F. (North-Holland, Amsterdam), pp. 697–812.

[10] Cunha, F, J. Heckman and S. M. Schennach (2010), "Estimating the Technology of Cognitive and Noncognitive Skill Formation," *Econometrica* 78, 883–931.

[11] Dubins, L. and Freedman (1981) "Machiavelli and the Gale-Shapley Algorithm," *The American Mathematical Monthly*, 88(7), p485-494.

[12] Gale, D. and Lloyd S. Shapley (1962) "College Admissions and the Stability of Marriage," *American Mathematical Monthly* 69, no. 1.

[13] Pathak, Parag (2011) "The Mechanism Design Approach to Student Assignment," *Annual Review of Economics*, Vol 3 (c).

[14] Roth, Alvin and Andrew Postlewaite (1977) "Weak versus Strong Domination in a Market with Invisible Goods." *Journal of Mathematical Economics*, August, 4(2), 131-137.

[15] Roth, Alvin (1982) "The Economics of Matching: Stability and Incentives," *Mathematics of Operations Research*, 7(4), p617-628.

[16] Roth, Alvin (2002) "The Economist as Engineer: Game Theory, Experimentation, and Computation as Tools for Design Economics," Fisher-Schultz Lecture, *Econometrica*, 70,4, July, 1341-1378.

[17] Roth, Alvin And Marilda A. O. Sotomayor (2002) "Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis," Cambridge University Press.

[18] Shapley, Lloyd, and Herbert Scarf (1974) "On Cores and Indivisibility," *Journal of Mathematical Economics*, 1(1), pp 23-37.