Treasure game*

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Abstract

We study a R&D race where the prize value is common knowledge, but the search costs are unknown ex ante. The race is modeled as a multistage game with observed previous actions. We provide a complete characterization of the efficient *symmetric Markov perfect equilibrium* in both single-player and multiple-player settings. There are two types of inefficiency in search for multiple players in comparison with a single player: a tragedy of the commons (for small races) and a free riding (for big races). We demonstrate that there is no monotonicity for 3 or more players: players can be better off if the race is longer even though such a race is more costly.
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1 Introduction

The R&D literature has grown up substantially in the recent years. It has three main directions.

(i) The classical papers, Loury (1979), Dasgupta and Stiglitz (1980a, b), Lee and Wilde (1980), assume that each firm in R&D competition makes once-and-for-all expenditure which determines the winner.

(ii) Reinganum (1981, 1982) considers a dynamic R&D race where each firm chooses a time path of expenditures. However, since the author uses the exponential success function the knowledge acquired in the past does not change the probability of

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the current success in the race. As a result, the equilibrium strategies are independent of time.[1]

(iii) Harris and Vickers (1985) analyze a race model where the winner is the first player to reach the finish line. Fershtman and Rubinstein (1997) consider an interactive model in which two players search for a single hidden treasure in one of a given set of labeled boxes. In both models the players know the upper bound of the costs: the distance in the first model and the number of boxes in the second model.

In this paper, we extend directions (ii) and (iii). In particular, we analyze a dynamic model where n players search for a treasure hidden somewhere on an island. The value of the treasure is common knowledge and searching is costly. Once the treasure is found the game ends. All players observe the current island size and make their search decisions simultaneously. If the treasure is not found in the current period, the unsearched island shrinks - its next-period size will be equal to the current island size minus the island part that has been searched by all players in the current period. The game we consider has Schelling's "conflict of partnership and competition" property: players are naturally competing against each other each period, but each player benefits from the other players' previous periods unsuccessful search, because it increases his chance to find the treasure in the current period.[2]

We assume that players are searching different parts of the island and only one player can obtain the treasure. If several players find the treasure simultaneously (search the same part of the island), each of them incurs costs but the treasure will be destroyed (players do not get any treasure). This assumption is standard in the R&D literature (see Chatterjee and Evans, 2004). It can be justified on the grounds that if several players discover the treasure simultaneously, they will be involved in a fierce competition afterwards and run out of any surplus. A good example of such a situation for just two players is 1960's Lockheed and Douglas jet development competition.[3] Many examples of simultaneous discoveries in science can be found in Merton (1973).

The game we analyze is stochastic where each state is described by its current island size. We restrict our attention to Markov strategies: individual search decisions depend only on the current island size. First, we describe a procedure how to construct a symmetric Markov perfect equilibrium (SMPE) for any $n \geq 1$. We present a Bellman equation for the problem and use the value iteration method to solve it. We find that each SMPE (for a fixed discount factor δ and a fixed number of players n) is a spline of degree one.[4] Our approach also describes the maximal number of search periods which is required to find the treasure. This number is equal to the number of spline pieces.

Since the monopolist $n = 1$ has the efficient search strategy, we can compare this strategy with the SMPE strategy for $n \geq 2$. Multi-player search is typically inefficient

[1] See Reinganum (1989) for more detail discussion about (i) and (ii).

[2] See Schelling (1971).

[3] For more detail see The Economist, 1985; and Chatterjee and Evans, 2004.

[4] A spline is a special function defined piecewise by polynomials, see for example Ahlberg, Nielson, and Walsh (1967).

except for very small islands when players behave as a cartel and search in just one period. In general there are three types of possible inefficiency. First, in the case of small islands, multiple players search too fast: the probability of finding a treasure is high which means players have incentives to overinvest in the current period. This is a standard *tragedy of the commons* effect. Second, in the case of large islands players search too slow: the probability of finding a treasure is low, so immediate payoff from investment is negative. Players want others to invest and incur current losses and hope that others will not find the treasure in the current period. Players have incentives to postpone their search for the future when it will be more profitable to search. This is a standard free riding effect. Finally, because search with multiple players is inefficient there are island sizes for which the monopolist finds searching profitable even though multiple players prefer not to search. It is interesting to note that when the size of unsearched island is exactly equal to the treasure value, the two mentioned above effects are absent and search with $n \geq 2$ players is efficient (Puzzle 2). This happens when the discount factor is low $\delta \leq 1/2$, which guarantees that players search the island in at most two periods. For this unique island size in the first period of search players get zero expected payoff and as a result leave the efficient island size for the second period of search.

Since in the SMPE all players make the same decisions simultaneously, they have equal probabilities to find the treasure in any period. Therefore, it looks natural to conjecture that a smaller island (smaller costs) is better than a bigger island for all players. It turns out that this conjecture is not correct. As we illustrate in the Example, players might be worse off with a smaller island size. This surprising observation, which we refer to as Puzzle 1, means that an increase in expected costs might make all the players better off. This puzzling behavior has the following intuitive explanation. If the island is small, the tragedy of the commons effect is strong and players oversearch the island. If the island size is increased, the tragedy of the commons effect decreases and players search the island more efficiently. It turns out that the improvement in efficiency when island size is increased could be higher than losses due to higher expected costs.

Our paper is related to the individual search literature; see Ross (1983) and Gittins (1989). However, in our model players are assumed to search strategically. Chatterjee and Evans (2004) analyze R&D model where two competing firms observe the other's past choices and search strategically. Their firms have to choose between two research projects. We have only one research project in their notation. Their model is complementary to our model. While agents in their model decide which area to search in (the size is predetermined), agents in our model decide how much area to search (the location has no importance).

Our model has the following main assumptions. First, $n \geq 1$ players (firms) are searching for the treasure. Second, the treasure (patent/vaccine/prize) has common value. Finally, the total search cost is unknown ex ante. There are many examples of this situation: detectives (police units) are looking for a criminal; journalists are looking for a movie star in the city hotels; researchers are looking for solutions of the six Millennium Prize Problems in mathematics. Another possible example is malaria.

Malaria is one of the most common infectious diseases and enormous public health problem which causes about 400 - 900 million cases of fever and approximately one to three million deaths annually - this represents at least one death every 30 seconds, see Breman (2001) and Greenwood, B., Bojang, K., Whitty., C., and G. Targett (2005). There is currently no vaccine that will prevent malaria (the search costs are unknown ex ante). Economic adviser Jeffrey Sachs estimates that malaria can be controlled for US3 billion in aid per year.[5] Therefore, the expected value of a malaria vaccine is at least US$3-12$ billion per year. Our results show that more firms (more research units) should (in expectations) find a vaccine faster. However, a search with more firms is less efficient (less profitable for the participants) and therefore firms might try to lobby for less "more efficient/productive" units.

The paper is organized as follows. The model is presented in Section 2. An illustrative example is described in details in Section 3. A general procedure of finding the SMPE is derived in Section 4. Properties of the SMPE are discussed in Section 5. Section 6 concludes.

2 The Model

There is an island of size $x(0) > 0$. There are $n \geq 1$ players who want to find a treasure which is hidden somewhere on the island. The value of the treasure is $R > 0$ for all players. The treasure has equal chances to be at any part of the island.[6]

In period $t = 0$ all players simultaneously choose how much to invest in search for the treasure. The search is costly. If player i searches $I^i(0)$ in period $t = 0$, his search cost is $-I^i(0)$. All players together search

$$I(0) = I^1(0) + ... + I^n(0)$$

in period $t = 0$. It is assumed similar to the Nash Demand game that if players search together more than the current island ($x(0) < I(0)$), the treasure is destroyed, all players incur their search costs (the payoff of player i is $-I^i(0)$), and the game ends.[7]

If $x(0) = I(0)$, player i has $\frac{I^i(0)}{x(0)}$ probability to find the treasure and the game ends after $t = 0$. Player i obtains the following expected payoff

$$\frac{I^i(0)}{x(0)} R - I^i(0).$$

[5]It has been argued that, in order to meet the Millennium Development Goals, money should be redirected from HIV/AIDS treatment to malaria prevention, which for the same amount of money would provide greater benefit to African economies.

[6]We focus our attention on uniform distribution because this is the most realistic situation when there is no information about the island.

[7]For simplicity it is also assumed that multiple players never invest at the same place. That could be rationalized by a similar assumption that investing at the same place is costly, but even if successful is not rewarded.

If $x(0) > I(0)$, player i has a $\frac{I^i(0)}{x(0)}$ probability to find the treasure and the game ends with probability $\frac{I(0)}{x(0)}$. If the treasure is not found in period $t = 0$ (this happens with probability $1 - \frac{I(0)}{x(0)}$), the island size shrinks to $x(1) = x(0) - I(0)$ and the game moves to the next period $t = 1$. The size of the island in the next period is equal to the size of the island in the previous period minus the searched part of the island.

In period $t = 1$, all players simultaneously choose how much to search for the treasure when the island is of size $x(1)$ and so on.

In general in period $t > 0$, each player knows the *history* $h(t) = (x(0); I(0), ..., I(t-1))$ and all players simultaneously choose how much to search for the treasure on the island of size $x(t)$.

If $I^1(t) + ... + I^n(t) = I(t) > x(t)$, the treasure is destroyed and player i gets payoff

$$-(I^i(0) + \delta I^i(1) + ... + \delta^t I^i(t)),$$

where δ is the common discount factor, and the game ends.[8]

If $I(t) = x(t)$, player i has $\frac{I^i(t)}{x(t)}$ probability to find the treasure. The expected payoff of player i is

$$\delta^t \frac{I^i(t)}{x(t)} R - (I^i(0) + \delta I^i(1) + ... + \delta^t I^i(t)),$$

and the game ends.

If $I(t) < x(t)$, player i has a $\frac{I^i(t)}{x(t)}$ probability to find the treasure and the game ends with probability $\frac{I(t)}{x(t)}$. If the treasure is not found in period t (this happens with probability $1 - \frac{I(t)}{x(t)}$), the island size shrinks to

$$x(t+1) = x(t) - I(t).$$

The new size of the island is equal to the previous island size minus the searched part.

We assume that each player can observe how much the other players have searched previously before making his searching plans for the next period. Note that all search costs are sunk, but only one player (if any) can find the treasure. Moreover, the value of the prize is known from the very beginning, but the search costs for each player will be determined only at the end of the game.

Player i's strategy is an infinite sequence of functions specifying how much to search in each period contingent upon any possible sequence of previous searches. Thus, the game we consider is stochastic and any history can be "summarized" by the "state" - the current island size. We will restrict our attention only to Markov strategies in which the past influences the current play only through its effect on the current island size. A pure Markov strategy for player i is a time-invariant map $I^i : X \to X$, where $X \in [0, x(0)]$. Therefore, the solution concept is a *symmetric*

[8]One possible motivation for a discount factor is that there is a $1 - \delta$ chance that the game terminates at the end of each period.

Markov perfect equilibrium (SMPE). Moreover, we will be looking for the SMPE with the highest total expected payoff. In SMPE, player i has to solve the following Bellman equation:

$$V(x) = \max_{0 \le I^i \le x - I^{-i}} -I^i + \frac{I^i}{x} R + \delta\Big(1 - \frac{I^1 + ... + I^n}{x}\Big) V(x - (I^1 + ... + I^n)), \qquad (1)$$

where $I^{-i} = I^1 + ... + I^{i-1} + I^{i+1} + ... + I^n$, $V(x)$ is a value function for each player (we use the symmetry assumption here). The first term in equation (1) describes player's searching cost in the current period. The second term is the player's expected value from finding the treasure in the current period. The last term is the player's expected value from the future periods. Denote the total value of this n-player game as

$$W_n(x) = nV(x).$$

In the next section we illustrate the derivation of the SMPE in the case of a specific example.

3 Example

Suppose that the value of the treasure is $R = 1$, the discount factor is $\delta = 0.25$, the number of players is $n = 3$ and the initial island size is $x(0) = 1.05$. In this section we answer the following questions: What is the SMPE? What is the maximal number of periods (the worst case scenario) when the players find the treasure for sure in the SMPE?

We use the value-iteration method to derive the SMPE. To make the exposition simple, as a start, let us assume that the players can search only once. Denote the value function of each player in this case by $V_1(x)$. How much should each player search in the SMPE? Note that player 1's expected value from the search if he is allowed to search only once is

$$V_1(x) = \max_{0 \le I^1 \le x - I^{-1}} -I^1 + R\frac{I^1}{x} = \max_{0 \le I^1 \le x - I^{-1}} \left(\frac{R}{x} - 1\right) I^1,$$

where $I^{-1} = I^2 + I^3$. It is evident that each player wants to search as much as possible if $R \ge x$ and does not want to search at all if $R < x$.[9] Since in the example $R = 1$ each player i in the SMPE searches

$$I^i = \begin{cases} x/3, & \text{if } x \le 1, \\ 0, & \text{if } x > 1. \end{cases}$$

[9] When $x = R$, the players are indifferent between searching and not searching. For simplicity we assume that whenever players are indifferent they choose to search.

Therefore,

$$V_1(x) = \begin{cases} \frac{1-x}{3}, & \text{if } x \le 1, \\ 0, & \text{if } x > 1. \end{cases} \quad (2)$$

Note that $3V_1(x) = W_3(x)$ and $V_1(x) = 0$ for $x > 1$.

Now suppose that players can search for at most two periods. Denote the value function of each player in this case by $V_2(x)$. How much is it optimal for each player to search in the first period and in the second period in the SMPE? Player 1's expected value from the search if he is allowed to search for at most two periods is

$$V_2(x) = \max_{0 \le I^1 \le x - I^{-1}} \left(\frac{I^1}{x} - I^1 \right) + \delta \left(1 - \frac{I^1 + I^{-1}}{x} \right) V_1(x - I^1 - I^{-1}),$$

where the first bracket is the expected value of finding the treasure in the first period and the second term is the expected value of finding the treasure in the second period. Note that if the treasure is not found in the first period, the island shrinks to $(x - I^1 - I^{-1})$ and player 1 obtains the expected value $V_1(x - I^1 - I^{-1})$ which is described in (2). Consequently if $x - I^1 - I^{-1} \le 1$,

$$V_2(x) = \max_{0 \le I^1 \le x - I^{-1}} \left(\frac{I^1}{x} - I^1 \right) + \delta \left(1 - \frac{I^1 + I^{-1}}{x} \right) \frac{1 - (x - I^1 - I^{-1})}{3}. \quad (3)$$

The optimal search in the first period I^1 satisfies the first order condition

$$\left(\frac{1}{x} - 1 \right) + \delta \left(-\frac{1}{x} \right) \frac{1 - x + I^1 + I^{-1}}{3} + \delta \left(1 - \frac{I^1 + I^{-1}}{x} \right) \frac{1}{3} = 0,$$

or

$$3(1 - x) - \delta \left(1 - x + I^1 + I^{-1} \right) + \delta \left(x - I^1 - I^{-1} \right) = 0.$$

In the symmetric equilibrium $I^{-1} = 2I^1$ and $3I^1 \le x$. Therefore,

$$I^1 = \left(\frac{3 - 2\delta}{6\delta} \right) (1 - x) + \frac{1}{6} = \frac{5}{3}(1 - x) + \frac{1}{6}, \quad (4)$$

$$I^1 + I^{-1} = 5(1 - x) + \frac{1}{2}$$

and

$$0 \le \left(x - I^1 - I^{-1} \right) \le 1,$$

or

$$\frac{11}{12} \le x \le \frac{13}{12}.$$

In addition to the above constraints there is a constraint on the value function to be positive, players always have an option of not searching. Let us substitute (4) into (3) to derive $V_2(x) = \left(-4(1 - x)^2 + \frac{1}{2}(1 - x) + \frac{1}{16} \right) \frac{1}{3x}$. The largest root of quadratic equation $V_2(x) = 0$ is equal to $x = \frac{15 + \sqrt{5}}{16} < \frac{13}{12}$. Since $\frac{15+\sqrt{5}}{16} \approx 1.08 < 1.05$, player i

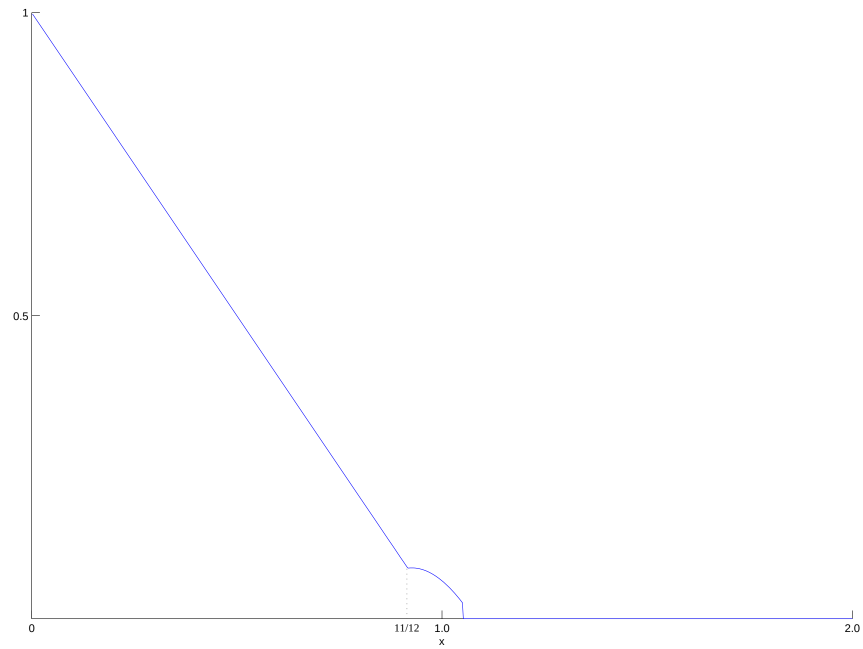

Figure 1: $W_3(x) = 3V_2(x)$.

searches

$$I^i = \begin{cases} x/3, & \text{if } x \leq \frac{11}{12}, \\ \frac{5}{3}(1-x) + \frac{1}{6}, & \text{if } \frac{11}{12} < x \leq 1.05. \end{cases} \tag{5}$$

The value function is

$$V_2(x) = \begin{cases} V_1(x) = (1-x)/3, & \text{if } x \leq \frac{11}{12}, \\ \left(-4(1-x)^2 + \frac{1}{2}(1-x) + \frac{1}{16}\right)\frac{1}{3x}, & \text{if } \frac{11}{12} < x \leq 1.05. \end{cases} \tag{6}$$

Point $x = \frac{11}{12}$ is a knot of the value function. The value function is described by two different functions on the left and on the right of the knot, but both these functions have the same value at the knot. Note that if $x = 1$, then $3I^1 = 1/2$ and $3V_2(1) = W_3(x) = \frac{1}{16}$. Figure 1 illustrates the total value in this case.

Now suppose that the players can search for at most three periods. Denote the value function of each player in this case by $V_3(x)$. How much is optimal for each player to search in the first, second, and third periods in the SMPE?

Player 1's expected value from the search if he is allowed to search for at most three periods is

$$V_3(x) = \max_{0 \leq I^1 \leq x - I^{-1}} \left(\frac{I^1}{x} - I^1\right) + \delta\left(1 - \frac{I^1 + I^{-1}}{x}\right) V_2(x - I^1 - I^{-1}),$$

where the first bracket is the expected value of finding the treasure in the first period and the second term is the expected value of finding the treasure after the first period. Note that if the treasure is not found in the first period, the unsearched island shrinks to $(x - I^1 - I^{-1})$ and the player obtains the expected value of $V_2(x - I^1 - I^{-1})$ which is described in (6).

Suppose that in the equilibrium, $\frac{11}{12} < x - I^1 - I^{-1} \leq 1.05$. Then using (6) we get

$$V_3(x) = \max_{0 \leq I^1 \leq x - I^{-1}} \frac{I^1}{x} - I^1 + \frac{\delta}{48x}\left(-64\left(1 - (x - I^1 - I^{-1})\right)^2 + 8\left(1 - (x - I^1 - I^{-1})\right) + 1\right). \tag{7}$$

The optimal search in the first period I^1 satisfies the first order condition

$$\left(\frac{1}{x} - 1\right) + \frac{\delta}{6x}\left(-16\left(1 - (x - I^1 - I^{-1})\right) + 1\right) = 0.$$

In the symmetric equilibrium $I^{-1} = 2I^1$ and $3I^1 \leq x$. Therefore,

$$24(1 - x) - 16\left(1 - (x - 3I^1)\right) + 1 = 0,$$

or

$$I^1 = \frac{1}{6}(1 - x) + \frac{1}{48}. \tag{8}$$

The three-period search takes place only if

$$V_3(x) > V_2(x).$$

Substituting optimal I^1 from (8) into the value function (7) gives

$$V_3(x) = -\frac{1}{768x}\left(448x^2 - 880x + 427\right).$$

Remember that

$$V_2(x) = -\frac{1}{48x}\left(64x^2 - 120x + 55\right).$$

One can show that $V_2(x) - V_3(x) > 0$ for $\frac{11}{12} < x \leq 1.05$, which means 3-period search is not optimal. Consequently,

$$I^i = \begin{cases} x/3, & \text{if } x \leq \frac{11}{12}, \\ \frac{5}{3}(1 - x) + \frac{1}{6}, & \text{if } \frac{11}{12} < x \leq 1.05; \end{cases} \tag{9}$$

and the value function is

$$V(x) = \begin{cases} V_1(x) = (1 - x)/3, & \text{if } x \leq \frac{11}{12}, \\ V_2(x) = \left(-4(1 - x)^2 + \frac{1}{2}(1 - x) + \frac{1}{16}\right)\frac{1}{3x}, & \text{if } \frac{11}{12} < x \leq 1.05. \end{cases} \tag{10}$$

It is always optimal to search for at most (in the worst case scenario) two periods in this example even if players' search is not restricted. The two-period search procedure that has been described is the SMPE.

This example illustrates the approach which will be applied in the general model. Using this example we present two puzzles which will be discussed in more details in Section 5.

Puzzle 1. Non-monotonicity If $n = 3$ a larger island could make all players better off.

It seems natural to expect that a larger island should make all players worse off. Consider two island sizes

$$\widetilde{x} = 0.9167 < 0.9168 = \widehat{x}.$$

Using (10) we get

$$V(0.9167) \approx 0.027779 < 0.027782 \approx V(0.9168). \tag{11}$$

It means the original intuition was incorrect and a larger island can make all players better off. This puzzling result could be explained by players' less efficient behavior when the island size is $\widetilde{x}$. Players search the island $\widetilde{x}$ too fast, in other words the tragedy of the commons effect is strong when $x = \widetilde{x}$. If the size of the island is increased, players search the island slower (more efficiently) - tragedy of the commons effect is not as strong as before. This example shows that the improvement in efficiency when island size is increased could be higher than losses due to higher expected costs.

Puzzle 2. Special island size If $x = R = 1$, then $nV(1) = W_n(1) = W_1(1) = \frac{1}{16}$ for any n.

It seems obvious that the monopolist searches efficiently. Therefore, if the island is small (enough), players search efficiently in just one period. Yet, if the island size increased, the tragedy of the commons effect leads to multiple players searching too fast. It turns out that there is a unique island size $x = R$ when players search efficiently, because when $x = R$ there is no tragedy of the common effect and there is no free riding effect. Note that from expression (10) we get

$$V(1) = \frac{1}{48} \text{ for } n = 3.$$

If one derives the value function when $n = 1$, one can show that $V(1) = \frac{1}{48}$ for $n = 1$ as well.

4 Analysis of the Model: SPME

Define the part of the island which player i does not search in the current period by

$$y := x - I_i$$

and the part of the island neither player searches in the current period (the remaining part of the island) by

$$z := x - (I_1 + ... + I_n) = x - (I_i + I_{-i}).$$

Equation (1) can be rewritten in the following way

$$V(x) = \max_{I_{-i} \leq y \leq x} \{-(x-y) + R(x-y)/x + \delta z V(z)/x\}. \tag{12}$$

Note, x, y, z, R and $V(x)$ are of the same unit measure. For convenience, we make the following substitution

$$x := x/R, \;\; y := y/R, \;\; z := z/R, \;\; V := V/R \tag{13}$$

to work with unit free variables. Equation (12) transforms into

$$V(x) = \max_{I_{-i} \leq y \leq x} \{-(x-y) + (x-y)/x + \delta z V(z)/x\}. \tag{14}$$

Let us derive player's value of the game $V(x)$. To simplify the exposition it is convenient to introduce the following function

$$\Psi(x) := xV(x). \tag{15}$$

From definition (15) it follows that

$$\Psi(x) \geq 0 \;\text{ for any x.} \tag{16}$$

Note also that in the symmetric equilibrium, $I_1 = ... = I_n = I$ and $I_{-i} = (n-1)I$. Equation (14) in terms of $\Psi(x)$ can be rewritten as

$$\Psi(x) = \max_{(n-1)I \leq y \leq x} \{(1-x)(x-y) + \delta\Psi(z)\} =: B\Psi(z). \tag{17}$$

The following Lemma follows from the contraction mapping theorem, see for example Stokey, Lucas and Prescott (1989).

Lemma 1. *If $\delta < 1$, the operator on the right hand side of equation (17) is a contraction operator. Therefore, equation (17) has a unique solution Ψ, that can be obtained as the limit of the following sequence $\{\Psi_k\}$, where*

$$\Psi_0 \equiv 0, \qquad \Psi_k := B\Psi_{k-1} \quad k = 1, 2, ... \tag{18}$$

4.1 Construction of sequences $\{\Psi_k\}$ and $\{V_k\}$

Note that with the help of Lemma 1 one can construct sequence $\{\Psi_k\}$. This procedure is called the value-iteration procedure. It is equivalent to using the backward induction argument and was already demonstrated in the Example.

4.1.1 Construction of Ψ_1 and V_1

Let us start from the end of the search process. What will be the value of the game, if players could only search at most one period? Equation (17) transforms into

$$\Psi_1(x) = \max_{(n-1)I \le y \le x} \{(1-x)(x-y)\}. \tag{19}$$

It is evident that the optimal y can be described in the following way[10]

$$y = \begin{cases} x, & \text{if} \quad x > 1, \\ (n-1)I, & \text{if} \quad x \le 1. \end{cases} \tag{20}$$

If $x < 1$, then in the SMPE players search the whole island, $I_1 + ... + I_n = nI = x$. Consequently,

$$y = \frac{(n-1)x}{n} \quad \text{and} \quad z = 0 \quad \text{if} \quad x < 1. \tag{21}$$

Therefore, (19) can be rewritten in the following way

$$\Psi_1(x) = \begin{cases} P_1(x), & \text{if} \quad x \le u_1 = 1, \\ 0, & \text{if} \quad x > u_1 = 1, \end{cases} \tag{22}$$

where $u_1 > 0$ is the largest positive root of polynomial $P_1(x) = x(1-x)/n$. It is evident that $u_1 = 1$. For future references note that

$$P_1(x) = \frac{a_1}{n}(1-x)^2 + \frac{b_1}{n}(1-x) + \frac{c_1}{n}, \tag{23}$$

where

$$a_1 = -1, \quad b_1 = 1, \quad c_1 = 0.$$

If the players can search the island in at most one period, then the SMPE is $(I(x), ..., I(x))$, where

$$I(x) = \begin{cases} \frac{x}{n}, & \text{if} \quad x \le u_1, \\ 0, & \text{if} \quad x > u_1. \end{cases} \tag{24}$$

Note that the optimal first-period search is independent from the discount factor because there is no delay.

Define the value of the game for each player (if the players can search the island in at most k periods) as $V_k(x) := \Psi_k(x)/x$, for any $x \ge 0$. The value of the game V can be obtained as the limit of the sequence $\{V_k\}$. From the above definition it follows

$$V_1(x) = \begin{cases} (1-x)/n, & \text{if} \quad x \le u_1, \\ 0, & \text{if} \quad x > u_1. \end{cases} \tag{25}$$

[10] Note that if $x = 1$, then *any* $y \in [(n-1)I, x]$ is optimal. We assume that players choose $y = (n-1)I$ in this case.

4.1.2 Construction of Ψ_2 and V_2

What will be the value of the game, if players can search the whole island in at most two periods? In general there could be three possibilities depending on the island size. The first possibility is that the players search the whole island in just one period. Intuitively this happens for small values of x because it is too costly to wait for another period when the island is very small. The second possibility is that the players finish the island in two periods. This happens for middle values of x. Finally, players can find searching to be too costly and abstain from searching at all. This happens when the initial island is too big (costs are very large).

We have already considered the first possibility in the previous subsection. Now we formally analyze the situation when players search for exactly two periods in the worst case. The first step is to construct $\Psi_2(x)$. Equation (17) in this case transforms into

$$\Psi_2(x) = \max_{(n-1)I \leq y \leq x} \{(1-x)(x-y) + \delta\Psi_1(z)\}. \tag{26}$$

Necessary conditions for y to be the optimal value in the interior of $[0, x]$ is

$$-(1-x) + \delta\Psi_1'(z) = 0 \tag{27}$$

and

$$0 < z \leq u_1. \tag{28}$$

The sufficient condition for y to be the optimal value in the interior of $[0, x]$ is satisfied because

$$\Psi_1''(z) = a_1 < 0. \tag{29}$$

The way to proceed is to use condition (27) in order to to construct an equilibrium and then show that the derived equilibrium satisfies condition (28).

From condition (27) and expression (22), it follows

$$-(1-x) + \delta\left(\frac{1-2z}{n}\right) = 0.$$

Consequently,

$$z(x) = \frac{n(x-1)+\delta}{2\delta}. \tag{30}$$

It is straightforward now that

$$y = \frac{x(n-1)+z(x)}{n} = \frac{2\delta(n-1)x + n(x-1)+\delta}{2\delta n}. \tag{31}$$

Plugging (31) into equation (26) and using (16) one obtains a *spline* of degree two on the interval $[0, u_2]$

$$\Psi_2(x) = \begin{cases} \Psi_1(x), & \text{if} \quad 0 \leq x \leq t_1, \\ P_2(x), & \text{if} \quad t_1 < x \leq u_2, \\ 0, & \text{if} \quad x > u_2, \end{cases} \tag{32}$$

where $u_2 > 0$ is the largest positive root of polynomial

$$P_2(x) = \frac{a_2}{n}(1-x)^2 + \frac{b_2}{n}(1-x) + \frac{c_2}{n}, \tag{33}$$

with

$$a_2 = -1 - s, \quad b_2 = \frac{1}{2}, \quad c_2 = \frac{\delta}{4},$$

and

$$s = \frac{n(n-2)}{4\delta}. \tag{34}$$

To find u_2 one needs to solve quadratic equation $P_2(u_2) = 0$. It is easy to check that

$$u_2 = 1 + \frac{\sqrt{4\delta(s+1)+1} - 1}{4(s+1)}.$$

Point $x = t_1$ is the first knot of the spline. The knot t_1 is an initial island size such that players are indifferent between searching the island in two periods or in one period:

$$\Psi_1(t_1) = \Psi_2(t_1). \tag{35}$$

From (23) and (33) one finds[11]

$$t_1 = 1 - \frac{\delta}{n}. \tag{36}$$

Note that all our calculations so far are valid for any $n \geq 1$. Consider parameter s now. From expression (34) it follows

$$s \begin{cases} < 0, & \text{if} \quad n = 1, \\ = 0, & \text{if} \quad n = 2, \\ > 0, & \text{if} \quad n \geq 3. \end{cases} \tag{37}$$

Condition (37) characterizes three different types of behavior in SMPE. There are three cases: $n = 1$ (a monopolist); $n = 2$ (two players); and $n \geq 3$ (many players).

It is straightforward to check that condition (28) holds for any initial island size $x \in [t_1, u_2]$ in the expression (32). Therefore, if the players can search the island in at most two periods, then $y(x)$ is a spline of degree one on the interval $[0, u_2]$ with one knot $x = t_1$

$$y(x) = \begin{cases} \frac{(n-1)x}{n}, & \text{if} \qquad x \leq t_1, \\ \frac{2\delta(n-1)x + n(x-1) + \delta}{2\delta n}, & \text{if} \quad t_1 < x \leq u_2, \\ x, & \text{if} \qquad x > u_2. \end{cases} \tag{38}$$

The SMPE (if the players can search the island in at most 2 periods) is also a spline

[11] Note that condition $t_1 \leq u_1 = 1$ must hold.

of degree one on the interval $[0, u_2]$ with one knot $x = t_1$

$$I(x) = \begin{cases} \frac{x}{n}, & \text{if} \quad x \le t_1, \\ \frac{(2\delta - n)x + n - \delta}{2\delta n}, & \text{if} \quad t_1 < x \le u_2, \\ 0, & \text{if} \quad x > u_2, \end{cases} \tag{39}$$

and the value function is

$$V_2(x) = \begin{cases} V_1(x), & \text{if} \quad x \le t_1, \\ P_2(x)/x, & \text{if} \quad t_1 < x \le u_2, \\ 0, & \text{if} \quad x > u_2. \end{cases} \tag{40}$$

4.1.3 Construction of Ψ_k and V_k

What will be the value of the game, if players can search the whole island in at most $k \ge 3$ periods? In general there could be $k + 1$ possibilities depending on the island size. The players can search the island in $1, 2, ..., k$ periods or don't search at all.

First, let us construct $\Psi_k(x)$. Equation (17) in this case transforms into

$$\Psi_k(x) = \max_{(n-1)I \le y \le x} \{(1 - x)(x - y) + \delta \Psi_{k-1}(z)\} =: B\Psi_{k-1}(z). \tag{41}$$

A necessary condition for y to be the optimal value in the interior of $[0, x]$ is

$$(1 - x) = \delta \Psi'_{k-1}(z). \tag{42}$$

and

$$t_{k-2} < z \le u_{k-1}, \tag{43}$$

where $t_0 := 0$. The sufficient condition for y to be the optimal value in the interior of $[0, x]$ is satisfied if

$$\Psi''_{k-1}(z) < 0. \tag{44}$$

As before the way to proceed is to use condition (42) to construct the equilibrium and then show that it satisfies conditions (43) and (44).

Note that if function $\Psi_{k-1}(x)$ in (41) is a quadratic polynomial, $\Psi_k(x) = B\Psi_{k-1}(x)$ has to be a quadratic polynomial as well. Since from (23) and (33), $P_1(x)$ and $P_2(x)$ are quadratic polynomials, any $P_k(x)$ can be represented in the following form:

$$P_k(x) = \frac{a_k}{n}(1 - x)^2 + \frac{b_k}{n}(1 - x) + \frac{c_k}{n}, \quad k \ge 1. \tag{45}$$

From condition (42) and expression (45), it follows

$$z(x) = 1 + \frac{\delta b_{k-1} + (1 - x)n}{2\delta a_{k-1}}. \tag{46}$$

It is straightforward now that

$$y = x + \frac{z - x}{n} = x + \frac{(1 - x)(n + 2\delta a_{k-1})}{2\delta n a_{k-1}} + \frac{b_{k-1}}{2na_{k-1}}. \tag{47}$$

Hence,

$$\Psi_k(x) = -(1 - x)\left(\frac{(1 - x)(n + 2a_{k-1}\delta)}{2na_{k-1}\delta} + \frac{b_{k-1}}{2na_{k-1}}\right) + \delta\Psi_{k-1}(z). \tag{48}$$

Define the largest root of polynomial $P_k(x)$ as u_k and an initial island size such that players are indifferent between searching the island in k periods or in $k-1$ periods as knot t_{k-1},

$$\Psi_{k-1}(t_{k-1}) = \Psi_k(t_{k-1}). \tag{49}$$

Plugging (47) into equation (41) and using (16), one obtains a spline of degree two on the interval $[0, u_k]$ with knots $t_1, ..., t_{k-1}$

$$\Psi_k(x) = \begin{cases} \Psi_{k-1}(x), & \text{if} \quad 0 \le x \le t_{k-1}, \\ P_k(x), & \text{if} \quad t_{k-1} < x \le u_k, \\ 0, & \text{if} \quad x > u_k, \end{cases} \tag{50}$$

where $P_k(x)$ is defined in (45). Our description will be complete if conditions (43) and (44) hold. The following lemma proves condition (43).

Lemma 2. *Solution (50) satisfies condition (43) for any $n \ge 1$ and any initial island size x.*

The proof of condition (44) will be presented at the end of the next subsection. Its exposition is simpler with the help of some of the preliminary results.

4.2 Preliminary results

Let us find a_k, b_k, and c_k for any $k \ge 2$ now. Using (41), (45) and (46) one can get the following result.

Theorem 1.

$$a_k = -1 + \frac{s}{a_{k-1}}, \quad b_k = -\frac{b_{k-1}}{2a_{k-1}}, \quad c_k = \delta\left(c_{k-1} - \frac{b_{k-1}^2}{4a_{k-1}}\right), \quad k \ge 2, \tag{51}$$

where s is defined in (34) and

$$a_1 = -1, \quad b_1 = 1, \quad c_1 = 0. \tag{52}$$

Theorem 1 describes all coefficients of the quadratic polynomials $P_k(x)$. Note that the dynamics of coefficients a_k depends on parameter s, which for different values of n can be negative, zero and positive. Let us consider separately the following three cases: $n = 1$ (a monopolist); $n = 2$ (two players); and $n \ge 3$ (many players).

4.2.1 n = 1

Let us start with the monopolist case. The following results characterize the splines in (50) and their knots when $n = 1$.

Proposition 1. *When $n = 1$ system (51) with intial conditions (52) has the following solution*

$$a_k = -\frac{\sin(k+1)\varphi}{2v\sin k\varphi}, \quad b_k = \frac{v^{k-1}\sin\varphi}{\sin k\varphi}, \quad c_k = \frac{v^{2k-1}\sin(k-1)\varphi}{2\sin k\varphi}, \tag{53}$$

$$t_k = 1 - v^k\cos k\varphi, \quad u_k = 1 + \frac{v^k(\sin k\varphi - \sin\varphi)}{\sin(k+1)\varphi}, \quad k \geq 1, \tag{54}$$

where $v = \sqrt{\delta}$ and $\varphi = \arccos v$.

Proof. See the Appendix.

4.2.2 n = 2

Next let us consider $n = 2$ case.

Proposition 2. *When $n = 2$ system (51) with intial conditions (52) has the following solution*

$$a_k = -1, \quad b_k = \frac{1}{2^{k-1}}, \quad c_k = \left(\frac{(4\delta)^{k-1} - 1}{4^{k-1}(4\delta - 1)}\right)\delta, \tag{55}$$

$$t_k = 1 - \frac{3\delta + (4\delta)^k(\delta - 1)}{2^k(4\delta - 1)}, \quad u_k = 1 + \frac{1}{2^k}\left(\sqrt{\frac{1 - (4\delta)^k}{1 - 4\delta}} - 1\right), \quad k \geq 1. \tag{56}$$

Proof. See the Appendix.

4.2.3 n ≥ 3

Finally, let us consider $n \geq 3$ case.

Proposition 3. *When $n \geq 3$ system (51) with intial conditions (52) has the following solution*

$$a_k = \frac{\left(\frac{\sqrt{1+4s}-1}{2}\right)^{k+1} - \left(\frac{-\sqrt{1+4s}-1}{2}\right)^{k+1}}{\left(\frac{\sqrt{1+4s}-1}{2}\right)^{k} - \left(\frac{-\sqrt{1+4s}-1}{2}\right)^{k}}, \quad b_k = \frac{(-\frac{1}{2})^{k-1}\sqrt{1+4s}}{\left(\frac{\sqrt{1+4s}-1}{2}\right)^{k} - \left(\frac{-\sqrt{1+4s}-1}{2}\right)^{k}}, \tag{57}$$

$$c_k = \sum_{i=2}^{k} \frac{-(1+4s)\delta^k}{(4\delta)^{i-1}\left[\left(\frac{\sqrt{1+4s}-1}{2}\right)^{i} - \left(\frac{-\sqrt{1+4s}-1}{2}\right)^{i}\right]\left[\left(\frac{\sqrt{1+4s}-1}{2}\right)^{i-1} - \left(\frac{-\sqrt{1+4s}-1}{2}\right)^{i-1}\right]},$$

$$t_k = 1 + \frac{b_{k+1} - b_k - \sqrt{(b_{k+1} - b_k)^2 - 4(a_{k+1} - a_k)(c_{k+1} - c_k)}}{2(a_{k+1} - a_k)}, \tag{58}$$

$$u_k = 1 + \frac{b_k - \sqrt{b_k^2 - 4a_k c_k}}{2a_k}. \tag{59}$$

Proof. See the Appendix.

Lemma 3. *Solution (50) satisfies condition (44) for any* $n \geq 1$ *and any initial island size* x*.*

4.3 SMPE

The SMPE can be described now. If players can search the island in at most k periods, then $y(x)$ is a spline of degree one on the interval $[0, u_k]$ with knots $t_1, ..., t_{k-1}$

$$y(x) = \begin{cases} \frac{(n-1)x}{n}, & \text{if} \quad x \leq t_1, \\ x + \frac{(1-x)(n+2\delta a_1)}{2\delta n a_1} + \frac{b_1}{2na_1}, & \text{if} \quad t_1 < x \leq t_2, \\ \vdots & \\ x + \frac{(1-x)(n+2\delta a_{k-2})}{2\delta n a_{k-2}} + \frac{b_{k-2}}{2na_{k-2}}, & \text{if} \quad t_{k-2} \leq x \leq t_{k-1}, \\ x + \frac{(1-x)(n+2\delta a_{k-1})}{2\delta n a_{k-1}} + \frac{b_{k-1}}{2na_{k-1}}, & \text{if} \quad t_{k-1} < x \leq u_k, \\ x, & \text{if} \quad x > u_k. \end{cases} \tag{60}$$

The SMPE (if the players can search the island in at most k periods) is also a spline of degree one on the interval $[0, u_k]$ with knots $t_1, ..., t_{k-1}$

$$I(x) = \begin{cases} \frac{x}{n}, & \text{if} \quad x \leq t_1, \\ -\frac{(1-x)(n+2\delta a_1)}{2\delta n a_1} - \frac{b_1}{2na_1}, & \text{if} \quad t_1 < x \leq t_2, \\ \vdots & \\ -\frac{(1-x)(n+2\delta a_{k-2})}{2\delta n a_{k-2}} - \frac{b_{k-2}}{2na_{k-2}}, & \text{if} \quad t_{k-2} \leq x \leq t_{k-1}, \\ -\frac{(1-x)(n+2\delta a_{k-1})}{2\delta n a_{k-1}} - \frac{b_{k-1}}{2na_{k-1}}, & \text{if} \quad t_{k-1} < x \leq u_k, \\ 0, & \text{if} \quad x > u_k, \end{cases} \tag{61}$$

and the value function (if the players can search the island in at most k periods) is

$$V_k(x) = \begin{cases} P_1(x)/x, & \text{if} \quad x \leq t_1, \\ P_2(x)/x, & \text{if} \quad t_1 < x \leq t_2, \\ \vdots & \\ P_{k-1}(x)/x, & \text{if} \quad t_{k-2} \leq x \leq t_{k-1}, \\ P_k(x)/x, & \text{if} \quad t_{k-1} < x \leq u_k, \\ 0, & \text{if} \quad x > u_k, \end{cases} \tag{62}$$

or

$$V_k(x) = \begin{cases} V_{k-1}(x), & \text{if} \quad 0 \leq x \leq t_{k-1}, \\ P_k(x)/x, & \text{if} \quad t_{k-1} < x \leq u_k, \\ 0, & \text{if} \quad x > u_k. \end{cases} \tag{63}$$

5 Properties of SMPE

The SMPE was described in the previous section. Let us discuss its properties now.

5.1 Maximum number of periods

Let us fix the number of players n and ask the following question: what is the minimum number k that $V(x) \equiv V_k(x)$. In other words, what is the maximum number of periods (the worst case scenario) in which the treasure will be found for sure. The answer in general is expected to depend on δ.

Note that from (22) and (36) it follows that $u_1 > t_1$. It means that there exists an island size x such that, in the worst case scenario, the treasure is found in *at least* two periods. This observation is true for any $n \geq 1$.

One way to answer the above question is to write the condition that the largest positive root of quadratic polynomial $P_k(x)$ coincides with the largest positive root of quadratic polynomial $P_{k+1}(x)$. It means that $\Psi_k(x) \equiv \Psi_{k+1}(x)$ or $V(x) \equiv V_k(x)$. Since such k depends on δ, let us define for each n a knot discount factor $\delta_k(n)$ which is the solution to the following equation

$$u_k(\delta_k(n)) = u_{k+1}(\delta_k(n)), \quad k \geq 2. \tag{64}$$

The knot discount factor $\delta_k(n)$ "connects" two regions: $V(x) \equiv V_k(x) \equiv V_{k+1}(x)$ for $0 < \delta < \delta_k(n)$, and $V(x) \not\equiv V_k(x)$ for $\delta_k(n) < \delta < 1$. The following theorem characterizes the knot discount factors for $n = 1$ and $n = 2$.

Theorem 2. *If $n = 1$, equation (64) has the following unique solution*

$$\delta_k(1) = \cos^2 \frac{\pi}{k+1}, \qquad k \geq 2. \tag{65}$$

If $n = 2$, equation (64) can be simplified to

$$(1 - \delta_k(2))^2 (1 - (4\delta_k(2))^k) = 1 - 4\delta_k(2), \qquad k \geq 2. \tag{66}$$

Proof. See the Appendix.

Figure 2 illustrates $\delta_k(n)$ for $n = 1$ and $n = 2$. One can see that there is a monotonic convergence of $\delta_k \to 1$ when $k \to \infty$. This convergence is easy to prove by taking a limit $k \to \infty$ and applying it to (65) and (66). One possible interpretation of this result is that when $n = 1$ and $n = 2$ players always search the island for a *finite* number of periods or not at all.

5.2 Tragedy of the Commons or Free Riding?

Let us look again at equation (17):

$$\Psi(x) = \max_{(n-1)I \leq y \leq x} \{(1 - x)(x - y) + \delta \Psi(z)\} =: B\Psi(z).$$

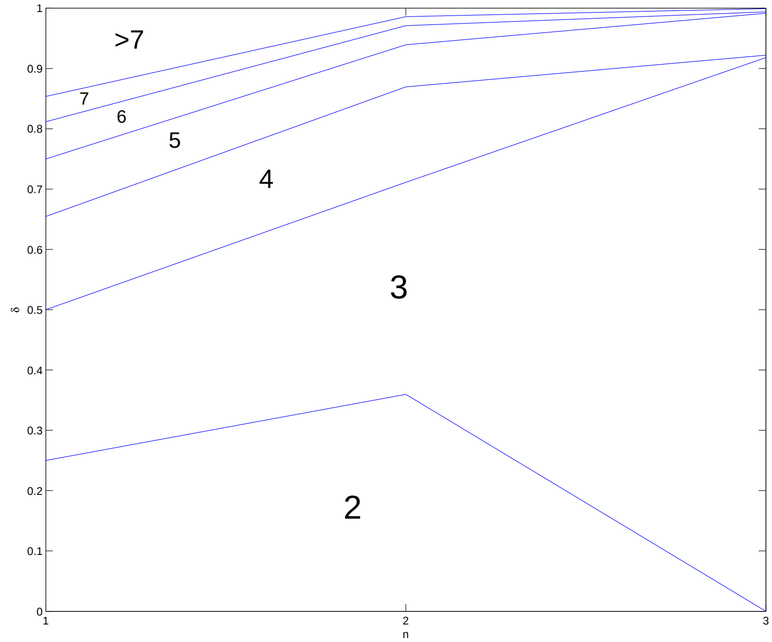

Figure 2: Different zones

Note that $I_i = (x - y)$ is the current search of player i. There are two main effects in the model: tragedy of the commons and free riding. The first effect works if the island is small, $0 < x < 1$. In this case the first term in equation (17) is positive and each player wants to take advange of the situation and as the result players search too fast which is a standard tragedy of the commons. The second effect is present for large islands, namely when $x > 1$. In this case the first term in equation (17) is negative and players want to decrease their current losses and as a result players search too slow, in other words there is a free riding effect. If the island size is $x = 1$, then none of the effects is present and that explains Puzzle 2. Figure 5 illustrates these two effects: tragedy of the commons (for $x < 1$) and free riding (for $x > 1$) for small discount factors. At the first interval, the aggregate search is the same for $n = 1$, $n = 2$, and $n = 3$: players search the whole island in just one period. At this part of the graph all three curves coincide. At the second interval curves for $n = 2$ and $n = 3$ are above the curve for $n = 1$. It means that the monopolist, $n = 1$, searches the island in one or two periods, but in the case of multiple players, $n = 2$ or $n = 3$, tragedy of the commons takes place and players search the whole island too fast in just one period. This is the region where $x < 1$. Finally, we can see that the curve for $n = 1$ is above the curves for $n = 2$ and $n = 3$ for $x > 1$. Free riding takes place on this interval and multiple players search the island too slow relative to the monopolist.

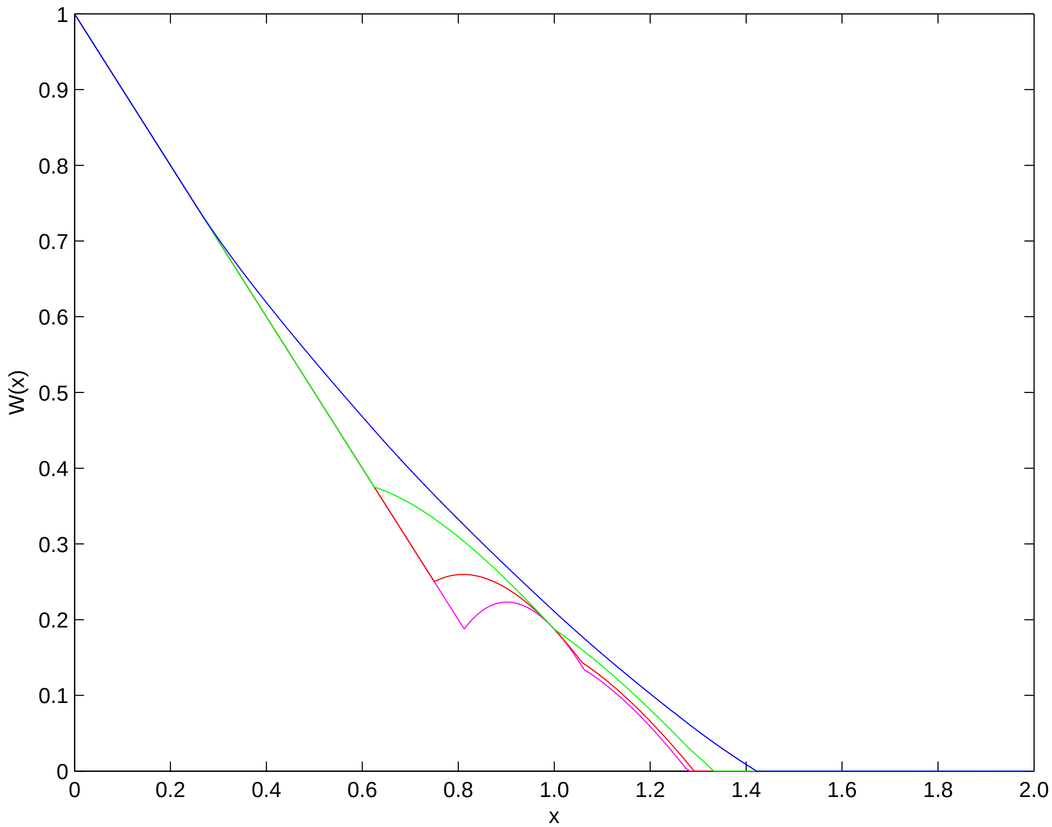

Figure 3: $W_n(x)$: $n = 1$ is a blue line, $n = 2$ is a green line, $n = 3$ is a red line; $\delta = 0.75$.

5.3 Puzzle 1.

Now let us look at puzzle 1 which was introduced in the Example. It seems intuitive that if the treasure is hidden on a smaller island, all players are expected to be better off in the SPME. One could expect that costs go down (smaller island), but the probability to find the treasure does not change in the SPME. However, as it was shown in the Example, this intuition does not take into account the fact that players search too fast (tragedy of the commons) a smaller island. It turns out that sometimes this tragedy of the commons effect can be so strong that all players are better off searching bigger island. The following proposition demonstrates that this result holds whenever there are at least $n = 3$ players.

Proposition 4. *For any $n \geq 3$, there exists $x > t_1$ such that $V(x) > V(t_1)$.*

Proof. See the Appendix.

5.4 Puzzle 2.

It is intuitive that aggregate search is independent from the number of players when the island size is small: players search the whole island in just one period. In this sense players search efficiently. It is also intuitive that the total value of the game does not increase with the number of players n. However, the efficient behavior of players is not limited only to small sizes of the island. When island size $x = 1$ and $\delta \leq 0.5$, players also search efficiently. Formally,

Proposition 5. *For any $0 < \delta \leq 0.5$, $W_n(1) = W_1(1)$ for any $n \geq 2$.*

Figure 4 illustrates Puzzle 2. It shows the total value function when one, two, three, or four players search the island and $\delta = 0.5$. The total value function at $x = 1$ is the same in all four cases.

If the discount factor is slightly greater than 0.5, the monopolist will have an incentive to search slower, while in the multi-player cases players will still search as fast as before.

Proposition 6. *For any $0 < \delta \leq 0.75$, $W_n(1) = W_2(1)$ for any $n \geq 3$.*

Figure 3 illustrates Proposition 6. It shows that the value function of the monopolist at $x = 1$ is greater than the total value for two, three, or four players at $x = 1$. The total values are the same at $x = 1$ for $n = 2, 3, 4$.

6 Conclusion

In this paper, we introduce a new dynamic search model and develop new methods in order to analyze it. Our symmetry assumption makes the analysis simpler and transparent. We construct the SMPE for any number of players and demonstrate that search is typically inefficient if $n \geq 2$.

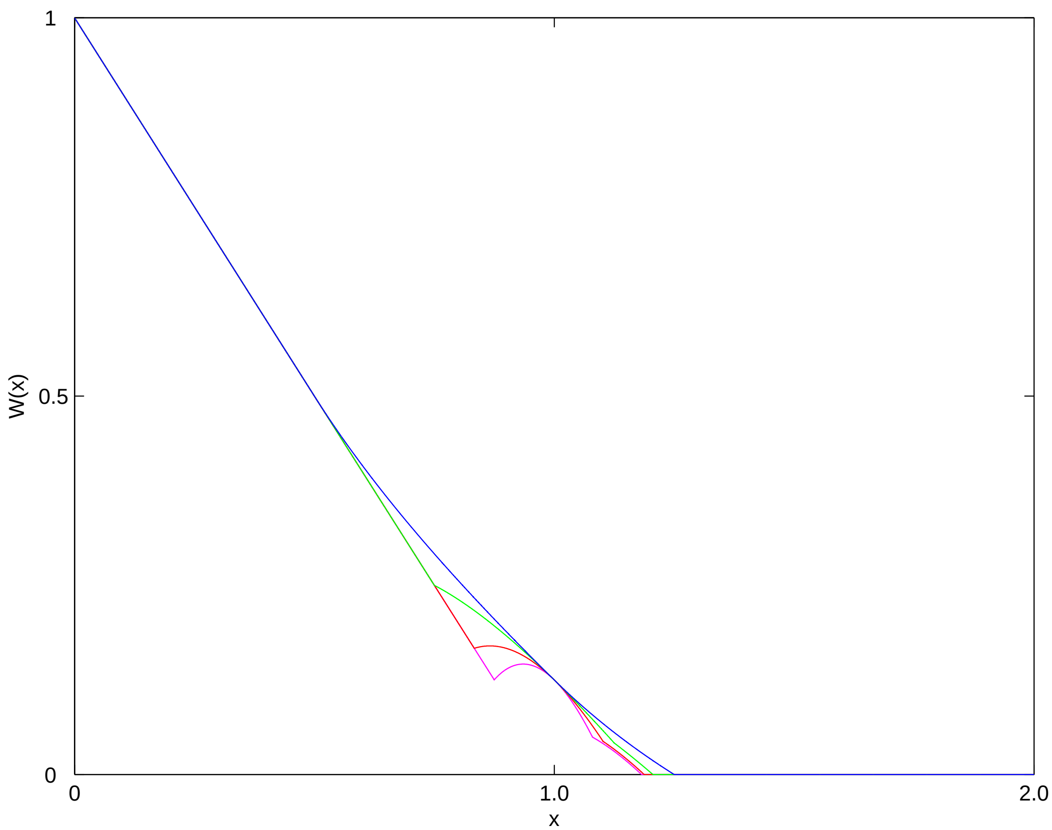

Figure 4: $W_1(x)$ is a blue line, $W_2(x)$ is a green line, $n = 3$ is a red line; $\delta = 0.5$.

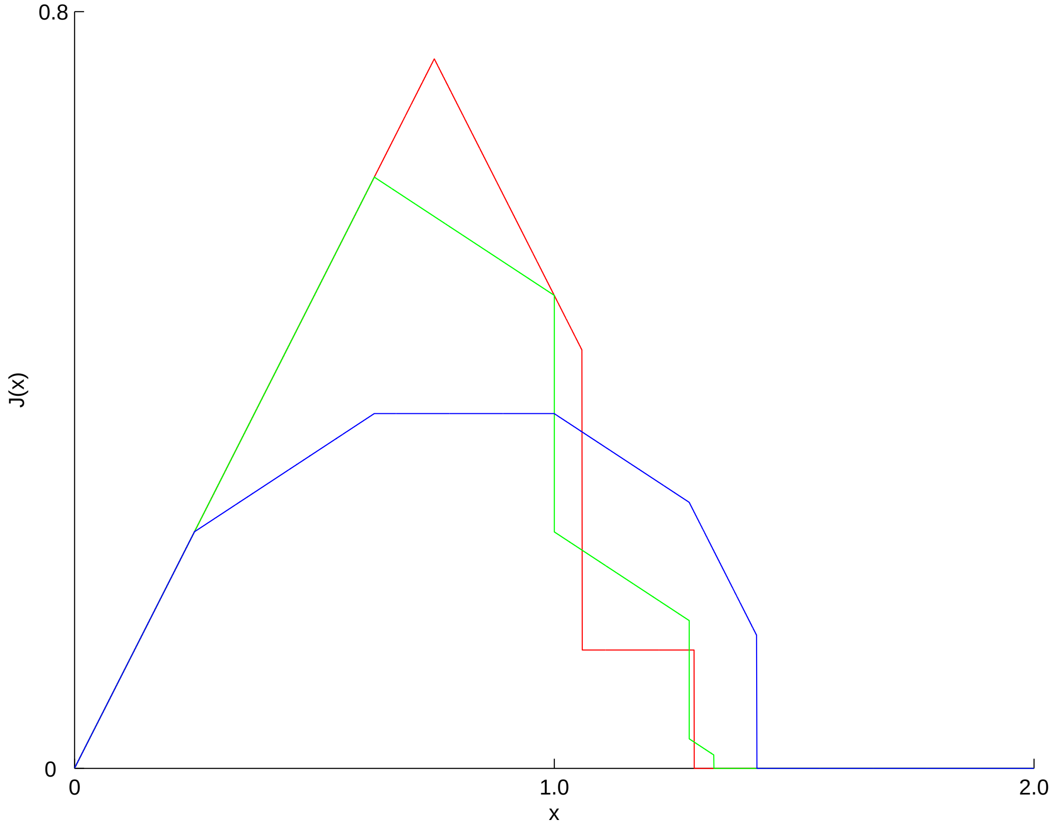

Figure 5: $J_n(x)$: $n = 1$ is a blue line, $n = 2$ is a green line, $n = 3$ is a red line; $\delta = 0.75$.

There are several natural extensions of our project. The first one is to allow players to have a positive externality on each other. For example, all players can benefit from the treasure in some way. The question is how this will affect the SMPE and our results.

It will be interesting to test our predictions in the experimental laboratory. For example, check whether subjects' behavior is consistent with the SMPE and either Puzzles 1 and 2 hold in experiments.

Appendix

Proof of Lemma 2

Firstly, let us show that $z \leq u_{k-1}$. Let us prove by contradiction assuming that $z > u_{k-1}$. Refer to equation (41) which could be seen below

$$\Psi_k(x) = \max_{(n-1)I \leq y \leq x} \{(1-x)(x-y) + \delta\Psi_{k-1}(z)\}.$$

Given $x \geq y \geq z > u_{k-1} \geq \ldots \geq u_2 \geq u_1 = 1$ it follows that the first term on the right $(1-x)(x-y)$ has to be non-positive. If $z > u_{k-1}$ then the second term on the right $\delta\Psi_{k-1}(z)$ is negative. That means the whole expression on the right of equation (41) has to be negative. Obviously that could not be an optimal choice for a player because by choosing $y = x$, i.e. making no investment, a player can get the value of zero. Consequently, there is a contradiction and condition $z \le u_{k-1}$ is proved.

Now let us show that $t_{k-2} < z$. Again let us prove by contradiction assuming that $z \le t_{k-2}$. Note that by construction when $z \le t_{k-2}$ the following condition holds $P_{k-1}(z) \le P_{k-2}(z)$. That implies that instead of using the original k-period path (invest $x - y$ in the first period and make further $k-1$ investments according to $P_{k-1}(z)$) one could use $k-1$ period path (invest the same amount $x-y$ in the first period and make further $k-2$ investments according to $P_{k-2}(z)$) and increase the value. Refer to equation (41), both paths have the same first term while the second term is larger for the $k-1$ period path. That implies the k period path does not improve the value in comparison with the optimal $k-1$ period path, which means whenever $z \le t_{k-2}$ the k period path is not optimal. Condition $t_{k-2} < z$ is proved. □

Proof of Proposition 1

Derivation of a_k, b_k and c_k

Let us show that when $n = 1$ formula (53) describes the solution to the system of difference equations (51).

Define

$$R_k := v^k \cdot \prod_{j=1}^{k} a_j \qquad k = 1, 2, \ldots \tag{67}$$

Using (51) one gets the following second-order difference equation

$$R_{k+1} = vR_k \cdot \left(-1 - \frac{1}{4\delta a_k}\right) = -vR_k - \frac{1}{4}R_{k-1} \quad k \ge 2. \tag{68}$$

The initial conditions are $R_0 = 1$ and $R_1 := -v$. The characteristic equation $4z^2 + 4vz + 1 = 0$ has two complex roots

$$z_1 = \frac{-v + ir}{2}, \quad z_2 = \frac{-v - ir}{2}, \qquad r := \sqrt{1 - v^2} > 0. \tag{69}$$

Denote $\varphi := \{\arg z_1 \in [0, \pi/2]\} = \arccos v$, then $z_{1,2} = -\frac{e^{\pm i\varphi}}{2}$. Further, write solutions to equation (68) in form $R_k = Az_1^{k+1} - Bz_2^{k+1}$ and use initial conditions to get $A = B = -\frac{i}{\sin\varphi}$. Consequently

$$R_k = -\frac{i}{(-2)^{k+1}\sin\varphi}(e^{i(k+1)\varphi} - e^{-i(k+1)\varphi}) = -\frac{\sin[(k+1)\varphi]}{(-2)^k \sin\varphi}. \tag{70}$$

Apply (67) and (51) to get

$$a_k = \frac{R_k}{vR_{k-1}} = -\frac{\sin{(k+1)\varphi}}{2v\sin k\varphi}, \tag{71}$$

$$b_k = -\frac{b_{k-1}}{2a_{k-1}} = \frac{v^{k-1}\sin\varphi}{\sin k\varphi}, \tag{72}$$

$$c_k = \delta\left[c_{k-1} - \frac{b_{k-1}^2}{4a_{k-1}}\right] = \frac{v^{2k-1}\sin{(k-1)\varphi}}{2\sin k\varphi}. \tag{73}$$

Derivation of t_k

To find t_k one needs to solve quadratic equation $P_k(t_k) = P_{k+1}(t_k)$, namely

$$(a_{k+1} - a_k)(1-t_k)^2 + (b_{k+1} - b_k)(1-t_k) + c_{k+1} - c_k = 0, \quad k \geq 1. \tag{74}$$

Substitute a_k from (71) to derive

$$a_{k+1} - a_k = \frac{\sin{(k+1)\varphi}}{2v\sin k\varphi} - \frac{\sin{(k+2)\varphi}}{2v\sin{(k+1)\varphi}} = \frac{\sin^2{(k+1)\varphi} - \sin k\varphi\sin{(k+2)\varphi}}{2v\sin k\varphi\sin{(k+1)\varphi}} = \frac{\sin^2\varphi}{2v\sin k\varphi\sin{(k+1)\varphi}}. \tag{75}$$

Substitute b_k from (72) and note that $v = \cos\varphi$ to derive

$$b_{k+1} - b_k = \frac{v^k\sin\varphi}{\sin{(k+1)\varphi}} - \frac{v^{k-1}\sin\varphi}{\sin k\varphi} = \frac{2v^k\sin\varphi(\cos\varphi\sin k\varphi - \sin{(k+1)\varphi})}{2v\sin k\varphi\sin{(k+1)\varphi}} = \frac{-2v^k\sin^2\varphi\cos k\varphi}{2v\sin k\varphi\sin{(k+1)\varphi}}. \tag{76}$$

Substitute c_k from (73) and note that $v = \cos\varphi$ to derive

$$c_{k+1} - c_k = \frac{v^{2k+1}\sin k\varphi}{2\sin{(k+1)\varphi}} - \frac{v^{2k-1}\sin{(k-1)\varphi}}{2\sin k\varphi} = \frac{v^{2k}(\cos^2\varphi\sin^2 k\varphi - \sin{(k+1)\varphi}\sin{(k-1)\varphi})}{2v\sin k\varphi\sin{(k+1)\varphi}} = \frac{v^{2k}\sin^2\varphi\cos^2 k\varphi}{2v\sin k\varphi\sin{(k+1)\varphi}}. \tag{77}$$

Substitute the above relationships into (74) and cancel non-zero common term $\frac{\sin^2\varphi}{2v\sin k\varphi\sin{(k+1)\varphi}}$ to derive

$$(1-t_k)^2 - 2v^k\cos k\varphi(1-t_k) + v^{2k}\cos^2 k\varphi = (1 - t_k - v^k\cos k\varphi)^2 = 0. \tag{78}$$

Consequently,

$$t_k = 1 - v^k\cos k\varphi. \tag{79}$$

Derivation of u_k

To find u_k one needs to solve quadratic equation $P_k(u_k) = 0$, namely

$$a_k(1-u_k)^2 + b_k(1-u_k) + c_k = 0, \quad k \geq 1. \tag{80}$$

Substitute (71), (72) and (73) into (80) to get

$$-(1-u_k)^2\sin{(k+1)\varphi} + 2(1-u_k)v^k\sin\varphi + v^{2k}\sin{(k-1)\varphi}. \tag{81}$$

Solving this quadratic equation results in

$$u_k = 1 + \frac{v^k(\sin k\varphi - \sin\varphi)}{\sin(k+1)\varphi}. \tag{82}$$

This concludes the proof. □

Proof of Proposition 2

Derivation of a_k, b_k and c_k

Let us show that when $n = 2$ formula (55) describes the solution to the system of difference equations (51). It is straightforward to derive $a_k = -1$ and $b_k = \frac{1}{2^{k-1}}$. The expression for c_k in (51) can be simplified to

$$c_k = \delta(c_{k-1} + 1/4^{k-1}). \tag{83}$$

Introduce new variable $e_k = c_k 4^k$. Equation (83) transforms to

$$e_k = 4\delta(e_{k-1} + 1), \tag{84}$$

where $e_1 = 0$. The solution to this linear difference equation is $e_k = \frac{4\delta-(4\delta)^k}{1-4\delta}$. Substitute $c_k = e_k/4^k$ to derive

$$c_k = \frac{4\delta - (4\delta)^k}{(1-4\delta)4^k}. \tag{85}$$

Derivation of t_k

To find t_k one needs to solve quadratic equation $P_k(t_k) = P_{k+1}(t_k)$, namely

$$a_k(1-t_k)^2 + b_k(1-t_k) + c_k = a_{k+1}(1-t_k)^2 + b_{k+1}(1-t_k) + c_{k+1}, \quad k \geq 1.$$

From equation (55) $a_k = a_{k+1} = -1$; consequently,

$$t_k = 1 + \frac{c_{k+1} - c_k}{b_{k+1} - b_k}. \tag{86}$$

Substitute b_k and c_k from equation (55) to derive the following indifference points

$$t_k = 1 - \frac{3\delta + (4\delta)^k(\delta - 1)}{2^k(4\delta - 1)}. \tag{87}$$

Derivation of u_k

To find u_k one needs to solve quadratic equation $P_k(u_k) = 0$, namely

$$a_k(1-u_k)^2 + b_k(1-u_k) + c_k = 0, \quad k \geq 1.$$

Substituting $a_k = -1$ from equation (55) and solving the above quadratic equation gives

$$u_k = 1 + \frac{\sqrt{b_k^2 + 4c_k} - b_k}{2}. \tag{88}$$

Note that with the help of (55) one can simplify

$$b_k^2 + 4c_k = \frac{(4\delta)^k - 1}{4^{k-1}(4\delta - 1)}. \tag{89}$$

Substitute equation (89) into equation (88) to get

$$u_k = 1 + \frac{\sqrt{\frac{1-(4\delta)^k}{1-4\delta}} - 1}{2^k}. \tag{90}$$

This concludes the proof. □

Proof of Proposition 3

Derivation of a_k, b_k and c_k

Let us show that when $n \geq 3$ the solution to the system of difference equations (51) is described by (57).

Define

$$R_k := \prod_{j=1}^{k} a_j \qquad k = 1, 2, \ldots \tag{91}$$

Using (51) one gets the following second-order difference equation

$$R_k + R_{k-1} - sR_{k-2} = 0 \quad k \geq 2. \tag{92}$$

The initial conditions are $R_1 = -1$ and $R_2 := 1 + s$. The characteristic equation $z^2 + z - s = 0$ has two real roots

$$z_1 = \frac{\sqrt{1+4s} - 1}{2}, \quad z_2 = \frac{-\sqrt{1+4s} - 1}{2}. \tag{93}$$

The solutions to equation (92) have to be in form $R_k = Az_1^{k+1} - Bz_2^{k+1}$. Applying initial conditions gives $A = B = \frac{1}{\sqrt{1+4s}}$. Next from $a_k = \frac{R_k}{R_{k-1}}$ one can derive a_k in (57).

Note that from (51) it follows that $a_{k-1} = -\frac{b_{k-1}}{2b_k}$. On the other hand, $a_{k-1} = \frac{R_{k-1}}{R_{k-2}}$. Consequently, b_k is inversely proportional to R_{k-1}. Find b_k in (57) by substitution.

c_k has to satisfy the initial condition $c_1 = 0$. Introducing $d_k = \frac{c_k}{\delta^k}$ and rewriting the difference equation gives

$$d_k = d_{k-1} + \frac{b_k b_{k-1}}{2\delta^{k-1}}.$$

Substituting the initial condition $d_1 = 0$ gives

$$d_k = d_1 + \sum_{i=2}^{k}(d_i - d_{i-1}) = \sum_{i=2}^{k} \frac{b_i b_{i-1}}{2\delta^{i-1}}.$$

Finally it follows that

$$c_k = \delta^k d_k = \delta^k \sum_{i=2}^{k} \frac{b_i b_{i-1}}{2\delta^{i-1}}.$$

Substitute b_k from (57) to derive c_k in (57).

Derivation of t_k and u_k

To find t_k one needs to solve quadratic equation $P_k(t_k) = P_{k+1}(t_k)$, namely

$$a_k(1-t_k)^2 + b_k(1-t_k) + c_k = a_{k+1}(1-t_k)^2 + b_{k+1}(1-t_k) + c_{k+1}, \quad k \geq 1.$$

The players' indifference points are

$$t_k = 1 + \frac{b_{k+1} - b_k - \sqrt{(b_{k+1}-b_k)^2 - 4(a_{k+1}-a_k)(c_{k+1}-c_k)}}{2(a_{k+1}-a_k)}. \tag{94}$$

To find u_k one needs to solve quadratic equation $P_k(u_k) = 0$, namely

$$a_k(1-u_k)^2 + b_k(1-u_k) + c_k = 0, \quad k \geq 1.$$

$\Psi_k(x)$ is strictly positive for any $t_{k-1} < x < u_k$ and zero for any $x \geq u_k$, where

$$u_k = 1 + \frac{b_k - \sqrt{b_k^2 - 4a_k c_k}}{2a_k}. \tag{95}$$

This concludes the proof. □

Proof of Lemma 3

Let us show that $\Psi''_{k-1}(z) < 0$. From equation (50) it is clear that the sufficient condition for $\Psi''_{k-1}(z) < 0$ is that $P''_{i-1}(z) < 0 \; \forall \; i = 2, \ldots, k-1$. From equation (45) it is easy to see that the above condition is equivalent to $a_{i-1} < 0 \; \forall \; i = 2, \ldots, k-1$. For $n \geq 2$ it is straightforward that $s \geq 0$. From (51) one can see that a_k is a sum of two negative numbers, consequently it has to be negative.

Now let us prove this condition for $n = 1$. Substitute t_k from (54) into (45) to get

$$P_k(t_k) = \frac{v^{2k-1}}{2n \sin k\varphi}(-\sin{(k+1)\varphi} \cos^2 k\varphi + 2\sin\varphi \cos k\varphi + \sin{(k-1)\varphi}). \tag{96}$$

Substitute the following equality

$$\sin{(k+1)\varphi} = \sin{(k-1)\varphi} + 2\sin{\varphi}\cos{k\varphi} \tag{97}$$

into (96) to derive

$$P_k(t_k) = \frac{-\delta^k a_k \sin^2 k\varphi}{n}. \tag{98}$$

Wherever the value function at t_{k-1} is positive, a_{k-1} has to be negative. □

Proof of Theorem 2

Firstly let us prove the $n = 1$ case. Equation (64) (which defines δ_k) is equivalent to condition $R_k = 0$ (R_k is defined in (67)). Apply (70) to get

$$\varphi_k \quad : \qquad (k+1)\varphi = n\pi, \tag{99}$$

where $n \geq 1$ is some integer which can be different for different values of k, i.e. $n = n_k$. Let us prove by induction that $n_k = 1 \ \forall \ k$. It is easy to see that for $k = 2$ the statement is correct, i.e. $\varphi_2 = \pi/3$ and $\delta_2(1) = \cos^2 \pi/3 = 1/4$. Substitute $k' := k+1$ in (99) to get (using the assumption of induction that $n_k = 1$ and the fact that $\varphi_k = \arccos\sqrt{\delta_k}$ is monotonically decreasing in k)

$$n_{k+1}\pi = (k+1)\varphi_{k+1} \leq (k+1)\varphi_k = \frac{k+1}{k}\pi < 2\pi.$$

Given that n_{k+1} is an integer, it must be that $n_{k+1} = 1$. Substitute $n = 1$ in (99) to get $\varphi_k = \pi/(k+1)$, which means $\delta_k(1) = \cos^2 \frac{\pi}{k+1}, \ k \geq 2$.

Now let us prove the $n = 2$ case. Substitute t_k from equation (56) and u_k from equation (90) into equation (64) to get

$$1 + \frac{\sqrt{\frac{1-(4\delta)^k}{1-4\delta}} - 1}{2^k} = 1 - \frac{3\delta + (4\delta)^k(\delta - 1)}{2^k(4\delta - 1)}. \tag{100}$$

Simplify the above expression to

$$\sqrt{\frac{1-(4\delta)^k}{1-4\delta}} - 1 = \frac{3\delta + (4\delta)^k(\delta-1)}{(4\delta-1)} \tag{101}$$

Further simplifications give

$$\sqrt{\frac{(4\delta)^k - 1}{4\delta - 1}} = \frac{((4\delta)^k - 1)(1-\delta)}{(4\delta - 1)} \tag{102}$$

and

$$\sqrt{\frac{(4\delta)^k - 1}{4\delta - 1}}(1-\delta) = 1. \tag{103}$$

Take squares of both sides of equation (103) to derive equation (66). This concludes the proof. □

Proof of Proposition 4

When $x = t_1$ players are indifferent between searching the island in two periods or in one period:

$$V_1(t_1) = V_2(t_1).$$

Let us show that $V_2'(t_1) > 0$ for $n \geq 3$, which means that there exists an island size x which is "slightly" bigger than island size t_1, i.e. $x > t_1$, such that $V(x) = V_2(x) > V(t_1)$:

$$V_2'(t_1) = \left(\frac{P_2(x)}{x}\right)'_{x=t_1} > 0. \tag{104}$$

Applying (33) and (36) transforms inequality (104) to

$$V_2'(t_1) = a_2 - \frac{a_2 + b_2 + c_2}{(1-\delta/n)^2} > 0. \tag{105}$$

With the help of (33) and (34), inequality (105) simplifies to

$$n^2(2n - 6 - 2\delta) + \delta(10n - 4\delta) > 0. \tag{106}$$

When $n \geq 4$ the above inequality always holds because both terms on the left are positive. When $n = 3$ inequity (106) simplifies to

$$\delta(12 - 4\delta) > 0,$$

which is also valid. The proposition is proved. □

Proof of Proposition 5

Let us prove this proposition in two steps. Firstly, let us show that when $0 < \delta \leq 0.5$, knot $t_2 \geq 1$ for any $n \geq 1$, which implies $V(1) = V_2(1)$ for any n. Applying (54) when $n = 1$ gives $t_2 = 1 - v^2 \cos 2\varphi = 1 - \delta(2\cos^2\varphi - 1) = 1 + \delta(1 - 2\delta)$. It is easy to see that $t_2 \geq 1$ when $0 < \delta \leq 0.5$. Applying (56) when $n = 2$ gives $t_2 = 1 - \frac{3\delta + 16\delta^2(\delta-1)}{4(4\delta-1)} = 1 + \frac{3}{4}\delta - \delta^2$. It is easy to see that $t_2 \geq 1$ when $0 < \delta \leq \frac{3}{4}$. Applying (51) when $n \geq 3$ one can derive $a_2 = -1 - s,\ b_2 = \frac{1}{2},\ c_2 = \frac{\delta}{4},\ a_3 = -\frac{1+2s}{1+s},\ b_3 = \frac{1}{4(1+s)},\ c_3 = \frac{\delta^2}{4} + \frac{\delta}{16(1+s)}$. When $n \geq 3$ it is easy to see that $a_3 - a_2 = 1 + s - \frac{1+2s}{1+s} = \frac{s^2}{1+s} > 0$, $b_3 - b_2 = \frac{1}{4(1+s)} - \frac{1}{2} < 0$ and finally $c_3 - c_2 = \frac{\delta}{16(1+s)} - \frac{\delta}{4}(1-\delta) = \frac{\delta(4\delta^2 - 3\delta - (1-\delta)n(n-2))}{4(4\delta + n(n-2))} < 0$ when $0 < \delta \leq \frac{3}{4}$. Applying (58) one can see that $t_2 > 1$ when $0 < \delta \leq \frac{3}{4}$.

Secondly, one needs to show that $nV_2(1)$ is the same for any n. Applying (45) gives $nV_2(1) = c_2 = \frac{\delta}{4}$. The proposition is proved. □

Proof of Proposition 6

Proof of this proposition follows directly from the proof of proposition 5. □

References

[1] H., Ahlberg, E., Nielson, and J. Walsh (1967): The Theory of Splines and Their Applications. New York: Academic Press.

[2] Breman, J. (2001): "The ears of the hippopotamus: manifestations, determinants, and estimates of the malaria burden," *Am J Trop Med Hyg*, 64, 1-11.

[3] Kalayan Chatterjee, and Robert Evans (2004): "Rivals Search for Buried Treasure: Competition and Duplication in R&D," *Rand Journal of Economics,* 35, 1, 160-183.

[4] Dasgupta and Stiglitz (1980a): "Industrial Structure and the Nature of Innovative Activity," *The Economic Journal*, 90, 266-293.

[5] Dasgupta and Stiglitz (1980b): "Uncertainty, Industrial Structure and the Speed of R and D," *Bell Journal of Economics*, 11, 1-28.

[6] The Economist (1985): "Aircraft Industry: The Big Six: A Survey," June 1, 8.

[7] Greenwood, B., Bojang, K., Whitty., C., and G. Targett (2005): "Malaria," *Lancet*, 365, 1487 - 1498.

[8] Gene Grossman, and Carl Shapiro (1986): "Optimal dynamic P&D programs," *Rand Journal of Economics,* 17, 4, 581-593.

[9] Chaim Fershtman and Ariel Rubinstein (1997): "A Simple Model of Equilibrium in Search Procedures," *Journal of Economic Theory,* 72, 432-441.

[10] J. Gittins (1989): *Multi-armed Bandit Allocation Indices*. New York: Wiley.

[11] Harris and Vickers (1985) "Perfect equilibrium in a model of a race," *Review of Economics Studies,* 52:193-209.

[12] Lee and Wilde (1980) "Market structure and innovation: A reformulation," *Quarterly Journal of Economics,* 94: 429-436.

[13] Loury (1979) "Market structure and innovation," *Quarterly Journal of Economics,* 93:395-410.

[14] R. Merton (1973): "Singletons and Multiples in Scientific Discovery." In R. Merton and N. Storer, eds. *The Sociology of Science.* Chicago: University of Chicago Press.

[15] D. Neher (1999): "Staged Financing: An Agency Perspective," *Review of Economic Studies*, Vol. 66, no. 2, 255-74.

[16] Reinganum (1981): "Dynamic Games of Innovation," *Journal of Economic Theory,* 25, 21-41.

[17] Reinganum (1982): "A Dynamic Game of R and D: Patent Protection and Competitive Behavior," *Econometrica,* vol 50, no. 3, 671-688.

[18] Reinganum (1989): "The Timing of Innovation: Research, Development, and Diffusion," *Handbook of Industrial Organization*, Vol. 1, 849-908.

[19] S. Ross (1983): *Introduction to Stochastic Dynamic Programming*. New York: Academic Press.

[20] Sachs, J. and P., Malaney (2002): "The economic and social burden of malaria," *Nature* 415, 680 - 685.

[21] Schelling (1971): "Dynamic Models of Segregation," *Journal of Mathematical Sociology* 1, 143-186.

[22] N. Stokey, R. Lucas and E. Prescott (1989) *Recursive Methods in Economic Dynamics*. Harvard University Press.