A STRATEGIC THEORY OF MARKETS

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This paper extends the strategic foundations of the Walrasian price mechanism using a game-theoretic model of double auctions in the interdependent value environment among buyers and sellers with multidimensional signals and multiple units of demand and supply. If players' signals are independently and nonidentically distributed across players conditional on the state with the monotone likelihood ratio property and values are interdependent with common value and private value elements, then, every perfect equilibrium price in uniform price double auctions with a discrete set of bids is a consistent and asymptotically normal estimator of the unknown value.

Keywords: Information Aggregation, Double Auction, Rational Expectations, Asymptotic Price Distribution.

1. INTRODUCTION

Informational efficiency of the price system is one of the central premises of modern economics. For example, the efficient market hypothesis claims that security prices in financial markets reflect all available information. But there are circumstances such as bubbles and crashes that prices in the market do not seem to aggregate information and these price fluctuations have large impacts on the aggregate economy. Therefore, it is significant to understand when the price system aggregates information dispersed in the economy and generates an efficient outcome.

Our approach to this problem is a game theoretic analysis of double auctions. Double auctions are some of the most popular mechanisms for price formation.[1] In this paper, we study the conditions under which strategic equilibria of double auctions aggregate private information as the number of players increases.

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[1]See Wilson (2008) for a survey on exchange mechanisms.

The literature about double auctions builds on pioneering works about one-sided auctions (Wilson (1977), Milgrom (1979, 81), Milgrom and Weber (1982), Pesendorfer and Swinkels (1997), Swinkels (2001), and Hong and Shum (2004), for example), and starts with private value models. Wilson (1985) shows that, if players have iid private values, a Bayesian-Nash equilibrium of double auctions maximizes the social welfare among all incentive compatible mechanisms when there are sufficiently many players. McAfee (1992) designs a double auction mechanism where buyers and sellers have dominant strategies. Rustichini, Satterthwaite, and Williams (1994) show that a Bayesian-Nash equilibrium strategy converges to price taking behavior at a rate $1/n$ as the number of players n increases. Jackson and Swinkels (2005) show that a wide class of private value double auctions have a nondegenerate mixed strategy equilibrium. Cripps and Swinkels (2006) show that, even when players have multiple units of demand and supply and the distributions of private values are asymmetric and dependent, if there are no gaps nor atoms in the aggregate distribution of values and the dependence is not too strong ("z-independence"), then, as the number of players increases, gaps among equilibrium bids vanish and every nondegenerate equilibrium is asymptotically efficient. Fudenberg, Mobius, and Szeidl (2007) show existence of a monotone pure strategy equilibrium of double auctions in correlated private value environments when there are many players.

These papers establish that double auctions aggregate information in a large class of private value environments. But many markets including financial markets are often characterized by interdependence of information and values among players, and these interdependences have practical consequences. Therefore, we would be interested in understanding the performance of the double auction mechanism in the interdependent value environment.

The state of the art in the study of double auctions in the interdependent value environment is Reny and Perry (2006). The Reny and Perry (2006) theorem shows that, if a player has a unit of demand and supply, one dimensional conditional iid signal with the monotone likelihood ratio property, and a symmetric value function with common value and private value elements, then, generically, there exist monotone pure strategy equilibria in double auctions with a finite set of bids among sufficiently many players, and these equilibria converge to the fully revealing rational expectations equilibrium in the limit economy.

Then, the research question of the paper is to extend and strengthen the information aggregation result of Reny and Perry (2006) by considering general environments where players have multidimensional signals, multiple units of demand and supply, and heterogeneous information, while maintaining their assumptions of conditional independence of signals across players with the monotone likelihood ratio property, symmetric value functions with a private value element and a common value element,[2]

[2]The assumption of symmetric value functions simplifies the formula of the asymptotic distribution of equilibrium prices. It is possible to allow asymmetries in value functions as long as the distributions of values do not have gaps.

and a finite set of possible bids.

Our results are as follows:

- Consistency: every perfect strategy equilibrium of the uniform price double auction with a finite set of bids converges to the fully revealing rational expectations equilibrium in the limit economy at a rate $1/n$ as the number of players increases and the bid grid size goes to zero.
- Asymptotic normality: every perfect equilibrium price is an asymptotically normal estimator of the unknown value.

Let us discuss the method of the proof used in the paper. The ingenious method of proof of Reny and Perry (2006) is to show existence of a monotone pure strategy equilibrium of double auctions in the limit economy with a sufficiently fine grid of bids that converges to the fully revealing rational expectations equilibrium as the grid size goes to zero, and then to construct a monotone pure strategy equilibrium with sufficiently many players that converges to the rational expectations equilibrium of the limit economy. In contrast to Reny and Perry (2006) that assume one-dimensional signal, single unit of demand and supply, and symmetric information among players, our model allows multidimensional signals, multiple units of demand and supply, and heterogeneous information among players. Furthermore, since the Reny and Perry (2006) method is based on the Athey (2001) theorem on single crossing conditions and existence of a monotone pure strategy equilibrium, the Reny and Perry (2006) method does not apply to a larger class of equilibria that we study in this paper.[3]

Our idea of proof is based on the crucial observation made by Cripps and Swinkels (2006) and Reny and Perry (2006) that shrinking gaps among bids is the key to information aggregation.[4] Let us explain the idea in a simple example. If there are no gaps in the distribution of bids, then, when there are many players, even if a buyer shades the bid, the buyer will be likely to lose to another buyer who is bidding just below that bid and will not be able to affect the price. Therefore, the buyer has to bid just like a price taker. Then, the bid reveals the private information and an equilibrium aggregates information.[5]

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[3]Notably, Pesendorfer and Swinkels (1997) show that every mixed strategy equilibrium in one-sided uniform price auctions among symmetric players is indeed monotone, pure, and aggregates information. But their method does not extend to double auctions where buyers and sellers are inherently asymmetric. See Reny and Perry (2006).

[4]An important precursor is Swinkels (2001).

[5]Neverthless, the argument of Cripps and Swinkels (2006) does not apply to our interdependent value environment since we cannot ensure that a mixed strategy equilibrium has shrinking gaps among bids because of the difficulty of establishing single crossing conditions in double auctions with interdependent value environments (see the counterexample in Reny and Perry (2006)).

Reny and Perry (2006) consider this problem in the context of "vanishing prices." If market clearing prices arise with positive but vanishing probabilities, then, even if the number of players increases, a player's ability to affect the price will not vanish. Then, Reny and Perry (2006) show that, since buyers' and sellers' strategies are asymmetric in double auction, single crossing conditions may not hold. Reny and Perry (2006) overcome this problem by first restricting players' strategies to have "ε-step size restriction" such that every bid is used by at least ε of players, and then showing

Specifically, our method of proof is as follows:

- We first construct nondegenerate perfect equilibria[6] in double auctions with a finite sets of bids among a finite number of players from a sequence of nondegenerate equilibria in perturbed double auctions where players tremble with positive probabilities.[7][8]
- We then study information aggregation in perturbed double auctions as the number of players increases and the bid grid size goes to zero. Since players' realized strategies with trembles are totally mixed, there are no gaps in the distribution of equilibrium bids. Using a first order condition, we show that equilibrium bids in perturbed double auctions converge to price taking behavior at a rate $1/n$.[9] Then we take the probability of trembles to zero and conclude that a perfect equilibrium also converges to price taking behavior at that rate.[10]
- Since the equilibrium bids converge to price taking behavior at a rate $1/n$, strategic behavior does not affect the asymptotic behavior of equilibrium prices, and the asymptotic distribution of equilibrium prices is normal.[11]

The contribution of the paper is that we provide a next step in the literature on double auctions as a foundation of the Walrasian price mechanism by extending and strengthening the Reny and Perry (2006) theorem in the following dimensions:

- The model extends the Reny and Perry (2006) model by allowing multidimensional signals and multiple units of demands and supply, in addition to heteroge-

that this restriction is not binding for sufficiently large n. Our method of proof can be regarded as a generalization of the Reny and Perry (2006) approach.

[6]Selten (1975), Myerson (1977), and Kreps and Wilson (1982). See Govindan and Wilson (2008) for a survey on refinements.

Milgrom (1981), Back and Zender (1993), Milgrom (2000), and Perry, Wolfstetter, and Zamir (2000) use perfection in the auction literature. These papers do not discuss perfection in terms of information aggregation.

[7]Since our model maintains the assumption of a finite set of bids of Reny and Perry (2006), we do not deal with existence of an equilibrium in double auctions with a continuum of bids among a finite number of players. Also we do not establish monotonicity of an equilibrium in double auctions among a finite number of players. Therefore, the counterexamples in Jackson (2009), Reny and Perry (2006), and Reny and Zamir (2004) do not apply to our model and results.

See Govindan and Wilson (2010a,b), Quah and Strulovici (2010), and Reny (2010) for recent progresses in the issue of existence of a monotone pure strategy equilibrium in auctions.

[8]The core of our proof procedure for nondegenerateness of a perfect equilibrium is a technique of Jackson and Swinkels (2005) that introduces noise players and shows that a significant number of players wants to bid seriously even when the probability of noise trading is very small.

[9]The first order condition approach is used by Wilson (1985), Rustichini, Satterthwaite, and Williams (1994), Cripps and Swinkels (2006), and Fudenberg, Mobius, and Szeidl (2007) in private value models.

[10]Since our model has a finite set of possible bids, the issue of discontinuity at the limit that occurs in auctions with a continnum set of bids does not take place in this step.

[11]The result is based on the central limit theorem of order statistics from dependent heterogeneous distributions by Sen (1968). Meirowitz and Shott (2009) is another application of the Sen (1968)'s theorem.

nous information among players.[12]

- The result shows convergence to the fully revealing REE not only for monotone pure strategy equilibria constructed by Reny and Perry (2006), but for every perfect equilibrium that exists even with a small number of players.[13]
- The result provides a rate of convergence and characterizes the asymptotic distribution of equilibrium prices.[14] These information are useful for applications:
  - A fast rate of convergence is consistent with the experimental evidence about information aggregation (Plott and Sunder (1988)).
  - A rate of convergence result will allow us to compare mechanisms (Satterthwaite and Williams (2002) and Hong and Shum (2004)).
  - With the knowledge of asymptotic normality, we can test the theory using empirical data (Hong, Paarsch, and Xu (2010)).
  - The asymptotic normality result connects the auction literature with the asset pricing literature where the price process follows a Brownian motion (Duffie (2001)).[15]

In summary, the keys to information aggregation are:

- Signals are sufficiently informative to identify the underlying state (the monotone likelihood ratio property).[16]
- Players are not too asymmetric (information smallness in Gul and Postlewaite (1992) and McLean and Postlewaite (2002), and the assumption of no gaps in the distribution of signals by Cripps and Swinkels (2006)).
- Players use credible strategies (perfect equilibrium).

[12] If there are further asymmetries in distribution of signals or value functions, the supports of the distribution of values may become disjoint. Then, as the number of players increases and players' bids become symmetric and get close to price taking behavior, there will eventually be gaps in the distribution of bids. Existence of gaps implies that players' ability to affect the price will not vanish even when there are many players ("vanishing prices" of Reny and Perry (2006)). Then, convergence may not take place and information aggregation will not occur. See an example in Section 3.1.

In other words, if there are gaps in the distribution of values between strong players and weak players, winner's curse (Wilson (1969)) is so strong that strong players may keep the ability of affecting the price and will not become small even if there are many strong and weak players. Then the market may not aggregate information.

[13] A mixed strategy equilibrium may have gaps in the distribution of bids. Then, "vanishing prices" may occur and the equilibrium in the finite economy may not converge.

[14] The asymptotic normality result extends Theorem 2 of Hong and Shum (2004) to a case where players have multi-dimensional signals and multiple units of demand and supply, and the trading mechanism is a uniform price double auction. This generalization is not trivial since equilibrium strategies of double auctions do not have a closed-form expression in contrast to one-sided auctions they studied that have explicit formula for equilibrium strategies.

[15] Our results also extend Cripps and Swinkels (2006) by allowing interdependent values and by characterizing the asymptotic distribution of equilibrium prices, while maintaining their assumptions of multiple units of demand and supply and asymmetric information among players.

[16] Without the MLRP, the different states may have the same quantile and the equilibrium in the limit market may not be able to distinguish the states.

- The price formation process is such that players' equilibrium strategies reveal private information when there are many players (uniform price double auctions).

When these conditions are met, even if players have multidimensional signals, multiple units of demand and supply, and heterogeneous information, double auction equilibrium prices aggregate information and the distribution of prices is asymptotically normal, consistent with the standard asset pricing model.

The organization of the paper is as follows. Section 2 describes the model. Section 3 presents the main result of the paper. Section 4 provides an overview of the proof. Section 5 deals with consistency. Section 6 derives asymptotic normality. Section 7 contains concluding remarks. The appendix provides the details of the proof.

2. THE MODEL

2.1. *The Basic Setup*

2.1.1. *Players*

We consider an economy that consists of heterogeneous groups of buyers and sellers. We assume that there are $G_B \geq 1$ groups of buyers and $G_S \geq 1$ groups of sellers where G_B and G_S are fixed throughout the paper. It follows that there are $G = G_B + G_S$ groups in the economy.

Without loss of generality, we assume that each group has the same number $n \geq 1$ of players. Then, the ratio of buyers in the economy is given by $\alpha = G_B/G$. For example, $\alpha = 1/2$ implies that the ratio of buyers is $1/2$ and there are equal numbers of buyers and sellers.

We assume that each buyer has fixed $M \geq 1$ units of demand and each seller has the same M units of supply. This assumption of equal units of demand and supply among players is made to simplify the notations and it is possible to allow heterogeneities in endowments among buyers and sellers.

2.1.2. *Information Structure*

Let $\theta \in [0, 1]$ be the state variable.[17] Let $h(\cdot)$ be the density function of a prior distribution held by each player.

A private information of player i is $X_i = (X_{i,1}, ..., X_{i,M})$ where $X_{i,m} \in [0, 1]$ is the information relevant for the marginal value of the mth unit. [18] We assume that a player's signals are nonincreasing in the number of units ($X_{i,1} \geq ... \geq X_{i,M}$) almost surely.

Let $f_{i,m}(\cdot|\theta)$ be the conditional marginal density function of $X_{i,m}$ and let $f_{i,m,m'}(\cdot, \cdot|\theta)$ be the conditional joint density function of $X_{i,m}$ and $X_{i,m'}$.

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[17] We assume that the state is one-dimensional so that we can identify the state from the equilibrium price.

[18] Pesendorfer and Swinkels (2000) analyze one-sided uniform price auctions where players receive two dimensional information for a unit of the good (a signal about the quality and a taste parameter).

We assume that players in the different groups may have different distributions of information, and that players in the same group receive different realizations of signals from the distribution common to all players in the same group. That is, for each group g, units m and m', there exist $f_{g,m}(\cdot|\theta)$ and $f_{g,m,m'}(\cdot,\cdot|\theta)$ such that, for every player i in the gth group, $f_{i,m}(\cdot|\theta) = f_{g,m}(\cdot|\theta)$ and $f_{i,m,m'}(\cdot,\cdot|\theta) = f_{g,m,m'}(\cdot,\cdot|\theta)$.

We make the following assumptions on the information structure.

Assumption 2.1: *(a). (Conditional Independence of Signals Across Players) Conditional on θ, signals are independent across players. (b). (No Gaps in the Distribution of Signals) For every interval $I \subset [0,1]$, there exist $X_{g,m}$ and $X_{g',m'}$ with $g \in G_B$, $1 \leq m \leq M$, $g' \in G_S$, and $1 \leq m' \leq M$ such that their supports include I. (c). (Uniformly Bounded and Smooth Densities) There exist $0 < \underline{h} < \overline{h} < \infty$ such that $\underline{h} < h(\cdot) < \overline{h}$. Also there exist $0 < \underline{f} < \overline{f} < \infty$ such that, for every $f_{i,m}$, $\underline{f} < f_{i,m}(\cdot) < \overline{f}$ in the support of $f_{i,m}$. Furthermore, $f_{i,m}$ and h are C^2. (d). (Strict MLRP) For every i and m, for every $(x_{i,m}, \theta)$ in the interior of the support of $f_{i,m}$, $\frac{\partial^2 \ln f_{i,m}(x_{i,m}|\theta)}{\partial x_{i,m} \partial \theta} > 0$.*

Assumptions 2.1(a)-(d) correspond to Assumption A.1 and A.2 in Reny and Perry (2006). Assumption 2.1(a) implies that signals across players are conditionally independent[19] and that signals within players may be dependent.[20] Formally, signals are m-dependent conditional on θ.[21] Assumption 2.1(b) implies that asymmetries of signals among players are not too strong.[22] Assumption 2.1(d) means that a high signal is a good news about the state.

We now define an order statistics of bids. Since there are G_B groups of buyers and G_S groups of sellers ($G = G_B + G_S$ groups in total), each group has n players, and each player has M units of demand and supply, there are $G_S \cdot M \cdot n$ units of goods and there are $G \cdot M \cdot n$ signals in the economy. We define $x_{l:G \cdot M \cdot n}$ to denote the lth highest signal among $G \cdot M \cdot n$ signals. We say that a signal $x_{i,m}$ is *pivotal* if $x_{i,m} = x_{G_S \cdot M \cdot n : G \cdot M \cdot n}$.

The average distribution of signals is defined by $\overline{F}(\cdot|\theta) = \frac{1}{G \cdot M} \sum_{g=1,m=1}^{G,M} F_{g,m}(\cdot|\theta)$

[19]The assumption of conditional independence across players (Wilson (1969)) is a standard assumption in the information aggregation literature (Wilson (1977), Milgrom (1979, 81), Pesendorfer and Swinkels (1997), Hong and Shum (2004), and Reny and Perry (2006)).

When the distribution of signals is symmetric, it would be possible to relax the assumption of conditional independence to association (affiliation) by the result of Cai and Roussas (1997). Wang, Hu, and Yang (2010) is a recent development.

[20]The assumption made above that the signals are nonincreasing in the units provides a necessary structure on the distribution of signals within players.

[21]Random variables X_1, X_2...are m-dependent if random vectors $(X_1, ..., X_i)$ and $(X_j, X_{j+1}, ..)$ are independent if $j - i \geq m$. That is, $m-$dependent random variables behave independently if they are sufficiently separated (i.e, belong to different players).

By Assumption 2.1, Lemma A.6 (Sen (1968)) holds and the sample quantile of players' signals converges to the population quantile of the average distribution of signals and the asymptotic distribution is normal.

[22]This condition is based on Milgrom (1981, p.933) and Cripps and Swinkels (2006).

and the average density is denoted by $\overline{f}(\cdot|\theta)$. Let $x(\theta)$ denote the α-quantile (counted from below) function of $\overline{F}$. That is, $\overline{F}(x(\theta)|\theta) = \alpha$.[23] By Assumption 2.1, the quantile function $x(\theta)$ is well defined and $x'(\theta) > 0$. Let $\theta(x_{i,m})$ denote the state where the α-quantile of $\overline{F}(\cdot|\theta)$ is closest to $x_{i,m}$.

2.1.2.1. *Example*

Let us explain the above formulation using a following example. Consider an economy where there are a $G_B = 1$ group of buyers and a $G_S = 1$ group of sellers. Let us assume that each group has $n = 2$ players and that each player has $M = 2$ units of demand or supply.

In this economy, there are $G_B \cdot n = 1 \cdot 2 = 2$ buyers and $G_S \cdot n = 1 \cdot 2 = 2$ sellers. In total, there are $G = G_B + G_S = 2$ groups and $G \cdot n = 2 \cdot 2 = 4$ players. Then, the ratio of buyers is $\alpha = G_B/G = 1/2$. Since each of 2 sellers has 2 units of goods, there are $G_S \cdot M \cdot n = 1 \cdot 2 \cdot 2 = 4$ units in the economy. Since each of 4 players has $M = 2$ signals, the total number of signals in the economy is $G \cdot M \cdot n = 2 \cdot 2 \cdot 2 = 8$.

Then, the pivotal signal is the 4th highest signal out of 8 signals and it is $x_{4:8}$. The average distribution of signals is $\overline{F}(\cdot|\theta) = \frac{1}{4}\sum_{g=1,m=1}^{2,2} F_{g,m}(\cdot|\theta)$. $x(\theta)$ is the median signal of $\overline{F}(\cdot|\theta)$ when the state is θ. When the state is $\theta(x_{i,m})$, $x_{i,m}$ is the median of the average distribution of signals.

2.1.3. *Values*

Let $v(\theta, x_{i,m})$ be the marginal value of the mth unit of the good when the state is θ and player i's signal is $x_i = (x_{i,1}, ..., x_{i,m}, ..., x_{i,M})$.[24][25]

We make the following assumptions on the value.

ASSUMPTION 2.2: *(a). (Nonnegative and Smooth Values)* $v(\theta, x_{i,m}) \geq 0$ *is* C^1. *(b). (Weakly Increasing in the State)* $v_\theta(\theta, x_{i,m}) \geq 0$. *(c). (Strictly Increasing in the Private Signal)* $v_{x_{i,m}}(\theta, x_{i,m}) > 0$. *(d). (Gains from Trade)* $v(1, 0) < v(0, 1)$.

Assumptions 2.2(a)-(c) are also made by Reny and Perry (2006). Assumption 2.2(d) implies that there are gains from trade between a seller with the lowest signal at the highest state (the seller's value in this case is $v(1, 0)$) and a buyer with the highest

[23] α is the ratio of buyers in the economy.

[24] Here we assume that player's marginal value for the mth unit depends on a player's private information through its mth signal. This formulation follows Cripps and Swinkels (2006), Jackson and Swinkels (2005), and Reny and Perry (2006) where they represent private information as information concerning the marginal value(s) of the good.

[25] Since we allow heterogeneous distributions of signals, players' values can be heterogeneous across units and players. Examples of this type of heterogeneity are in Maskin and Riley (2000) and Milgrom (2004).

signal at the lowest state (the buyer's value in this case is $v(0, 1)$).[26][27][28] The only role of Assumption 2.2(d) is to ensure nondegeneratedness of an equilibrium and this assumption does not play any further role in the development.

2.1.4. *Utilities*

Player i's von Neumann-Morgenstein utility is the sum of utilities from units $m = 1, ..., M$. Let $q_{i,m} \in \{0, 1\}$ denote player i's allocation of the mth unit ($q_{i,m} = 1$ if player i is allocated the mth unit and $= 0$ otherwise). Also let p be the transaction price. Then, if player i is a buyer, player i's ex post utility is

$$\sum_{m=1}^{M} (v(\theta, x_{i,m}) - p) q_{i,m}.$$

Similarly, if player i is a seller, the utility is $\sum_{m=1}^{M} (p - v(\theta, x_{i,m}))(1 - q_{i,m})$.

2.2. *The Limit Economy*

Let us first consider the limit economy where there is a unit mass of players, each group has the same measure of players, and the set of possible prices is a continuum.

We first view this situation as an exchange economy. A rational expectations equilibrium in this exchange economy is a price function P^{REE} from the state to the price such that every player chooses demand or supply to maximize the expected utility taking the price and the information contained in the price as given, and that the market clears. It is fully revealing if P^{REE} is a one-to-one mapping between the price and the state.

Lemma 2.1 states that there exists a unique fully revealing rational expectations equilibrium in the limit economy.

LEMMA 2.1: *If the limit economy satisfies Assumption 2.1 and 2.2, then, there exists a unique fully revealing rational expectations equilibrium with*

$$P^{REE}(\theta) = v(\theta, x(\theta)).$$

According to lemma 2.1, the rational expectations equilibrium price at state θ is equal to the value of the unit of the good where the signal for the unit is $x(\theta)$. The proof is identical to that of Reny and Perry (2006), Proposition 3.1.(i). Suppose the

[26] We thank for a referee for suggesting this condition.

[27] Under this assumption, players who observe extreme signals can still have large uncertainty in values due to a common value component. Since these players maximize the expected utility, they are not noise traders.

[28] Assumption 2.2(d) ensures nondegenerateness of every perfect equilibrium even with a small number of players. In contrast, in Reny and Perry (2006), their Assumption A.4 (boundary condition) is used to establish existence of a monotone pure strategy equilibrium among sufficiently many players. Thus it is reasonable that Assumption 2.2(d) is somewhat stronger than Assumption A.4 of Reny and Perry (2006).

price function $P^{REE}(\theta)$ is fully revealing. Then a player can identify the state from the price. Since each seller has M units of supply and the ratio of sellers is $1-\alpha$, there are $M \cdot (1-\alpha)$ of supply. Therefore, by the market clearing condition, there are $M \cdot (1-\alpha)$ of demands at the equilibrium price. Since a player identifies the state, by Assumption 2.1 and 2.2, the value for each unit is increasing in the signal for that unit. It follows that the equilibrium price must be $v(\theta, x(\theta))$.

We also note that, as in Reny and Perry (2006), when players trade through double auctions in the limit economy, price-taking bidding behavior of player i with signal $x_i = (x_{i,1}, ..., x_{i,M})$ for the mth unit is to bid $v(\theta(x_{i,m}), x_{i,m})$. To check, suppose every player other than i bids as above. By Assumption 2.1 and 2.2, the bidding function $v(\theta(x_{i,m}), x_{i,m})$ is strictly increasing in $x_{i,m}$. Therefore, the transaction price is determined by the bid with signal $x(\theta)$. It follows that the transaction price is $v(\theta, x(\theta))$. If player i bids $v(\theta(x_{i,m}), x_{i,m})$ for the mth unit, player i ends up with the mth unit if and only if the ex post value of the mth unit exceeds the transaction price. Furthermore, since there is a continuum of bids, player i's bid for the mth unit cannot influence the transaction price and the allocation for other units of player i. Therefore, player i's bid for the mth unit does not affect utilities from other units for player i. Consequently, it is a best response for player i with signal x_i to bid $v(\theta(x_{i,m}), x_{i,m})$ for the mth unit.

2.3. *The Double Auction Mechanism*

We now return to the finite economy where there are n players in each group and we define the price formation process via the uniform price double auction mechanism.

2.3.1. *Bids*

Each player submits bids for M units of goods after observing the signal.[29] Player i's bid profile is denoted by $b_i = (b_{i,1}, ..., b_{i,M})$ where $b_{i,m}$ is the bid for the mth unit of the good. We assume that each unit bid is from a finite set $B_\Delta = \{0, \Delta, 2\Delta, ..., \overline{v}\}$ where $\Delta > 0$ is the finiteness of the grid and $\overline{v}$ is such that $v(1,1) < \overline{v} < \infty$. Also bids are assumed to be nonincreasing in units: $b_{i,1} \geq b_{i,2} \geq ... \geq b_{i,M}$. Let b denote the bid profiles by all the players in the economy ($b = (b_i, b_{-i})$).

The order statistics of bids is defined in the same way as the order statistics of signals was defined above. Since there are G_B groups of buyers and G_S groups of sellers and each group has n players with M units of supply, there are $G_S \cdot M \cdot n$ units of supply. Since each player submits M bids, there are $G \cdot M \cdot n$ bids in the market. Let $b_{l:G \cdot M \cdot n}$ denote the lth highest bid out of $G \cdot M \cdot n$ bids. We say that the bid $b_{i,m}$ is *pivotal* if it is the α-quantile of bids: $b_{i,m} = b_{G_S \cdot M \cdot n:G \cdot M \cdot n}$.

[29]Following the convention, a seller's offer is also called as "a bid."

2.3.2. *The Transaction Price*

The transaction price is determined by the standard $k \in [0,1]$-double auction pricing rule. That is, the price is a convex combination of the pivotal bid and the bid just below the pivotal bid. Formally, given the bid profile b, the pricing rule P is defined by

$$P(b) = (1-k) \cdot b_{G_S \cdot M \cdot n+1 : G \cdot M \cdot n} + k \cdot b_{G_S \cdot M \cdot n : G \cdot M \cdot n}. \tag{1}$$

2.3.3. *The Allocation*

The allocation rule is also standard:

- If a bid $b_{i,m}$ is strictly above the pivotal bid $b_{G_S \cdot M \cdot n : G \cdot M \cdot n}$, then, player i is assigned the mth unit. It means that, if player i is a buyer, player i obtains the mth unit, and if player i is a seller, player i does not sell the mth unit.
- If a bid $b_{i,m}$ is strictly below the pivotal bid, then, player i is not assigned the mth unit.
- If a bid $b_{i,m}$ is pivotal and tied with other bids, ties are broken uniformly among all bids that are equal to pivotal bids (symmetrically between buyers and sellers) after having allocated the units to bids strictly above the pivotal bid.[30]

Formally, let $q_{i,m}(b_i, b_{-i})$ be player i's allocation of the mth unit when player i's bid profile is b_i and other players' bid profiles are b_{-i}. Then, for every i, m, and (b_i, b_{-i}), we have

$$q_{i,m}(b_i, b_{-i}) = \begin{cases} 1 & \text{if } b_{i,m} > b_{G_S \cdot M \cdot n : G \cdot M \cdot n} \\ \text{1 with probability } \frac{G_S \cdot M \cdot n - \#\{b_{i',m'} : b_{i',m'} > b_{G_S \cdot M \cdot n : G \cdot M \cdot n}\}}{\#\{b_{i',m'} : b_{i',m'} = b_{G_S \cdot M \cdot n : G \cdot M \cdot n}\}} & \text{if } b_{i,m} = b_{G_S \cdot M \cdot n : G \cdot M \cdot n} \\ 0 & \text{else.} \end{cases} \tag{2}$$

2.3.3.1. *An Example*

Let us consider a simple example to illustrate the mechanics of the double auction mechanism. Consider an economy consisting of one group of buyers ($G_B = 1$) and one group of sellers ($G_S = 1$). We assume that each group has $n = 2$ players and each player has $M = 2$ units of demand or supply. Let us assume that players trade through a $k = 0.5$ uniform price double auction. In this economy, there are $G = G_B + G_S = 2$ groups and there are $G_S \cdot M \cdot n = 1 \cdot 2 \cdot 2 = 4$ units of supply.

Suppose that players' bids are as follows: buyer 1 and 2 bid 4 and 3 ($b_1 = b_2 = (4,3)$), seller 1 offers 6 and 5 ($b_3 = (6,5)$), and seller 2 offers 4 and 3 ($b_4 = (4,3)$). Then, $b = (4,3,4,3,6,5,4,3)$. In the descending order, bids are $(6,5,4,4,4,3,3,3)$.

Let us first calculate the price according to (1). Since there are 4 units of good, the pivotal bid is $b_{4:8} = 4$. The bid below the pivotal bid (5th highest bid) is $b_{5:8} = 4$.

[30] This tie-breaking rule is adopted by Reny and Perry (2006) and simplifies the proof of Lemma 2.2.

Then, the transaction price is 4.

We now compute the allocation according to (2).

- For buyer 1, his bid for the first unit 4 is equal to the pivotal bid 4. Since there are 2 bids (2 offers by seller 1) strictly above the pivotal bid 4, there are 2 units (2 units of supply by seller 2) left out of total supply of 4 units after having allocated 2 units to these bids (2 offers by seller 1). Also note there are 3 bids at price 4 (the first unit bids by buyer 1, buyer 2, and seller 2). Then the tie-breaking rule is to allocate these 2 units equally among 3 bids. It follows that buyer 1's first unit bid is allocated (wins) the unit with probability 2/3. For buyer 1's bid for the 2nd unit, since the bid 3 is strictly below the pivotal bid 4, buyer 1 will not be assigned the $2nd$ unit.
- For buyer 2, the allocation can be calculated as in the case of buyer 1.
- For seller 1, both offers are strictly above 4. Therefore, seller 1 will be allocated these two units. In other words, seller 1 will not sell any of the 2 units.
- For seller 2, the offer for the first unit is 4 and equal to the pivotal bid. As we discussed in the allocation of buyer 1, seller 2's first unit bid is allocated the unit with probability 2/3. In other words, seller 2 sells the first unit with probability 1/3. For seller 2's offer for the second unit, since the offer 3 is below the pivotal bid 4, seller 2 will not be allocated the second unit. In other words, seller 2 will sell the second unit.

2.4. *Bayesian Game*

We formulate the price formation process described above as a Bayesian game.

2.4.1. *Strategy*

A mixed strategy of player i with signal x_i is a random variable $\beta_{n,\Delta,i}(x_i)$ over the set of nonincreasing vectors in $\underbrace{B_\Delta \times ... \times B_\Delta}_{M \text{ times}}$. According to this formulation, every player has the same set of possible strategies. Let us denote this set of possible strategies by $\mathcal{B}_\Delta$.

We consider an equilibrium where every player in the same group uses the same strategy. Let $\beta_{n,\Delta,g}$ be a strategy used by every player in the gth group. Let $\beta_{n,\Delta}$ denote the strategy profile of all G groups ($\beta_{n,\Delta} = (\beta_{n,\Delta,1}, ..., \beta_{n,\Delta,G})$).

2.4.2. *Payoffs*

Let $u_i(x_i, b_i, \beta_{n,\Delta})$ be the interim expected payoff of player i whose signal is x_i when the player bids b_i and all other players employ strategies $\beta_{n,\Delta}$.

Also let $U_i(\beta_{n,\Delta,i}, \beta_{n,\Delta})$ be the ex ante expected payoff of player i when player i employs a strategy $\beta_{n,\Delta,i}$ and all other players employ strategies $\beta_{n,\Delta}$.

Since players in the same group are symmetric, every player in the same group evaluates the payoff using the same payoff function. Let U_g be the payoff function common to every player in the gth group. Let us denote the profile of payoff functions of all the groups by $U = (U_1,...., U_G)$.

2.4.2.1. *An Example*

Consider an economy with G_B groups of buyers and G_S groups of sellers. Suppose there are $n = 2$ players in each group, and that each player has $M = 2$ units of demand or supply. Then, there are $G = G_B + G_S$ groups, $G \cdot 2 \cdot 2$ players, and $G_S \cdot 2 \cdot 2$ units of supply in the economy. The pivotal bid is $b_{G_S \cdot 2 \cdot 2 : G \cdot 2 \cdot 2}$.

Take buyer i in the gth group with signal $x_i = (x_{i,1}, x_{i,2})$ and the bid $b_i = (b_{i,1}, b_{i,2})$. Let us assume $b_{i,1} > b_{i,2}$. Suppose other players' strategies are $\beta_{n,\Delta}$.

Let us calculate buyer i's expected utilities when buyer i wins 2 units. According to (2), there are 2 cases where buyer i wins 2 units.

- If $b_{i,2} > b_{G_S \cdot 2 \cdot 2 : G \cdot 2 \cdot 2}$, then, buyer i wins 2 units. According to (1), buyer i pays $P(b_i, b_{-i})$ for each unit. Then, the expected utility is the sum of marginal utilities $\{v(\theta, x_{i,1}) - P(b_i, b_{-i})\} + \{v(\theta, x_{i,2}) - P(b_i, b_{-i})\}$ conditional on the signal x_i and information that $b_{i,2} > b_{G_S \cdot 2 \cdot 2 : G \cdot 2 \cdot 2}$.
- If $b_{i,2} = b_{G_S \cdot 2 \cdot 2 : G \cdot 2 \cdot 2}$, then, buyer i wins 2 units when buyer i wins the tie-breaking for the 2nd unit. The price is $b_{i,2}$. The expected utility is $\{v(\theta, x_{i,1}) - b_{i,2}\} + \{v(\theta, x_{i,2}) - b_{i,2}\}$ conditional on the information x_i, $b_{i,2} = b_{G_S \cdot 2 \cdot 2 : G \cdot 2 \cdot 2}$, and $b_{i,2}$ wins the tie.

2.4.3. *The Double Auction Game*

Let $\Gamma(n, \mathcal{B}_\Delta, U)$ denote the double auction game with n players in each group where the set of strategies is $\mathcal{B}_\Delta$ and the profile of utility functions is U.

2.4.4. *Bayesian Nash Equilibrium*

A Bayesian Nash equilibrium of $\Gamma(n, \mathcal{B}_\Delta, U)$ is a strategy profile $\widehat{\beta}_{n,\Delta} = (\widehat{\beta}_{n,\Delta,1}, ..., \widehat{\beta}_{n,\Delta,G})$ such that, for each player i in the gth group, $\widehat{\beta}_{n,\Delta,g}$ is a best response for player i when all other players use $\widehat{\beta}_{n,\Delta}$. An equilibrium $\widehat{\beta}_{n,\Delta}$ is *nondegenerate* if there is a positive probability of trade when players play $\widehat{\beta}_{n,\Delta}$.

We are interested in the asymptotic behavior of the equilibrium price $P_{n,\Delta}(\widehat{\beta}_{n,\Delta})$ as n increases and Δ goes to 0.[31]

2.4.5. *Perfect Equilibrium*

A perfect equilibrium of $\Gamma(n, \mathcal{B}_\Delta, U)$ is defined as a limit of equilibria in perturbed games $\Gamma(n, \mathcal{B}_\Delta, U, \varepsilon)$ as $\varepsilon \to 0$ where ε is the probability that a player trembles. As we will explain in Section 3.1, we introduce a perfect equilibrium in our analysis to exclude outcomes based on noncredible threats.

[31] We assume that each replica $\Gamma(n, B_\Delta, U)$ is independent.

2.4.5.1. *Perturbed Game*

Let us define a perturbed game $\Gamma(n, \mathcal{B}_\Delta, U, \varepsilon)$ of $\Gamma(n, \mathcal{B}_\Delta, U)$. Player i chooses a strategy $\beta_{n,\Delta,i}$ from $\mathcal{B}_\Delta$. Let $H_{n,\Delta,i}(b_i|x_i)$ be the distribution function of bids according to $\beta_{n,\Delta,i}$.

Player i's payoff in $\Gamma(n, \mathcal{B}_\Delta, U, \varepsilon)$ is determined as follows. In a perturbed game, with probability ε, player i trembles and chooses a bid randomly instead of following $\beta_{n,\Delta,i}$. To simplify the presentation, we assume that, in case of a tremble, a player chooses a bid for each unit according to the uniform distribution on $Q \cup B_\Delta$, where $Q < 0$ implies nonparticipation, and that the player chooses a bid for a unit independently from bids for other units (independent uniform trembles).[32][33] Then, the distribution of realized bids is $H^\varepsilon_{n,\Delta,i}(b_i|x_i) = (1-\varepsilon) \cdot H_{n,\Delta,i}(b_i|x_i) + \varepsilon \cdot F_{unif,\Delta}(b_i)$ where $F_{unif,\Delta}$ is an independent multivariate uniform distribution of bids when the player trembles. Let $\beta^\varepsilon_{n,\Delta,i}$ represent the "perturbed strategy" whose bid distribution is $H^\varepsilon_{n,\Delta,i}(b_i|x_i)$. Then, player i's expected payoff in $\Gamma(n, \mathcal{B}_\Delta, U, \varepsilon)$ when player i chooses a strategy $\beta_{n,\Delta,i}$ and other players choose strategies $\beta_{n,\Delta}$ is defined to be $U_i(\beta^\varepsilon_{n,\Delta,i}, \beta^\varepsilon_{n,\Delta})$.

A Bayesian Nash equilibrium of the perturbed game $\Gamma(n, \mathcal{B}_\Delta, U, \varepsilon)$ is defined by $\widehat{\beta}_{n,\Delta,\varepsilon} = (\widehat{\beta}_{n,\Delta,\varepsilon,1}, ..., \widehat{\beta}_{n,\Delta,\varepsilon,G})$ such that for every player i in the gth group, $\widehat{\beta}_{n,\Delta,\varepsilon,g}$ is a best response for player i to strategies of other players $\widehat{\beta}_{n,\Delta,\varepsilon}$.

2.4.5.2. *Perfect Equilibrium*

We define that $\widehat{\beta}_{n,\Delta} = (\widehat{\beta}_{n,\Delta,1}, ..., \widehat{\beta}_{n,\Delta,G})$ is a perfect equilibrium of $\Gamma(n, \mathcal{B}_\Delta, U)$ if $\widehat{\beta}_{n,\Delta}$ is a Bayesian Nash equilibrium of $\Gamma(n, \mathcal{B}_\Delta, U)$ and there is a sequence of Bayesian Nash equilibria $\{\widehat{\beta}_{n,\Delta,1/q}\}_{q=1,2,...}$ of perturbed games $\{\Gamma(n,\mathcal{B}_\Delta, U, 1/q)\}_{q=1,2,...}$ such that $\widehat{\beta}_{n,\Delta,1/q} \to \widehat{\beta}_{n,\Delta}$ as $q \to \infty$.

2.5. *Existence of a Perfect Equilibrium*

We conclude this section with a lemma about existence of a nondegenerate perfect equilibrium of $\Gamma(n, \mathcal{B}_\Delta, U)$.

Lemma 2.2: *Consider a double auction game $\Gamma(n, \mathcal{B}_\Delta, U)$ that satisfies Assumption 2.1 and 2.2. For every $n \geq 1$, for sufficiently small $\Delta > 0$, $\Gamma(n, B_\Delta, U)$ has a nondegenerate perfect equilibrium $\widehat{\beta}_{n,\Delta}$.*

Proof Sketch: First fix $q \geq 1$ and consider a perturbed game $\Gamma(n, \mathcal{B}_\Delta, U, 1/q)$. Since the set of bids is finite, there exists a mixed strategy equilibrium $\widehat{\beta}_{n,\Delta,1/q}$ of

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[32] The key property is that each bid will be chosen with probability bounded away from zero.

[33] The introduction of Q implies that the tremble puts some mass weights at the boundary of the set of possible bids.

We further simplify the notation by assuming that a player chooses a bid from $\{Q, Q + \Delta, ..., 0, \Delta, ..., \overline{v}\}$ and interpret that a bid less than 0 is nonparticipation.

$\Gamma(n, \mathcal{B}_\Delta, U, 1/q)$. By construction, every player trembles with probability $1/q$ and bids randomly in $\Gamma(n, \mathcal{B}_\Delta, U, 1/q)$. Therefore, players trade with a positive probability when playing $\widehat{\beta}_{n,\Delta,1/q}$ in $\Gamma(n, \mathcal{B}_\Delta, U, 1/q)$.

Consequently, there exists a sequence of nondegenerate mixed strategy equilibria $\{\widehat{\beta}_{n,\Delta,1/q}\}_{q=1,\ldots}$ of $\{\Gamma(n, \mathcal{B}_\Delta, U, 1/q)\}_{q=1,\ldots}$. Since the set of bids is finite, the set of mixed strategies is a compact set. Therefore there exists a subsequence limit $\widehat{\beta}_{n,\Delta}$ of $\{\widehat{\beta}_{n,\Delta,1/q}\}_{q=1,\ldots}$. Since the payoff functions are continuous, $\widehat{\beta}_{n,\Delta}$ is a perfect equilibrium of $\Gamma(n, B_\Delta, U)$.

It remains to show that $\widehat{\beta}_{n,\Delta}$ is nondegenerate. By Assumption 2.2(d), there are gains from trade between buyers and sellers. It follows that, even if $1/q \to 0$, the numbers of players who wants to trade and bids seriously do not vanish. Then, the probability of trade cannot go to zero.

The details of the proof are in Appendix A.1. *Q.E.D.*

3. THE MAIN RESULT

The main result of the paper is the following:

PROPOSITION 1: *Consider a double auction game $\Gamma(n, \mathcal{B}_\Delta, U)$ that satisfies Assumption 2.1 and 2.2. (a). (Consistency) Every perfect equilibrium strategy $\widehat{\beta}_{n,\Delta}$ converges to price taking behavior in the limit economy at a rate $1/n$ as $n \to \infty$ and $\Delta \to 0$ simultaneously.*[34] *(b). (Asymptotic Normality) For each θ_0, $\sqrt{n}(P_{n,\Delta}(\widehat{\beta}_{n,\Delta}) - v(\theta_0, x(\theta_0)))$ is normal with mean 0 and variance*

$$\begin{aligned}(3)\qquad & \frac{1}{\left(\sum_{g=1,m=1}^{G,M} f_{g,m}(x(\theta_0)|\theta_0)\right)^2} \\ & \cdot \left[\sum_{g=1,m=1}^{G,M} (F_{g,m}(x(\theta_0)|\theta_0) \cdot (1 - F_{g,m}(x(\theta_0)|\theta_0)) \right. \\ & \left. +2 \sum_{g=1,m=1,l=1}^{G,M-1,M-m} \begin{pmatrix} F_{g,m,m+l}(x(\theta_0), x(\theta_0)|\theta_0) \\ -F_{g,m}(x(\theta_0)|\theta_0) \cdot F_{g,m+l}(x(\theta_0)|\theta_0) \end{pmatrix}\right] \\ & \cdot \left(\frac{\partial v(\theta_0, x(\theta_0))}{\partial \theta} \frac{\partial \theta(x(\theta_0))}{\partial x} + \frac{\partial v(\theta_0, x(\theta_0))}{\partial x} \right)^2 .\end{aligned}$$

Intuitively, Part (a) says that every nondegenerate perfect equilibrium bid converges to price taking behavior as the number of players increases and the bid grid size goes to zero. Part (b) shows that the distribution of prices is asymptotically normal. The asymptotic variance is the sum of the terms concerning the variance of $F_{g,m}$ around $x(\theta_0)$ and the terms that account for the correlation of signals within players around $x(\theta_0)$.

[34] We simplify the presentation by assuming that $O(\Delta)$ is $1/n$.

3.1. *Remark*

We now illustrate the role of perfection in Proposition 1 with a following example.

Consider an economy with n buyers and n sellers. Buyers have a unit demand and sellers have a unit supply. There are two states $\theta = 1$ and $\theta = 2$. If $\theta = 1$, buyers have private values 5 and sellers have private values 5, whereas if $\theta = 2$, buyers have private values 7 and sellers have private values 6. Therefore, there are strictly positive gains from trade between buyers and sellers when $\theta = 2$. We consider a $k = 0$ double auction with the trade-maximizing tie-breaking rule.

Consider a following strategy profile. Every buyer bids 5 regardless of the private value and every seller offers 5 when the private value is 5 and offers 7 when the private value is 6. In this strategy profile, buyers shade their bids from value 7 to 5 and sellers inflate their offers from 6 to 7 and these behavior are individually rational. This strategy profile is a monotone, pure, and nondegenerate Bayesian-Nash equilibrium for every finite n.[35]

But this equilibrium in a finite economy does not converge to an equilibrium in the limit economy. To see this, take a buyer with a private value 7. In a finite economy, if the buyer increases a bid from 5 to 7, according to the $k = 0$ double auction pricing rule, it will increase the price to 7. It implies that increasing the bid to 7 is not a profitable deviation. But in the limit economy, if the buyer increases a bid from 5 to 7, since a bid does not affect the price, the price is still 5. Therefore, increasing the bid to 7 is a profitable deviation. That is, although the above strategy profile is an equilibrium for every finite n, it is not an equilibrium in the limit economy. It implies that, in order for equilibria of finite economies to converge to an equilibrium in the limit economy, gaps among bids (in this case gaps between 5 and 7) must shrink as the number of players increases.

We now consider whether the equilibrium gaps shrink if we consider a more general

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[35] To see this, first consider a buyer with private value 5. The buyer can infer that the state is 5. So the buyer sees that every seller has value 5 and bids 5. When all other players use the above strategy, when the buyer bids 5, the buyer can win the good for sure (given the trade-maximizing rule) and the utility is 0. Even if the buyer increases or decreases the bid from 5, the utility does not change.

Now consider a buyer with value 7. The buyer can see that the state is 2 and sellers have private values 6 and offer 7. When all other players use the strategy specified above, the buyer with bid 5 does not trade. In order for the buyer to trade, the buyer has to increase the bid to 7. When the buyer increases the bid to 7, the buyer can win the good, but given the $k = 0$ pricing rule, the price is 7. That is, the payoff is zero. Therefore, the buyer does not have a profitable deviation.

Consider a seller with value 5. The seller can identify the state to be 1, so knows that buyers have value 5 and bid 5. When all other players follow the strategy specified above, the seller sells the good with price 5 with utility 0. Even if the seller reduces or increases the bid from 5, the utility is 0.

Finally, consider a seller with value 6. The seller identifies the state to be 2 and all buyers bid 5 and all other sellers offer 7. If the seller bids 7, the seller does not make a sale and the utility is 0. For the seller to make a sale, the seller has to reduce the bid to 5. Given the $k = 0$ pricing rule, the sales price is 5, and it is below the seller's value 6. Therefore, the seller does not have an incentive to reduce the bid.

model where the are no gaps in the distribution of signals of buyers and sellers between 5 and 7.

Previous studies identified conditions for gaps among bids to shrink. Cripps and Swinkels (2006) consider private value double auctions where the aggregate distribution of values does not have gaps (the "no asymptotic gaps" assumption). When n is large, there will be many players with private values between 5 and 7 and these players will bid between 5 and 7. Therefore, gaps will shrink. Pesendorfer and Swinkels (1997) consider a one-sided uniform price auction among symmetric players with the common value and show that any mixed strategy equilibrium satisfies the single crossing condition (thus indeed is a monotone pure strategy equilibrium). That is, buyers with signals higher than 5 will bid more than buyers with signal 5 and sellers with signals lower than 7 will prefer to bid less than sellers with signal 7. Consequently, the gaps will shrink and information aggregation holds. Even when players are asymmetric, Reny and Zamir (2004) show that a single crossing condition still holds in a first price auction.[36]

But our model of uniform price double auctions allows a higher degree of heterogeneities among players than any of these models: values are interdependent, buyers and sellers have multiple units of demand and supply, information are heterogeneous, and the payment depends on other players' bids. As a result, the techniques used in Pesendorfer and Swinkels (1997), Reny and Zamir (2004), and Cripps and Swinkels (2006) do not establish a single crossing condition in our model. Indeed, Reny and Perry (2006, Section 5.2) present a robust counterexample such that, even when players' signals are affiliated and players use monotone pure strategies, if buyers and sellers use asymmetric strategies, as they are expected to do in double auctions in the finite economy, the single crossing condition may not hold.[37] That is, a buyer with a signal higher than 5 may not want to bid more than a buyer with signal 5 does, and a seller with a signal lower than 7 may not wish to bid lower than a seller with signal 7 does. In other words, if we just consider a Bayesian-Nash equilibrium in the finite economy, even if there are no gaps in the distribution of signals, we cannot exclude the possibility that equilibrium gaps among bids may persist even as n increases.

But the gap and nonconvergence in the above example are driven by noncredible threats by sellers with value 6 offering 7 that forgo gains from trade. Perfection excludes this kind of noncredible threats and prevents "vanishing prices" of Reny and Perry (2006).

Let us explain the effect of perfection in the above example. A perfect equilibrium is a limit of Bayesian-Nash equilibria in perturbed games where players tremble with

[36]Reny and Zamir (2004)'s insight is that, in the first price auction, the payment is the player's bid, and it is independent from the inference on other players' signals and bids.

[37]In constrast to first price auctions studied by Reny and Zamir (2004), the player's payment in uniform price auctions depends on other players' bids. In Reny and Perry (2006)'s amazing counterexample, an increase in a player's signal actually lowers the transaction price because of asymmetries of buyers' and sellers' strategies, and a player with a higher signal does not have an incentive to bid higher.

a positive probability ε and their realized strategies are totally mixed. That is, in our auctions, every bid is chosen with a positive probability and the distribution of bids does not have any gaps. In a perturbed game, if a buyer with value 7 increases the bid to 7, then, due to trembles, the price will be below 7 with some positive probability. It implies that the buyer will get a positive expected utility from increasing the bid to 7. Therefore, increasing the bid to 7 is a profitable deviation and the above equilibrium will not be supported. Indeed, we will show that, for every $\varepsilon > 0$, equilibria in perturbed double auctions converge to price taking behavior in the limit economy. Then, a perfect equilibrium which is a limit of these equilibria as $\varepsilon \to 0$ also converges to price taking behavior.[38]

4. PROOF OVERVIEW

We now present the proof of Proposition 1. We begin with an overview of the proof.

4.1. *Part (a)*

In Part (a), we first consider perturbed games. From Lemma 2.2, there exists a sequence of nondegenerate equilibria $\{\widehat{\beta}_{n,\Delta,\varepsilon}\}_\varepsilon$ of perturbed games $\{\Gamma(n, \mathcal{B}_\Delta, U, \varepsilon)\}_\varepsilon$ that converges to $\widehat{\beta}_{n,\Delta}$ as $\varepsilon \to 0$. We show that, for every $\varepsilon > 0$, $\widehat{\beta}_{n,\Delta,\varepsilon}$ converges to price taking behavior as $n \to \infty$ and $\Delta \to 0$ simultaneously.

To show convergence, we consider a first order condition for $\widehat{\beta}_{n,\Delta,\varepsilon}$ in $\Gamma(n, \mathcal{B}_\Delta, U, \varepsilon)$. For example, take buyer i with signal x_i and the equilibrium bid $b_{i,m}$ for the mth unit that is in the support of $\widehat{\beta}_{n,\Delta,\varepsilon}$.[39] Suppose that buyer i increases the bid for the mth unit by d while keeping the bids for other units constant. Then, when $b_{i,m}$ is the pivotal bid, if there are gaps among the bids, increasing the bid by d raises the price and incurs the cost. On the other hand, if $b_{i,m}$ is less than the pivotal bid but $b_{i,m} + d$ surpasses the pivotal bid, buyer i wins the mth unit and obtains a benefit. The first order condition for $\widehat{\beta}_{n,\Delta,\varepsilon}$ is that the expected benefit is less than the expected cost.

Let us consider the expected cost. In a perturbed game, a player's realized strategy is totally mixed. That is, a player bids every bid with a strictly positive probability (it may be due to $\widehat{\beta}_{n,\Delta,\varepsilon}$ or by trembles). Then, as the number of players increases and the bid grid size goes to zero while keeping ε fixed, the probability that there exists a bid between $b_{i,m}$ and $b_{i,m}+d$ grows and converges to 1. In other words, asymptotically, there will be no gaps among bids. Thus, the expected cost will converge to 0.

Therefore, by the first order condition, the expected benefit will also converge to 0. In other words, since a player's bid will not be able to affect the price asymptotically, a player's equilibrium bid converges to price taking behavior. Given the conditional independence, the rate is $1/n$.

[38] Since the distribution of bids when players follow price taking bidding behavior does not have gaps, "vanishing prices" will not occur when we take $\varepsilon \to 0$.

[39] The argument for a seller is similar.

Then we take $\varepsilon \to 0$. Since $\widehat{\beta}_{n,\Delta,\varepsilon}$ converge to price taking behavior at a rate $1/n$ for every $\varepsilon > 0$, the limit $\widehat{\beta}_{n,\Delta}$ also converges to price taking behavior at a rate $1/n$.

4.2. *Part (b)*

Part (b) derives the asymptotic distribution of perfect equilibrium prices $P_{n,\Delta}(\widehat{\beta}_{n,\Delta})$. $P_{n,\Delta}(\widehat{\beta}_{n,\Delta}) - v(\theta, x(\theta))$ is decomposed as the sum of (a) the sample size effect that is the change in the transaction prices as the number of players increases when players (hypothetically) adopt price taking behavior and (b) the strategic effect that is the difference in the transaction prices between price taking behavior and the equilibrium $\widehat{\beta}_{n,\Delta}$.[40]

For the sample size effect, by the central limit theorem (Lemma A.6), the asymptotic distribution is normal. Since $\widehat{\beta}_{n,\Delta}$ converges to price taking behavior at a rate $1/n$ by Proposition 1(a), the strategic effect is asymptotically negligible. Consequently, the equilibrium price is asymptotically normal.

5. PROOF OF PART (A)

Let $\widehat{\beta}_{n,\Delta}$ be a perfect equilibrium of $\Gamma(n, \mathcal{B}_\Delta, U)$. From Lemma 2.2, there exists a sequence of nondegenerate equilibria $\{\widehat{\beta}_{n,\Delta,\varepsilon}\}_\varepsilon$ of perturbed games of $\{\Gamma(n, \mathcal{B}_\Delta, U, \varepsilon)\}_\varepsilon$ that converge to $\widehat{\beta}_{n,\Delta}$ as $\varepsilon \to 0$. We show that, for any $\varepsilon > 0$, a sequence of equilibrium strategies $\{\widehat{\beta}_{n,\Delta,\varepsilon}\}_{n,\Delta}$ converges to price-taking behavior at a rate $1/n$ as $n \to \infty$ and $\Delta \to 0$. Then, part (a) follows by taking $\varepsilon \to 0$.

To simplify the presentation, we fix ε, take $\Delta = 1/n$, and consider a sequence of equilibria $\{\widehat{\beta}_{n,1/n,\varepsilon}\}_n$ of perturbed games $\{\Gamma(n, \mathcal{B}_{1/n}, U, \varepsilon)\}_n$ as $n \to \infty$. We further simplify the notation by writing $\{\widehat{\beta}_{n,1/n,\varepsilon}\}_n$ as $\{\widehat{\beta}_{n,\varepsilon}\}_n$.

5.1. *The First Order Condition*

Let us take an equilibrium $\widehat{\beta}_{n,\varepsilon}$ of a perturbed game $\Gamma(n, \mathcal{B}_{1/n}, U, \varepsilon)$ and consider the first order condition of buyer i with signal x_i in the gth group with an equilibrium bid $b_{n,\varepsilon,i,m}$.[41] Suppose that buyer i increases the bid for the mth unit from $b_{n,\varepsilon,i,m}$ to $b_{n,\varepsilon,i,m} + d$ while keeping the bids for other units constant. Let $b'_{n,\varepsilon,i} = (b_{n,\varepsilon,i,1}, ..., b_{n,\varepsilon,i,m} + d, ..., b_{n,\varepsilon,i,M})$ be a bid profile after buyer i increases the bid. Then the first order condition is

[40]This decomposition is used also in Hong and Shum (2004).

[41]Since we fix x_i in this part of the analysis, we simplify the notation by writing $b_{n,\varepsilon,i,m}$ instead of $b_{n,\varepsilon,i,m}(x_i)$

$$\begin{aligned}(4)\qquad & \sum_{g',m'} \Pr_g(W_{n,\varepsilon,g',m'}|x_i) \cdot \Big\{\mathbf{E}_g[v(\theta, x_{i,m}) - P(b'_{n,\varepsilon,i}, b_{n,\varepsilon,-i})|x_i, W_{n,\varepsilon,g',m'}]\Big\} \\ \leq\ & \sum_{g',m'} \Pr_g(W_{n,\varepsilon,g',m'}|x_i) \cdot (m-1) \cdot \mathbf{E}_g[\Delta P_{1,g\prime,m'}(b_{n,\varepsilon,i}, b'_{n,\varepsilon,i}, b_{n,\varepsilon,-i})|x_i, W_{n,\varepsilon,g',m'}] \\ & + \Pr_g(\overline{L}_{n,\varepsilon}|x_i) \cdot m \cdot \mathbf{E}_g[\Delta P_2(b_{n,\varepsilon,i}, b'_{n,\varepsilon,i}, b_{n,\varepsilon,-i})|x_i, \overline{L}_{n,\varepsilon}] \\ & + \Pr_g(\underline{L}_{n,\varepsilon}|x_i) \cdot (m-1) \cdot \mathbf{E}_g[\Delta P_3(b_{n,\varepsilon,i}, b'_{n,\varepsilon,i}, b_{n,\varepsilon,-i})|x_i, \underline{L}_{n,\varepsilon}],\end{aligned}$$

where

- $W_{n,\varepsilon,g',m'}$ = the event that $b_{n,\varepsilon,i,m}$ loses but $b_{n,\varepsilon,i,m} + d$ wins the mth unit by surpassing a bid for the m'th unit from player i' in the g'th group (we call that bid $b_{g',m'}$). Note that $W_{n,\varepsilon,g',m'}$ includes the event that $b_{n,\varepsilon,i,m}$ is tied with $b_{g',m'}$ and loses and the event that $b_{n,\varepsilon,i,m} + d$ is tied with $b_{g',m'}$ and wins.
- $\Delta P_{1,g',m'}(b_{n,\varepsilon,i}, b'_{n,\varepsilon,i}, b_{n,\varepsilon,-i})$=an increase in the transaction price in the event of $W_{n,\varepsilon,g',m'}$.
- $\overline{L}_{n,\varepsilon}$=the event that $b_{n,\varepsilon,i,m}$ is the pivotal bid and wins the mth unit.
- $\Delta P_2(b_{n,\varepsilon,i}, b'_{n,\varepsilon,i}, b_{n,\varepsilon,-i})$=an increase in the transaction price in the event of $\overline{L}_{n,\varepsilon}$.
- $\underline{L}_{n,\varepsilon}$=the event that $b_{n,\varepsilon,i,m} + d$ does not win but increases the transaction price by increasing the $G_S \cdot M \cdot n + 1$st highest bid.
- $\Delta P_3(b_{n,\varepsilon,i}, b'_{n,\varepsilon,i}, b_{n,\varepsilon,-i})$=an increase in the transaction price in the event of $\underline{L}_{n,\varepsilon}$.

The first order condition (4) is formally derived in Appendix A.2, along with the definitions of $\Delta P_{1,g',m'}$, ΔP_2, and ΔP_3 (see (14), (17), and (19)). The first order condition says that the benefit from winning the mth unit is less than the costs from paying more. Specifically, the left hand side of (4) is the expected benefit from winning the mth unit at the price $P(b'_{n,\varepsilon,i}, b_{n,\varepsilon,-i})$. The right hand side of (4) accounts for the expected costs from

- an increase in the price $\Delta P_{1,g',m'}$ for the $1, .., m-1$st units when $b_{n,\varepsilon,i,m}$ loses but $b_{n,\varepsilon,i,m} + d$ wins the mth unit.
- an increase in the price ΔP_2 for the $1, ..., m$th units when $b_{n,\varepsilon,i,m}$ is the pivotal bid and wins and $b_{n,\varepsilon,i,m} + d$ increases the price for the $1, ..., m$th unit.
- an increase in the price ΔP_3 for the $1, ..., m-1$st units when $b_{n,\varepsilon,i,m} + d$ does not win the mth unit but increases the price for the $1, ..., m-1$st unit.[42]

In comparison with the first order condition derived in Rustichini, Satterthwaite, and Williams (1994), the first order condition here takes into account of (1) the inference on the state, and (2) the impact of a change of a bid on mth unit on the payments for $1, ..., m-1$th units through the change in the transaction price.[43]

[42]It may be that player i wins less than $m-1$ units in this case, but the cost in this case is lower than the cost when the player wins $m-1$ units.

[43]Here we simplify the presentation by assuming that neither of $b_{n,\varepsilon,i,m}$ and $b_{n,\varepsilon,i,m} + d$ have ties with bids by player i for other units.

5.1.1. *An Example*

Let us illustrate the above first order condition in a following example. Suppose there are one group of buyers ($G_B = 1$) and one group of sellers ($G_S = 1$). Then $G = G_B + G_S = 2$. Assume that there are $n = 2$ players in each group and each player has $M = 2$ units of demand or supply. That is, there are $G \cdot n = 2 \cdot 2 = 4$ players, $G_s \cdot M \cdot n = 1 \cdot 2 \cdot 2 = 4$ units of supply, and $G \cdot M \cdot n = 2 \cdot 2 \cdot 2 = 8$ bids. The pivotal bid is the 4th highest bid out of 8 bids $b_{4:8}$.

Consider buyer 1 in the 1st group with signal $x_1 = (x_{1,1}, x_{1,2})$ who increases the bid for the 1st unit from $b_{2,\varepsilon,1,1}$ to $b_{2,\varepsilon,1,1} + d$ while keeping the bid for the 2nd unit constant. Let $b'_{2,\varepsilon,1} = (b_{2,\varepsilon,1,1} + d, b_{2,\varepsilon,1,2})$ denote a bid profile of buyer 1 after buyer 1 increases the bid for the 1st unit. There are 3 cases to be considered.

- Case 1: $b_{2,\varepsilon,1,1}$ is less than the 4th bid but $b_{2,\varepsilon,1,1} + d$ surpasses the 4th bid. Buyer 1 will be able to win its 1st unit as a result of increasing the bid. There are 4 subcases depending on the characteristics of the old pivotal bid surpassed by $b_{2,\varepsilon,1,1} + d$. One possibility is that the old pivotal bid is a bid by a buyer in the 2nd group for the 1st unit. This event is denoted by $W_{2,\varepsilon,2,1}$ where the last $(2,1)$ of $W_{2,\varepsilon,2,1}$ means that the old pivotal bid is from a bid by a player in the 2nd group for the 1st unit. The expected utility in this subcase is $\Pr_1(W_{2,\varepsilon,2,1}|x_1) \cdot \mathbf{E}_1[v(\theta, x_{1,1}) - P(b'_{2,\varepsilon,1}, b_{2,\varepsilon,-1})|x_1, W_{2,\varepsilon,2,1}]$.
- Case 2: $b_{2,\varepsilon,1,1}$ is the 4th highest bid (and wins the tie). Then,
  - if $b_{2,\varepsilon,1,1} + d$ is still the 4th highest bid, the price increases by $k \cdot d$.
  - if $b_{2,\varepsilon,1,1}+d$ surpasses the 3rd highest bid by other bidders, the price increases by $k \cdot (b_{3:8\setminus\{b_{2,\varepsilon,1,1}\}} - b_{2,\varepsilon,1,1})$.
- Case 3: If $b_{2,\varepsilon,1,1}$ was less than the 5th highest bid but $b_{2,\varepsilon,1,1} + d$ is the new 5th bid, it will increase the transaction price. Since buyer 1 will not be able to win any unit even before increasing the bid, it will not affect buyer 1's utility.

5.2. *Analysis of the First Order Condition*

Let us take a look at the first order condition (4). For (4) to hold, there have to be some g' and m' and a sequence (by taking a subsequence and relabelling the subsequence if necessary) $n = 1, 2, ...$ such that, for all n,

$$\begin{aligned}
(5) \qquad & \mathbf{E}_g[v(\theta, x_{i,m}) - P(b'_{n,\varepsilon,i}, b_{n,\varepsilon,-i})|x_i, W_{n,\varepsilon,g',m'}] \\
\leq\; & (m-1) \cdot \mathbf{E}_g[\Delta P_{1,g',m'}(b_{n,\varepsilon,i}, b'_{n,\varepsilon,i}, b_{n,\varepsilon,-i})|x_i, W_{n,\varepsilon,g',m'}] \\
& + \frac{\Pr_g(\overline{L}_{n,\varepsilon}|x_i)}{\Pr_g(W_{n,\varepsilon,g',m'}|x_i)} \cdot m \cdot \mathbf{E}_g[\Delta P_2(b_{n,\varepsilon,i}, b'_{n,\varepsilon,i}, b_{n,\varepsilon,-i})|x_i, \overline{L}_{n,\varepsilon}] \\
& + \frac{\Pr_g(\underline{L}_{n,\varepsilon}|x_i)}{\Pr_g(W_{n,\varepsilon,g',m'}|x_i)} \cdot (m-1) \cdot \mathbf{E}_g[\Delta P_3(b_{n,\varepsilon,i}, b'_{n,\varepsilon,i}, b_{n,\varepsilon,-i})|x_i, \underline{L}_{n,\varepsilon}].
\end{aligned}$$

Then,

Lemma 5.3: *For such g' and m', we have*

$$\mathbf{E}_g\left[v(\theta, x_{i,m}) - P(b'_{n,\varepsilon,i}, b_{n,\varepsilon,-i})|x_i, W_{n,\varepsilon,g',m'}\right] \to 0 \text{ at a rate } 1/n.$$

Intuition is as follows: when there are no gaps in the distribution of bids, as $n \to \infty$, increasing the bid can change the outcome ($\Pr_g(\underline{L}_{n,\varepsilon,g',m'}|x_i) > 0$) but it will not affect the price ($\mathbf{E}_g[\Delta P_{1,g',m'}|x_i, W_{n,\varepsilon,g',m'}] \to 0$, $\mathbf{E}_g[\Delta P_2|x_i, \overline{L}_{n,\varepsilon}] \to 0$, and $\mathbf{E}_g[\Delta P_3|x_i, \underline{L}_{n,\varepsilon}] \to 0$). Then, the equilibrium expected utility from the mth unit must converge to zero.

We now provide a more detailed proof that deals with a rate of convergence.

Step 1 - We need to deal with ties in the calculation of the RHS of (5). Let us redefine:

- $W_{n,\varepsilon,g',m'}$ = the event that $b_{n,\varepsilon,i,m}$ loses and $b_{n,\varepsilon,i,m} + d$ wins since $b_{g',m'}$ is strictly between $b_{n,\varepsilon,i,m}$ and $b_{n,\varepsilon,i,m} + d$.
- $\overline{L}_{n,\varepsilon}$=the event that $b_{n,\varepsilon,i,m}$ is the pivotal bid.

For $W_{n,\varepsilon,g',m'}$, we exclude the events that $b_{n,\varepsilon,i,m}$ is tied with $b_{g',m'}$ and loses and the event that $b_{n,\varepsilon,i,m} + d$ is tied $b_{g',m'}$ and wins. For $\overline{L}_{n,\varepsilon}$, we include the event of losing the tie at pivotal into $\overline{L}_{n,\varepsilon}$. These redefinitions in the RHS are legitimate since we want to bound the RHS from above.

Step 2 - Sample spacing of bids: Let us define the spacing between the $G_S \cdot M \cdot n - 1$st highest bid and the $G_S \cdot M \cdot n$th highest bid out of $G \cdot M \cdot n$ bids except for $b_{n,\varepsilon,i,m}$ by $T_{G_S \cdot M \cdot n-1:G \cdot M_S \cdot n} = b_{G_S \cdot M \cdot n-1:G \cdot M \cdot n \setminus \{b_{n,\varepsilon,i,m}\}} - b_{G_S \cdot M \cdot n:G_S \cdot M \cdot n \setminus \{b_{n,\varepsilon,i,m}\}}$.

By Lemma A.6, these sample spacings converge to zero. Furthermore, as in the case of uniform distributions,

$$\mathbf{E}_g[T_{G_S \cdot M \cdot n-1:G_S \cdot M \cdot n}|x_i, \overline{L}_{n\,\varepsilon}] \to 0 \text{ and } \mathbf{E}_g[T_{G_S \cdot M \cdot n:G_S \cdot M \cdot n+1}|x_i, \underline{L}_{n\,\varepsilon}] \to 0 \text{ at a rate } 1/n.$$

It follows that increases in the expected price vanish:

$$\begin{aligned} &\mathbf{E}_g[\Delta P_{1,g',m'}(b_{n,\varepsilon,i}, b'_{n,\varepsilon,i}, b_{n,\varepsilon,-i})|x_i, W_{n,\varepsilon,g',m'}] \to 0, \\ &\mathbf{E}_g[\Delta P_2(b_{n,\varepsilon,i}, b'_{n,\varepsilon,i}, b_{n,\varepsilon,-i})|x_i, \overline{L}_{n,\varepsilon}] \to 0, \\ &\text{and } \mathbf{E}_g[\Delta P_3(b_{n,\varepsilon,i}, b'_{n,\varepsilon,i}, b_{n,\varepsilon,-i})|x_i, \underline{L}_{n,\varepsilon}] \to 0 \\ &\text{each at a rate } 1/n. \end{aligned} \tag{6}$$

Step 3 - We now decompose $W_{n,\varepsilon,g',m'}$ into two subevents. Let

- $W'_{n,\varepsilon,g',m'}$ = there is a player (called i') in the g'th group has a bid for m'th unit strictly between $b_{n,\varepsilon,i,m}$ and $b_{n,\varepsilon,i,m} + d$.
- $W''_{n,\varepsilon,g',m'}$ = players other than i and i' have $G_S \cdot M \cdot n - m - m' + 1$ bids above $b_{n,\varepsilon,i,m}$.

Then $W_{n,\varepsilon,g',m'} = W'_{n,\varepsilon,g',m'} \cap W''_{n,\varepsilon,g',m'}$.[44] By conditional independence,

$$\frac{\Pr_g(\overline{L}_{n,\varepsilon}|x_i)}{\Pr_g(W_{n,\varepsilon,g',m'}|x_i)} = \frac{\int \Pr\left(\overline{L}_{n,\varepsilon}|\theta\right) f_g(\theta|x_i)d\theta}{\int \Pr\left(W'_{n,\varepsilon,g',m'}|\theta\right) \Pr\left(W''_{n,\varepsilon,g',m'}|\theta\right) f_g(\theta|x_i)d\theta}.$$

Step 4 - $\Pr(W'_{n,\varepsilon,g',m'}|\theta)$ is the probability that there exists a bid by a player in the g'th group for the m'th unit strictly between $b_{n,\varepsilon,i,m}$ and $b_{n,\varepsilon,i,m}+d$. For any player in the g'th group, the probability that the player will bid for the m'th unit in the interval between for $b_{n,\varepsilon,i,m}$ and $b_{n,\varepsilon,i,m}+d$ is more than the probability that the player does so due to a tremble. Since a player trembles uniformly independently from bids for other units, the probability that the player bids in this interval is more than $d\varepsilon/(\overline{v}+Q) > 0$ [45] and it is independent of n. The number of players in the g'th groups increases at a rate $O(n)$. Consequently,

$$\Pr(W'_{n,\varepsilon,g',m'}|\theta) \simeq O(n). \tag{7}$$

Step 5 - Next consider $\Pr\left(\overline{L}_{n,\varepsilon}|\theta\right)/\Pr\left(W''_{n,\varepsilon,g',m'}|\theta\right)$: Since a player's unit bids are nonincreasing, player i has $m-1$ bids above $b_{n,\varepsilon,i,m}$ and $M-m$ bids below $b_{n,\varepsilon,i,m}$. Therefore,

$$\begin{aligned} \overline{L}_{n,\varepsilon} &= \{\text{players have } G_S \cdot M \cdot n - 1 \text{ bids above } b_{n,\varepsilon,i,m}\} \\ &= \{\text{players other than } i \text{ have } G_S \cdot M \cdot n - m \text{ bids above } b_{n,\varepsilon,i,m}\}. \end{aligned} \tag{8}$$

Also since player i' has m' bids, including $b_{i',m'}$, above $b_{n,\varepsilon,i,m}$ and $M-m'$ bids below $b_{n,\varepsilon,i,m}$,

$$\begin{aligned} & W''_{n,\varepsilon,g',m'} \\ = & \left\{ \begin{array}{c} \text{players other than } i \text{ and } i' \text{ have} \\ G_S \cdot M \cdot n - m - m' + 1 \text{ bids above } b_{n,\varepsilon,i,m} \end{array} \right\}. \end{aligned} \tag{9}$$

Since the distributions of realized bids are uniformly bounded and independent across players conditional on θ, we can show that this ratio is bounded above and below by a constant. That is,[46]

$$\frac{\Pr\left(\overline{L}_{n,\varepsilon}|\theta\right)}{\Pr\left(W''_{n,\varepsilon,g',m'}|\theta\right)} \simeq O(1). \tag{10}$$

Step 6 - $\Pr(\underline{L}_{n,\varepsilon}|\theta)/\Pr(W_{n,\varepsilon,g',m'}|\theta)$ can be analyzed following a method used in

[44] $W_{n,\varepsilon,g',m'}$ implies that there are $G_S \cdot M \cdot n$ bids above $b_{n,\varepsilon,i,m}$. By construction, there are $m-1$ bids above $b_{n,\varepsilon,i,m}$ by player i. $W'_{n,\varepsilon,g',m'}$ implies that there are m' bids above $b_{n,\varepsilon,i,m}$ by player i'. Then, it has to be that players other than i and i' have $G_S \cdot M \cdot n - m - m' + 1$ bids above $b_{n,\varepsilon,i,m}$.

[45] That is, except for the adjustments due to finiteness of the grids.

[46] The details of the calculation are in Appendix A.3.

Step 5 and we have

$$\frac{\Pr(\underline{L}_{n,\varepsilon}|\theta)}{\Pr(W_{n,\varepsilon,g',m'}|\theta)} \simeq O(1). \tag{11}$$

Step 7 - Putting it all together: From (6), (7),(10), and (11), via Lemma 5 of Milgrom (1979), we have (the RHS of (5))$\simeq O(1/n)$. *Q.E.D.*

5.3. *Convergence to Price Taking Behavior*

Can there be a group g' and a unit m' such that (5) does not hold? If there are any such g' and m', from Lemma 5.3, the first order condition will not be satisfied. Therefore, Lemma 5.3 holds for every g and m and the equilibrium bid converges to $\mathbf{E}_g[v(\theta, x_{i,m})|x_i, b_{n,\varepsilon,i,m}$ is pivotal] at a rate $1/n$. It implies that bidding strategies are asymptotically symmetric among players and units. Then, the price will be set by the bid with a pivotal signal. Therefore, $b_{n,\varepsilon,i,m} \to v(\theta(x_{i,m}), x_{i,m})$ with a rate $1/n$.

6. PROOF OF PART (B)

Part (b) derives the asymptotic distribution of equilibrium prices $P_{n,\Delta}(\widehat{\beta}_{n,\Delta})$. First, let $P_{n,\Delta}(\theta_0)$ be the transaction price in $\Gamma(n, \mathcal{B}_\Delta, U)$ when every player bids $v(\theta(x_{i,m}), x_{i,m})$ for $m = 1, ..., M$ in the state θ_0.[47] Then,

$$\begin{aligned} &\sqrt{n}(P_{n,\Delta}(\widehat{\beta}_{n,\Delta}) - v(\theta_0, x(\theta_0))) \\ = \ &\underbrace{\sqrt{n}(P_{n,\Delta}(\theta_0) - v(\theta_0, x(\theta_0)))}_{\text{the sample size effect}} + \underbrace{\sqrt{n}(P_{n,\Delta}(\widehat{\beta}_{n,\Delta}) - P_{n,\Delta}(\theta_0))}_{\text{the strategic effect}}. \end{aligned}$$

The sample size effect measures the change in the transaction price as n increases when every player follows price taking behavior. The strategic effect is the difference between price taking behavior and equilibrium bidding behavior. We now examine these two effects.

6.1. *The Sample Size Effect*

Lemma 6.4: *The sample size effect is asymptotically normal with the distribution given in Proposition 1(b).*

Proof Sketch: We approximate $P_{n,\Delta}(\theta_0)$ by $P_n(\theta_0)$ that is the price when players bid $v(\theta(x_{i,m}), x_{i,m})$ instead of $v(\theta(x_i), x_i)_\Delta$. The rounding difference between $P_{n,\Delta}(\theta_0)$ and $P_n(\theta_0)$ is asymptotically negligible as $n \to \infty$.[48]

When every player adopts price taking behavior, $P_n(\theta_0)$ is determined by the pivotal signal. By Lemma A.6 in Appendix, the distribution of the pivotal signal is

[47] To be precise, it is the transaction price when every player chooses a bid $v(\theta(x_i), x_i)_\Delta$ that is in B_Δ and closest to the price taking behavior $v(\theta(x_i), x_i)$.

[48] Here we approximate price taking behavior in the finite set of possible bids with the one in the continuum set of possible bids. We do not approximate (perfect) equilibrium bidding behavior.

asymptotically normal. The result is then obtained from the application of the delta method. The calculation is in Appendix A.4. *Q.E.D.*

6.2. *The Strategic Effect*

Next we show that the strategic effect converges to zero fast enough so that it does not affect the asymptotic distribution of equilibrium prices.

LEMMA 6.5: *$\sqrt{n}(P_{n,\Delta}(\widehat{\beta}_{n,\Delta}) - v(\theta_0, x(\theta_0)))$ is asymptotically equivalent to the sample size effect in the sense that, for each a, as $n \to \infty$, $\Pr(\sqrt{n}(P_{n,\Delta}(\widehat{\beta}_{n,\Delta}) - v(\theta_0, x(\theta_0))) \leq a) - \Pr(\sqrt{n}(P_{n,\Delta}(\theta_0) - v(\theta_0, x(\theta_0))) \leq a) \to 0$.*

PROOF SKETCH: The first term is the probability that the equilibrium price is less than $v(\theta_0, x(\theta_0))) + a/\sqrt{n}$ and the second term is the probability that the price is less than $v(\theta_0, x(\theta_0))) + a/\sqrt{n}$ when players follow price taking behavior. Since the joint distribution of equilibrium bids converges to price taking behavior at a rate $1/n$, these two probabilities will be asymptotically equal. The calculation is in Appendix A.5. *Q.E.D.*

7. CONCLUSION

This paper considers a model of uniform price double auctions among buyers and sellers in general environments of multiple units of demand and supply, conditionally independent and nonidentically distributed signals with the monotone likelihood ratio property, and interdependent values with common and private value elements, which approximate existing double auctions (such as the NordPool market for electricity contracts). When there are many players, prices approximately reveal the state of the world, inefficiencies are small, and there is an explicit distribution of prices.

APPENDIX A:

A.1. *Proof of Lemma 2.2*

We first construct a perfect equilibrium and then show that it is nondegenerate.

STEP 1 - Existence of a Perfect Equilibrium: Consider perturbed games $\Gamma(n, \mathcal{B}_\Delta, U, 1/q)$ for $q = 1, 2, ...$ where $1/q$ is the probability of trembles. For each q, since the set of possible bids in $\Gamma(n, \mathcal{B}_\Delta, U, 1/q)$ is finite, there exists a mixed strategy equilibrium $\widehat{\beta}_{n,\Delta,1/q}$ of $\Gamma(n, \mathcal{B}_\Delta, U, 1/q)$. Since a player's realized strategy puts a positive probability weight on every bid in B_Δ, the probability of trade between buyers and sellers is positive. Therefore, $\widehat{\beta}_{n,\Delta,1/q}$ is nondegenerate.

Thus, there exists a sequence of nondegenerate mixed strategy equilibria $\{\widehat{\beta}_{n,\Delta,1/q}\}_{q=1,2,...}$ of $\{\Gamma(n, \mathcal{B}_\Delta, U, 1/q)\}_{q=1,2,...}$. Since the set of mixed strategies of $\Gamma(n, \mathcal{B}_\Delta, U, q)$ is compact, there exists a convergent subsequence of $\{\widehat{\beta}_{n,\Delta,1/q}\}_{q=1,2,...}$. Let its subsequence limit be

$\widehat{\beta}_{n,\Delta}$. Since the expected utility function is continuous, by taking the limit of the equilibrium conditions for $\widehat{\beta}_{n,\Delta,1/q}$ in $\Gamma(n, \mathcal{B}_\Delta, U, 1/q)$ as $q \to \infty$, $\widehat{\beta}_{n,\Delta}$ is an equilibrium of $\Gamma(n, \mathcal{B}_\Delta, U)$. That is, $\widehat{\beta}_{n,\Delta}$ is a perfect equilibrium of $\Gamma(n, \mathcal{B}_\Delta, U)$.

Step 2 - It remains to show that $\widehat{\beta}_{n,\Delta}$ is nondegenerate. The idea is as follows: suppose $\widehat{\beta}_{n,\Delta}$ is degenerate. It implies that the minimum equilibrium sell offer of $\widehat{\beta}_{n,\Delta}$ is high enough to deter a buyer with the highest signal from bidding to that level. But then, a seller with a low signal wants to lower the offer from the minimum equilibrium sell offer. We now elaborate the argument below.

Step 3 - Consider the original double auction game $\Gamma(n, \mathcal{B}_\Delta, U)$ and let $\underline{b}_i^s$ be the minimum offer out of all the sell offers in $\widehat{\beta}_{n,\Delta}$. Since $\widehat{\beta}_{n,\Delta}$ is degenerate, no buyer will want to increase the bid to $\underline{b}_i^s$ and trade. That is, a buyer's expected utility from increasing the bid to $\underline{b}_i^s$ is nonpositive. If buyer i with signal x_i increases the bid for the first unit to $\underline{b}_i^s$, buyer i wins the unit with a positive probability and pays $\underline{b}_i^s$.[49] Therefore, for every i, m, and x_i, $\mathbf{E}_g[v(\theta, x_{i,1})|x_i, b_{G_S \cdot M \cdot n : G \cdot M \cdot n} = \underline{b}_i^s, \underline{b}_i^s \text{ wins the tie}] - \underline{b}_i^s \leq 0$. Since this condition holds for a buyer who signal x_i where the signal for the first unit is 1, we have

$$\mathbf{E}_g[v(\theta, 1)|x_i, b_{G_S \cdot M \cdot n : G \cdot M \cdot n} = \underline{b}_i^s, \underline{b}_i^s \text{ wins the tie}] - \underline{b}_i^s \leq 0. \tag{12}$$

Step 4 - Let the maximum bid by buyers be $\overline{b}_i^b$. Since $\widehat{\beta}_{n,\Delta}$ is degenerate, $\overline{b}_i^b < \underline{b}_i^s$.

Step 5 - We now identify sellers who would want to deviate from $\widehat{\beta}_{n,\Delta,1/q}$ for large enough q. Observe

$$\underline{b}_i^s \geq \mathbf{E}_g[v(\theta, 1)|x_i, \underline{b}_i^s = b_{G_S \cdot M \cdot n : G \cdot M \cdot n}, \underline{b}_i^s \text{ wins the tie}] \geq v(0, 1) > v(1, 0).$$

The first inequality is from (12). The second inequality is from Assumption 2.2(b) (v nondecreasing in the state). The third inequality follows from Assumption 2.2(d) (gains from trade).

Then, by Assumption 2.2(a)(v is continuous), 2.1(b)(no gaps), 2.2(b), and 2.2(c) (v strictly increasing in the private signal), there exist $\Delta > 0$ and sellers with a set of signals D with a positive measure such that, for every $x_{i,M} \in D$,

$$v(\theta, x_{i,M}) \leq v(1, x_{i,M}) < \underline{b}_i^s - 6\Delta \text{ for every } \theta \in [0, 1]. \tag{13}$$

Intuitively, a seller with such $x_{i,M} \in D$ has a signal for the Mth unit so low that the value is strictly less than the equilibrium minimum sell offer $\underline{b}_i^s$ for every state θ. Note that D does not depend on q.

Step 6 - We now consider $\Gamma(n, \mathcal{B}_\Delta, U, 1/q)$ with Δ defined to satisfy (13) and

[49] It is because both of $b_{G_S \cdot M \cdot n : G \cdot M \cdot n}$ and $b_{G_S \cdot M \cdot n+1 : G \cdot M \cdot n}$ are $\underline{b}_i^s$.

suppose that seller i with a signal in D lowers the offer for the Mth unit to $\underline{b}_i^s - 2\Delta$ whenever the equilibrium offer is at least $\underline{b}_i^s$.

STEP 7 - We first consider the most probable cases that trade takes place for that seller with a bid $\underline{b}_i^s - 2\Delta$.

- The first reason is a tremble by a buyer.
- The second reason is a buyer's equilibrium bid. By definition of $\overline{b}_i^b$ and Step 4, all buyers will eventually bid less than $\overline{b}_i^b < \underline{b}_i^s$. Therefore, for sufficiently small Δ, for any $\rho > 0$, for q sufficiently large, the probability that one buyer's equilibrium bids are above $\underline{b}_i^s - 2\Delta$ will be at most ρ. In the following, we consider such Δ, ρ, and q.

STEP 8 - We now consider cases that a deviation $\underline{b}_i^s - 2\Delta$ will affect player i's allocation and payment. From Step 7, for sufficiently large q, it is suffice to consider the case where only one buyer bids above $\overline{b}_i^b$ (either due to trembles or by equilibrium behavior) rather than the case where multiple buyers bid above $\overline{b}_i^b$ or some sellers bid below $\underline{b}_i^s$. There are three cases that needs to be considered.

- Case 1: $b_{G_S \cdot M \cdot n : G \cdot M \cdot n \setminus \{b_{i,M}\}} = \underline{b}_i^s$.
- Case 2: $b_{G_S \cdot M \cdot n : G \cdot M \cdot n \setminus \{b_{i,M}\}} = \underline{b}_i^s - \Delta$.
- Case 3: $b_{G_S \cdot M \cdot n : G \cdot M \cdot n \setminus \{b_{i,M}\}} = \underline{b}_i^s - 2\Delta$.

STEP 9 - Case 1: $b_{G_S \cdot M \cdot n : G \cdot M \cdot n \setminus \{b_{i,M}\}} = \underline{b}_i^s$ implies that the $G_S \cdot M \cdot n$th highest bid among all bids other than seller i's Mth offer is $\underline{b}_i^s$. Since there are only $G_S \cdot M \cdot n - 1$ sell offers except for seller i's Mth offer, there is one bid by a buyer not less than $\underline{b}_i^s$. According to Step 7, there are two subcases.

- Subcase 1-1: this buy bid is a result of tremble. The probability of this subcase is $O(1/q)$.
- Subcase 1-2: this buy bid is a strategically chosen bid. The probability of this case is $O(\rho)$.

We now estimate the changes in the expected utility. We consider two subcases depending on whether seller i can make a sale at $\underline{b}_i^s$ or not.

- Subcase 1-1': a seller was not able to sell with offer $\underline{b}_i^s$. By lowering an offer to $\underline{b}_i^s - 2\Delta$, the seller can sell for sure. In both of subcase 1-1 and 1-2, the utility is at least $\underline{b}_i^s - 2\Delta - v(1, x_{i,M})$. From (13), the utility is at least 4Δ.
- Subcase 1-2': a seller was able to sell with offer $\underline{b}_i^s$. By lowering an offer, seller i reduces the transaction price. The reduction in the price is at most 2Δ. Thus the loss is at most -2Δ.

We need to estimate the relative probabilities of subcases 1-1' and 1-2'. When there is one buy bid at $\underline{b}_i^s$, by (2), the probability that the seller with the sell bid $\underline{b}_i^s$ can sell is at most 1/2: for example, if there is another sell offer at $\underline{b}_i^s$, then, there are two units of goods for three bids, and each bid will be allocated a unit with probability 2/3. That is, the probability of sales is 1/3. Intuitively, two sellers compete to fill one buy bid and the probability of sales decreases. Thus, the probability of the second case (sales) is at most 1/2. Consequently, the utility is at least $1/2 \cdot 4\Delta + 1/2(-2\Delta) \geq \Delta$. Therefore, the expected utility is at least $(O(1/q) + O(\rho))\Delta$.

STEP 10 - Case 2: In case 2, there will be one tremble or one strategic buy bid at $\underline{b}_i^s - \Delta$. Since a player chooses a bid from the uniform distribution by assumption, case 1 and case 2 occur with approximately equal probabilities.

The seller is able to sell as a result of lowering the sell bid to $\underline{b}_i^s - 2\Delta$ by hitting the buy bid at this price. The price is at least $\underline{b}_i^s - 2\Delta$. The expected utility is $\underline{b}_i^s - 2\Delta - \mathbf{E}[v(\theta, x_{i,M})|x_i, b_{-i,G_S \cdot M \cdot n:G \cdot M \cdot n} = \underline{b}_i^s - \Delta] > 4\Delta$. The expected utility is at least $(O(1/q) + O(\rho)) \cdot 4\Delta$.

STEP 11 - Case 3: In this case, the seller will be able to sell the good subject to the tie-breaking rule. The expected payoff from the sale is still positive.

STEP 12 - Collecting the results for Case 1-3, we have

$$U_i(x_i, \underline{b}_i^s - 2\Delta, \widehat{\beta}_{n,\Delta,i,1/q}) \geq \Delta \cdot (O(\varepsilon) + O(\rho)) + 4\Delta \cdot (O(\varepsilon) + O(\rho)) > 0.$$

This implies that, for all sufficiently large q, players with a signal in D strictly prefer to deviate from $\widehat{\beta}_{n,\Delta,1/q}$ by lowering offers to $v(1{,}1) - 2\Delta$. *Q.E.D.*

A.2. *Derivation of the First Order Condition*

We derive the first order condition (4).

STEP 1 - We consider the effect of increasing the bid from $b_{n,\varepsilon,i,m}$ to $b_{n,\varepsilon,i,m} + d$ on the allocation and the payment of player i. There are 3 cases that increasing the bid will affect the outcome for player i:

- Case 1: if $b_{n,\varepsilon,i,m}$ is below the pivotal bid and loses but $b_{n,\varepsilon,i,m} + d$ surpasses the pivotal bid, then, player i wins the mth unit as a result of increasing the bid and the price increases.
- Case 2: if $b_{n,\varepsilon,i,m}$ is equal to the pivotal bid and wins, then, increasing the bid by d does not affect the allocation for player i but increases the price.
- Case 3: if $b_{n,\varepsilon,i,m} + d$ does not surpass the pivotal bid but still surpasses the bid just below the pivotal bid, although player i still does not win the mth unit, the price for other units increases.

STEP 2 - Case 1: Let us define[50]

$$W_{n,\varepsilon} = \left\{ \begin{array}{c} b_{n,\varepsilon,i,m} \le b_{G_S \cdot M \cdot n : G \cdot M \cdot n \backslash \{b_{n,\varepsilon,i,m}\}} \le b_{n,\varepsilon,i,m} + d, \\ \text{if } b_{n,\varepsilon,i,m} \text{ is the old pivotal bid with a tie, } b_{n,\varepsilon,i,m} \text{ loses the tie,} \\ \text{and, if } b_{n,\varepsilon,i,m} + d \text{ is the new pivotal bid with a tie, } b_{n,\varepsilon,i,m} + d \text{ wins the tie} \end{array} \right\}.$$

Now let us consider subcases $W_{n,\varepsilon,g',m'}$ for $g' = 1, ..., G$ and $m' = 1, ..., M$ where the old pivotal bid is made by a player i' from the g'th group for the m' th unit. Let us denote this bid by $b_{g',m'}$. Then,

$$W_{n,\varepsilon,g',m'} = \left\{ \begin{array}{c} b_{n,\varepsilon,i,m} \le b_{G_S \cdot M \cdot n : G \cdot M \cdot n \backslash \{b_{n,\varepsilon,i,m}\}} = b_{g',m'} \le b_{n,\varepsilon,g,m} + d, \\ \text{if } b_{n,\varepsilon,i,m} = b_{g',m'},\ b_{n,\varepsilon,i,m} \text{ loses the tie,} \\ \text{and, if } b_{n,\varepsilon,i,m} + d = b_{g',m'},\ b_{n,\varepsilon,i,m} + d \text{ wins the tie} \end{array} \right\}.$$

By construction, $W_{n,\varepsilon} = \bigcup_{g'=1,...,G, m'=1,...,M} W_{n,\varepsilon,g',m'}$.

Let us calculate player i's expected payoff conditional on $W_{n,\varepsilon,g',m'}$. We first look at prices. The old price (the price before player i increases the bid) is, from (1),[51]

$$\begin{aligned} P(b_{n,\varepsilon,i}, b_{n,\varepsilon,-i}) &= (1-k) \cdot \max(b_{n,\varepsilon,i,m}, b_{G_S \cdot M \cdot n+1 : G \cdot M \cdot n \backslash \{b_{n,\varepsilon,i,m}\}}) \\ &\quad + k \cdot b_{G_S \cdot M \cdot n : G \cdot M \cdot n \backslash \{b_{n,\varepsilon,i,m}\}}. \end{aligned}$$

After the increase in the bid, the new price is,[52]

$$\begin{aligned} P(b'_{n,\varepsilon,i}, b_{n,\varepsilon,-i}) &= (1-k) \cdot b_{G_S \cdot M \cdot n : G \cdot M \cdot n \backslash \{b_{n,\varepsilon,i,m}\}} \\ &\quad + k \cdot \min(b_{G_S \cdot M \cdot n-1 : G \cdot M \cdot n \backslash \{b_{n,\varepsilon,i,m}\}}, b_{n,\varepsilon,i,m} + d). \end{aligned}$$

Therefore,

$$\begin{aligned} (14) \qquad & \Delta P_{1,g',m'}(b_{n,\varepsilon,i}, b'_{n,\varepsilon,i}, b_{n,\varepsilon,-i}) = P(b'_{n,\varepsilon,i}, b_{n,\varepsilon,-i}) - P(b_{n,\varepsilon,i}, b_{n,\varepsilon,-i}) \\ = \ & (1-k) \cdot (b_{G_S \cdot M \cdot n : G \cdot M \cdot n \backslash \{b_{n,\varepsilon,i,m}\}} - \max(b_{n,\varepsilon,i,m}, b_{G_S \cdot M \cdot n+1 : G \cdot M \cdot n \backslash \{b_{n,\varepsilon,i,m}\}})) \\ & + k \cdot (\min(b_{G_S \cdot M \cdot n-1 : G \cdot M \cdot n \backslash \{b_{n,\varepsilon,i,m}\}}, b_{n,\varepsilon,i,m} + d) - b_{G_S \cdot M \cdot n : G \cdot M \cdot n \backslash \{b_{n,\varepsilon,i,m}\}}). \end{aligned}$$

Then, the change in the expected utility is,

$$\begin{aligned} (15) \qquad & \Pr_g(W_{n,\varepsilon,g',m'} | x_i) \cdot \{ \mathbf{E}_g \left[v(\theta, x_{i,m}) - P(b'_{n,\varepsilon,i}, b_{n,\varepsilon,-i}) | x_i, W_{n,\varepsilon,g',m'} \right] \\ & -(m-1) \cdot \mathbf{E}_g [\Delta P_{1,g',m'}(b_{n,\varepsilon,i}, b'_{n,\varepsilon,i}, b_{n,\varepsilon,-i}) | x_i, W_{n,\varepsilon,g',m'}] \}. \end{aligned}$$

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[50] We mean "the old pivotal bid" by the pivotal bid before player i increases the bid for the mth unit. "The new pivotal bid" is the pivotal bid after player i increases the bid for the mth unit.

[51] Given that $b_{n,\varepsilon,i,m}$ is less than $b_{G_S \cdot M \cdot n : G \cdot M \cdot n \backslash \{b_{n,\varepsilon,i,m}\}}$, $b_{G_S \cdot M \cdot n : G \cdot M \cdot n \backslash \{b_{n,\varepsilon,i,m}\}}$ is the pivotal bid, since it is the $G_S \cdot M \cdot n$ the highest bids of all bids. The next highest bid is $\max(b_{n,\varepsilon,i,m}, b_{G_S \cdot M \cdot n+1 : G \cdot M \cdot n \backslash \{b_{n,\varepsilon,i,m}\}})$.

[52] After $b_{n,\varepsilon,i,m} + d$ exceeds $b_{G_S \cdot M \cdot n : G \cdot M \cdot n \backslash \{b_{n,\varepsilon,i,m}\}}$, $b_{G_S \cdot M \cdot n : G \cdot M \cdot n \backslash \{b_{n,\varepsilon,i,m}\}}$ is the $G_S \cdot M \cdot n + 1$st highest bid (bid below the pivotal bid). The next highest bid is the min of $b_{G_S \cdot M \cdot n-1 : G \cdot M \cdot n \backslash \{b_{n,\varepsilon,i,m}\}}$ and $b_{n,\varepsilon,i.m} + d$.

Step 3 - Case 2: Let

$$\overline{L}_{n,\varepsilon} = \left\{ \begin{array}{c} b_{G_S \cdot M \cdot n : G \cdot M \cdot n \setminus \{b_{n,\varepsilon,i,m}\}} \leq b_{n,\varepsilon,i,m} \leq b_{G_S \cdot M \cdot n-1 : G \cdot M \cdot n \setminus \{b_{n,\varepsilon,i,m}\}} \\ \text{if } b_{n,\varepsilon,i,m} = b_{G_S \cdot M \cdot n : G \cdot M \cdot n \setminus \{b_{n,\varepsilon,i,m}\}} \text{ or } b_{G_S \cdot M \cdot n-1 : G \cdot M \cdot n \setminus \{b_{n,\varepsilon,i,m}\}}, \\ b_{n,\varepsilon,i,m} \text{ wins the tie} \end{array} \right\}. \tag{16}$$

The price before the bid increase is[53]

$$P(b_{n,\varepsilon,i}, b_{n,\varepsilon,-i}) = (1-k) \cdot b_{G_S \cdot M \cdot n : G \cdot M \cdot n \setminus \{b_{n,\varepsilon,i,m}\}} + k \cdot b_{n,\varepsilon,i,m}.$$

The price after the bid increase is[54]

$$\begin{aligned} P(b'_{n,\varepsilon,i}, b_{n,\varepsilon,-i}) &= (1-k) \cdot b_{G_S \cdot M \cdot n : G \cdot M \cdot n \setminus \{b_{n,\varepsilon,i,m}\}} \\ &\quad + k \cdot \min(b_{n,\varepsilon,i,m} + d, b_{G_S \cdot M \cdot n-1 : G \cdot M \cdot n \setminus \{b_{n,\varepsilon,i,m}\}}). \end{aligned}$$

Consequently, the increase in the price is

$$\Delta P_2 = k(\min(b_{n,\varepsilon,i,m} + d, b_{G_S \cdot M \cdot n-1 : G \cdot M \cdot n \setminus \{b_{n,\varepsilon,i,m}\}}) - b_{n,\varepsilon,i,m}). \tag{17}$$

Then, the expected cost is

$$-\Pr_g(\overline{L}_{n,\varepsilon} | x_i) \cdot m \cdot \mathbf{E}_g \left[\Delta P_2 | x_i, \overline{L}_{n,\varepsilon} \right]. \tag{18}$$

Step 4 - Case 3: Let

$$\underline{L}_{n,\varepsilon} = \left\{ \begin{array}{c} b_{n,\varepsilon,i,m} \leq b_{G_S \cdot M \cdot n+1 : G \cdot M \cdot n \setminus \{b_{n,\varepsilon,i,m}\}} \leq b_{n,\varepsilon,i,m} + d \leq b_{G_S \cdot M \cdot n : G \cdot M \cdot n \setminus \{b_{n,\varepsilon,i,m}\}} \\ \text{if } b_{n,\varepsilon,i,m} + d = b_{G_S \cdot M \cdot n : G \cdot M \cdot n \setminus \{b_{n,\varepsilon,i,m}\}}, b_{n,\varepsilon,i,m} + d \text{ loses a tie} \end{array} \right\}.$$

For the price change, the old price is[55]

$$P(b_{n,\varepsilon,i}, b_{n,\varepsilon,-i}) = (1-k) \cdot b_{G_S \cdot M \cdot n+1 : G \cdot M \cdot n \setminus \{b_{n,\varepsilon,i,m}\}} + k \cdot b_{G_S \cdot M \cdot n : G \cdot M \cdot n \setminus \{b_{n,\varepsilon,i,m}\}}.$$

The new price is[56]

$$P(b'_{n,\varepsilon,i}, b_{n,\varepsilon,-i}) = (1-k) \cdot (b_{n,\varepsilon,i,m} + d) + k \cdot b_{G_S \cdot M \cdot n : G \cdot M \cdot n \setminus \{b_{n,\varepsilon,i,m}\}}.$$

[53]In this case, $b_{n,\varepsilon,i,m}$ is the pivotal bid and $b_{G_S \cdot M \cdot n : G \cdot M \cdot n \setminus \{b_{n,\varepsilon,i,m}\}}$ is the $G_S \cdot M \cdot n + 1$st highest bid among all bids.

[54]$b_{G_S \cdot M \cdot n : G \cdot M \cdot n \setminus \{b_{n,\varepsilon,i,m}\}}$ remains to be the $G_S \cdot M \cdot n + 1$st highest bid among all bids. The next highest bid, the pivotal bid, is the minimum of the increased bid $b_{n,\varepsilon,i,m} + d$ and $b_{G_S \cdot M \cdot n-1 : G \cdot M \cdot n \setminus \{b_{n,\varepsilon,i,m}\}}$.

[55]Since $b_{n,\varepsilon,i,m}$ is below $b_{G_S \cdot M \cdot n+1 : G \cdot M \cdot n \setminus \{b_{n,\varepsilon,i,m}\}}$, $b_{G_S \cdot M \cdot n+1 : G \cdot M \cdot n \setminus \{b_{n,\varepsilon,i,m}\}}$ is the $G_S \cdot M \cdot n + 1$st highest bid and $b_{G_S \cdot M \cdot n : G \cdot M \cdot n \setminus \{b_{n,\varepsilon,i,m}\}}$ is the pivotal bid.

[56]The $G_S \cdot M \cdot n + 1$st highest bid among all bids is now $b_{n,\varepsilon,i,m} + d$ instead of $b_{G_S \cdot M \cdot n+1 : G \cdot M \cdot n \setminus \{b_{n,\varepsilon,i,m}\}}$, and the pivotal bid.is still $b_{G_S \cdot M \cdot n : G \cdot M \cdot n \setminus \{b_{n,\varepsilon,i,m}\}}$.

Therefore, the price change is

$$\Delta P_3 = (1-k)(b_{n,\varepsilon,i,m} + d - b_{G_S\cdot M\cdot n+1:G\cdot M\cdot n\setminus\{b_{n,\varepsilon,i,m}\}}). \tag{19}$$

Then, the expected cost is

$$-\Pr_g(\underline{L}_{n,\varepsilon}|x_i)\cdot(m-1)\cdot \mathbf{E}_g[\Delta P_3|x_i,\underline{L}_{n,\varepsilon}]. \tag{20}$$

STEP 5 - Putting it all together: The first order condition of a buyer is obtained from (15), (18), and (20). *Q.E.D.*

A.3. *Proof of Lemma 5.3*

Given the discussion in the text, it remains to estimate $\frac{\Pr\left(\overline{L}_{n,\varepsilon}|\theta\right)}{\Pr_g\left(W''_{n,\varepsilon,g',m'}|\theta\right)}$.

STEP 1 - We decompose $\overline{L}_{n,\varepsilon}$ depending on the number of bids above $b_{n,\varepsilon,i,m}$ by player i'. From the discussion in the text,

- $\overline{L}_{n,\varepsilon}$ is the event that $G\cdot n-1$ players other than player i have $G_S\cdot M\cdot n-m$ bids above $b_{n,\varepsilon,i,m}$.
- $W''_{n,\varepsilon,g',m'}$ is the event that $G\cdot n-2$ players other than i and i' have $G_S\cdot M\cdot n-m-m'+1$ bids above $b_{n,\varepsilon,i,m}$.

Let

$$I'(m'') = \text{player } i' \text{ has } m'' \text{ bids above } b_{n,\varepsilon,i,m} \text{ and } M-m'' \text{ bids below } b_{n,\varepsilon,i,m},$$

and

$$Y_{n,\varepsilon,g',m'}(m'') = \left\{ \begin{array}{c} G\cdot n-2 \text{ players other than } i \text{ and } i' \text{ have} \\ G_S\cdot M\cdot n-m-m'' \text{ bids above } b_{n,\varepsilon,i,m} \end{array} \right\}. \tag{21}$$

Then, $I'(m'')\cap Y_{n,\varepsilon,g',m'}(m'')$ implies that there are $m''+(G_S\cdot M\cdot n-m-m'') = G_S\cdot M\cdot n-m$ bids above $b_{n,\varepsilon,i,m}$ out of $1+(G\cdot n-2)=G\cdot n-1$ players other than player i. Therefore, $I'(m'')\cap Y_{n,\varepsilon,g',m'}(m'')\subset \overline{L}_{n,\varepsilon}$. Since $m''=0,...,M$,

$$\overline{L}_{n,\varepsilon} = \bigcup_{0\le m''\le M}\{I'(m'')\cap Y_{n,\varepsilon,g',m'}(m'')\}.$$

Also, by inspection, $W''_{n,\varepsilon,g',m'} = Y_{n,\varepsilon,g',m'}(m'-1)$. Therefore,

$$\begin{aligned}\Pr(\overline{L}_{n,\varepsilon}|\theta) &= \sum_{0\le m''\le M}\Pr(I(m'')|\theta)\cdot\Pr(Y_{n,\varepsilon,g',m'}(m'')|\theta)\\ &= \sum_{0\le m''\le M, m''\neq m'-1}\Pr(I(m'')|\theta)\cdot\Pr(Y_{n,\varepsilon,g',m'}(m'')|\theta)\\ &\quad +\Pr(I(m'-1)|\theta)\cdot\Pr(W''_{n,\varepsilon,g',m'}|\theta).\end{aligned}$$

By rewriting the above relation, we have

$$\frac{\Pr(\overline{L}_{n,\varepsilon}|\theta)}{\Pr(W''_{n,\varepsilon,g',m'}|\theta)} = \sum_{0\le m''\le M, m''\neq m'-1} \Pr(I(m'')|\theta)\cdot\frac{\Pr(Y_{n,\varepsilon,g',m'}(m'')|\theta)}{\Pr(W''_{n,\varepsilon,g',m'}|\theta)} + \Pr(I(m'-1)|\theta). \tag{22}$$

Step 2 - An Example: Consider an economy that consists of one group of buyers ($G_B = 1$) and one group of sellers ($G_S = 1$). Then $G = 2$. Each group has $n = 2$ players and each player has $M = 2$ units of demand or supply. Then there are $G \cdot n = 2\cdot 2 = 4$ players and $G_S \cdot n \cdot M = 1\cdot 2\cdot 2 = 4$ units of goods.

Consider the first order condition of buyer 1 with signal x_1 in the 1st group. Take a bid $b_{2,\varepsilon,1,1}$ for the 1st unit ($i = 1$ and $m = 1$). From (8), $\overline{L}_{2,\varepsilon}$ is the event that there are $G_S \cdot M \cdot n - m = 1\cdot 2\cdot 2-1 = 3$ bids above $b_{2,\varepsilon,1,1}$ out of $2\cdot 2-1 = 3$ players other than player 1. Then, $b_{2,\varepsilon,1,1}$ becomes the pivotal bid, consistent with the definition of the event $\overline{L}_{2,\varepsilon}$.

We now consider the above decomposition of $\overline{L}_{2,\varepsilon}$ in terms of $W''_{2,\varepsilon,2,2}$ ($i' = 2, m' = 2$). Let

- $I''(0)$ =Player 2 has 0 bids above $b_{2,\varepsilon,1,1}$. This is the case where player 2's 1st unit bid $b_{2,1}$ is below $b_{2,\varepsilon,1,1}$.
- $I''(1)$ =Player 2 has 1 bids above $b_{2,\varepsilon,1,1}$. This is the case where player 2's 1st unit bid $b_{2,1}$ is above $b_{2,\varepsilon,1,1}$ and 2nd unit bid $b_{2,2}$ is below $b_{2,\varepsilon,1,1}$.
- $I''(2)$ =Player 2 has 2 bids above $b_{2,\varepsilon,1,1}$. This is the case where player 2's 2nd unit bid $b_{2,2}$ is above $b_{2,\varepsilon,1,1}$.

Let

- $Y''_{2,\varepsilon,2,2}(0)$ is the event that that there are $G_S\cdot M\cdot n-m-m'' = 1\cdot 2\cdot 2-1-0 = 3$ bids above $b_{2,\varepsilon,1,1}(x_1)$ out of $G\cdot n-2 = 2\cdot 2-2 = 2$ players other than 1 and 2.
- $W''_{2,\varepsilon,2,2}$ is the event that that there are $G_S\cdot M\cdot n-m-m'+1 = 1\cdot 2\cdot 2-1-2+1 = 2$ bids above $b_{2,\varepsilon,1,1}(x_1)$ out of 2 players other than player 1 and 2.
- $Y''_{2,\varepsilon,2,2}(2)$ is the event that that there are $G_S\cdot M\cdot n-m-m'' = 1\cdot 2\cdot 2-1-2 = 1$ bids above $b_{2,\varepsilon,1,1}(x_1)$ out of 2 players other than 1 and 2.

Then, $\overline{L}_{2,\varepsilon} = \{I''(0) \text{ and } Y''_{2,\varepsilon,2,2}(0)\}\cup\{I''(1) \text{ and } W''_{2,\varepsilon,2,2}\}\cup\{I''(2) \text{ and } Y''_{2,\varepsilon,2,2}(2)\}$.

Step 3 - We now examine (22). By definition, $\Pr(I(m'')|\theta)$ is a probability that there are m'' bids above $b_{n,\varepsilon,i,m}$ and $M - m''$ bids less than $b_{n,\varepsilon,i,m}$ by player i'. It is greater than the probability that there are m'' bids above $b_{n,\varepsilon,i,m}$ when player i' trembles. By construction, player i trembles with probability ε. By assumption, player i' chooses a bid for each unit independently according to uniform distribution in case of trembles. Consequently, the probability that player i chooses a unit bid above $b_{n,\varepsilon,i,m}$ is $(\frac{\overline{v}-b_{n,\varepsilon,i,m}}{\overline{v}+|Q|})$. Since player i chooses bids independently across units, the probability that there are m'' bids above $b_{n,\varepsilon,i,m}$ when player i' trembles is ${}_MP_{m''}(\frac{\overline{v}-b_{n,\varepsilon,i,m}}{\overline{v}+|Q|})^{m''}(1-$

$\frac{\overline{v}-b_{n,\varepsilon,i,m}}{\overline{v}+|Q|})^{M-m''}\varepsilon$. Observe

$$(\frac{\overline{v}-b_{n,\varepsilon,i,m}}{\overline{v}+|Q|})^{m''}(1-\frac{\overline{v}-b_{n,\varepsilon,i,m}}{\overline{v}+|Q|})^{M-m''}\varepsilon \geq (\frac{\overline{v}-v(1,1)}{\overline{v}+|Q|})^{m''}(\frac{|Q|}{\overline{v}+|Q|})^{M-m''}\varepsilon > 0.$$

This lower bound is independent of n for a fixed ε.

From a similar calculation, the probability that the number of bids that player i' above $b_{n,\varepsilon,i,m}$ is different from m'' is strictly positive. Therefore, the probability that there are m'' bids above $b_{n,\varepsilon,i,m}$ and $M-m''$ bids less than $b_{n,\varepsilon,i,m}$ by player i' is bounded above by a constant independent of n.

By combining the discussions in the above 2 paragraphs, we have, for every $1 \leq m'' \leq M$,

$$\Pr(I(m'')|\theta) \simeq O(1). \tag{23}$$

Step 4 - Next consider $\Pr(Y_{n,\varepsilon,g',m'}(m'')|\theta)/\Pr(W''_{n,\varepsilon,g',m'}|\theta)$ for $0 \leq m'' \leq M$. From (21) and (9),

- the numerator is the probability that $G \cdot n - 2$ players have $G_S \cdot M \cdot n - m - m''$ bids above $b_{n,\varepsilon,i,m}$.
- the denominator is the probability that $G \cdot n - 2$ players have $G_S \cdot M \cdot n - m - m' + 1$ bids above $b_{n,\varepsilon,i,m}$.

We show the likelihood ratio between $b_{n,\varepsilon,i,m}$ is the kth highest bid and $b_{n,\varepsilon,i,m}$ is the $k+1$st highest bid is $O(1)$ for each k. Then, since a player has fixed M units of demand and supply, we fix m' [57] and change $m'' = 0, ..., M$, $\Pr(Y_{n,\varepsilon,g',m'}(m'')|\theta)/\Pr(W''_{n,\varepsilon,g',m'}|\theta)$ is $O(1)$ for each m''. We proceed by extending the argument of Fudenberg, Mobius, and Szeidl (2007, Lemma 4) to our situation where players have multiple units of demand and supply from heterogeneous distributions of bids and signals.

Step 5 - First, let us consider the probability that $G \cdot n - 2$ players have $k-1$ bids above $b_{n,\varepsilon,i,m}$. Take player j and let l_j be the number of bids that player j has above $b_{n,\varepsilon,i,m}$. Let

$$C_{k-1} = \{(l_1, ..., l_{G\cdot n-2}) : \text{where each } l_j \in \mathbf{N}_+ \text{ and } \sum_{0 \leq l_j \leq M,\ j=1,...,G\cdot n-2} l_j = k-1\}.$$

That is, C_{k-1} is the set of assignments of the number of units for players $i = 1, ..., G \cdot n - 2$ such that, in total, there are $k-1$ bids above $b_{n,\varepsilon,i,m}$. Let $q_{j,l}$ be the conditional probability that player j will have l bids above $b_{n,\varepsilon,i,m}$ when player j follows an equilibrium strategy. Then, conditional on θ, the probability that there are $k-1$ bids above $b_{n,\varepsilon,i,m}$ is $\sum_{C_{k-1}} q_{1,l_1} \cdots q_{G\cdot n-2, l_{G\cdot n-2}}$.

Step 6 - Next we examine $q_{j,l}/q_{j,l-1}$. It is the likelihood ratio between the event

[57] Recall we take g' and m' fixed and estimate $\frac{\Pr(\overline{L}_{n,\varepsilon}|\theta)}{\Pr_g(W''_{n,\varepsilon,g',m'}|\theta)}$.

that player j places l bids above $b_{n,\varepsilon,i,m}$ over the event that player j places $l-1$ bids above $b_{n,\varepsilon,i,m}$. Since player j's unit bids are nonincreasing, in order for player j places l bids instead of $l-1$ bids, one possibility is that player j's lth bid increases below $b_{n,\varepsilon,i,m}$ to above $b_{n,\varepsilon,i,m}$. The probability that lth bid is above $b_{n,\varepsilon,i,m}$ is more than the probability that it is so by trembles, which is $(\overline{v}-b_{n,\varepsilon,i,m})/(\overline{v}+Q)\cdot\varepsilon$. This implies that the probability that the lth bid is below $b_{n,\varepsilon,i,m}$ is at most $1-(\overline{v}-b_{n,\varepsilon,i,m})/(\overline{v}+Q)\cdot\varepsilon$. Therefore,

$$0 < \frac{(\overline{v}-v(1,1))\varepsilon/(\overline{v}+|Q|)}{1-(\overline{v}-v(1,1))\varepsilon/(\overline{v}+|Q|)} \leq \frac{(\overline{v}-b_{n,\varepsilon,i,m})\varepsilon/(\overline{v}+|Q|)}{1-(\overline{v}-b_{n,\varepsilon,i,m})\varepsilon/(\overline{v}+|Q|)} \leq \frac{\Pr(l\text{th bid is above } b_{n,\varepsilon,i,m})}{\Pr(l\text{th bid is below } b_{n,\varepsilon,i,m})}. \tag{24}$$

This lower bound does not depend on n.

The event that player j places l bids above $b_{n,\varepsilon,i,m}$ includes the event that player j places $l-1$ bids above $b_{n,\varepsilon,i,m}$ and player j's lth bid switches from below $b_{n,\varepsilon,i,m}$ to above $b_{n,\varepsilon,i,m}$. Therefore, from (24), $\frac{(\overline{v}-v(1,1))\varepsilon/(\overline{v}+|Q|)}{1-(\overline{v}-v(1,1))\varepsilon)/(\overline{v}+|Q|)}\cdot q_{j,l-1}\leq q_{j,l}$. This implies that,

$$\frac{(\overline{v}-v(1,1))\varepsilon/(\overline{v}+|Q|)}{1-(\overline{v}-v(1,1))\varepsilon/(\overline{v}+|Q|)} \leq \frac{q_{j,l}}{q_{j,l-1}}. \tag{25}$$

Step 7 - We now consider the probability that there are k bids above $b_{n,\varepsilon,i,m}$. This event contains the event that, starting from $(l_1,....,l_{G\cdot n-2})\in C_{k-1}$, one player increases the number of bids above $b_{n,\varepsilon,i,m}$ by 1. Then, from (25),

$$\frac{(\overline{v}-v(1,1))\varepsilon/(\overline{v}+|Q|)}{1-(\overline{v}-v(1,1))\varepsilon/(\overline{v}+|Q|)} \leq \frac{\Pr(\text{there are } k \text{ bids above } b_{n,\varepsilon,i,m}|\theta)}{\Pr(\text{there are } k-1 \text{ bids above } b_{n,\varepsilon,i,m}|\theta)}.$$

That is, the likelihood ratio is bounded below by a constant independent of n. Applying a similar argument, we see that the likelihood ratio is bounded above by a constant independent of n. Then, by the argument in Step 4, for every $1\leq m''\leq M$,

$$\Pr(Y_{n,\varepsilon,g',m'}(m'')|\theta)/\Pr(W''_{n,\varepsilon,g',m'}|\theta)\simeq O(1). \tag{26}$$

Step 8 - Putting it all together: From (23) and (26), we see that every term of (22) is $O(1)$. Then the conclusion follows. *Q.E.D.*

A.4. *Proof of Lemma 6.4*

Step 1 - Approximation of Prices when Players Adopt Price Taking Behavior: Suppose that $k=0$. Other cases of $0<k\leq 1$ are similar. Let $P_n(\theta_0)$ be the price when every player bids $v(\theta(x_{i,m}),x_{i,m})$. Since $v(\theta(x_{i,m}),x_{i,m})$ is strictly increasing in

$x_{i,m}$, $P_n(\theta_0) = v(\theta(X_{G_S\cdot M\cdot n+1:G\cdot M\cdot n}), X_{G_S\cdot M\cdot n+1:G\cdot M\cdot n})$. Recall, from the text,

$$\begin{aligned} &\sqrt{n}(P_{n,\Delta}(\theta_0) - v(\theta_0, x(\theta_0))) \\ = \; &\underbrace{\sqrt{n}(P_{n,\Delta}(\theta_0) - P_n(\theta_0))}_{(A1)} + \underbrace{\sqrt{n}(P_n(\theta_0) - v(\theta_0, x(\theta_0)))}_{(A2)}. \end{aligned}$$

The difference between $v(\theta(x_i), x_i)_\Delta$ and $v(\theta(x_i), x_i)$ will shrink at the rate $O(1/n)$. Thus, (A1) will not affect the asymptotic behavior of (A). Thus it remains to consider (A2).

STEP 2 - Asymptotic Normality of Order Statistics: Due to Assumption 2.1, the distribution of signals satisfies the assumptions (smoothness and (27)) of Lemma A.6. Thus, using the notation of Lemma A.6,

$$\sqrt{n}\frac{\overline{f}(x(\theta_0))}{v_{n,m}(x(\theta_0))}(X_{G_S\cdot M\cdot n+1:G\cdot M\cdot n} - x(\theta_0))$$

converges to $N(0, 1)$.

STEP 3 - Delta Method: According to the delta method (van der Vaart (2000)), for a sequence of random variables $\{T_n\}$ such that $\sqrt{n}(T_n - \theta_0) \xrightarrow{d} N(0,\sigma^2)$, if ϕ is differentiable, $\sqrt{n}(\phi(T_n) - \phi(\theta_0)) \xrightarrow{d} N(0,\phi'(\theta_0)^2\sigma^2)$. We now apply this result. First, we have

$$\phi(X_{G_S\cdot M\cdot n+1:G\cdot M\cdot n}) = v(\theta(X_{G_S\cdot M\cdot n+1:G\cdot M\cdot n}), X_{G_S\cdot M\cdot n+1:G\cdot M\cdot n}),$$

and $\phi'(x(\theta_0)) = \frac{\partial v(\theta_0,x(\theta_0))}{\partial \theta}\frac{\partial \theta(x(\theta_0))}{\partial x} + \frac{\partial v(\theta_0,x(\theta_0))}{\partial x}$. From Step 2, σ^2 is given by Lemma A.6. Then, we obtain the expression of Proposition 1(b). *Q.E.D.*

A.5. *Proof of Lemma 6.5*

We want to show that for each $a > 0$, $\Pr(\sqrt{n}(P_{n,\Delta}(\widehat{\beta}_{n,\Delta}) - v(\theta_0, x(\theta_0))) \leq a) - \Pr(\sqrt{n}(P_{n,\Delta}(\theta_0) - v(\theta_0, x(\theta_0))) \leq a) \to 0$ as $n \to \infty$. As in the proof of Lemma 6.4, we can approximate $P_{n,\Delta}(\theta_0)$ by $P_n(\theta_0)$. Then, the condition is asymptotically equivalent to

$$\eta_n(a) = \Pr(P_{n,\Delta}(\widehat{\beta}_{n,\Delta}) \leq v(\theta_0, x(\theta_0)) + \frac{a}{\sqrt{n}}) - \Pr(P_n(\theta_0) \leq v(\theta_0, x(\theta_0)) + \frac{a}{\sqrt{n}}) \to 0.$$

We first study $\Pr(P_n(\widehat{\beta}_{n,\Delta}) \leq v(\theta_0, x(\theta_0)) + \frac{a}{\sqrt{n}})$. From Proposition 1(a), $\widehat{\beta}_{n,\Delta}$ converges to price taking behavior at a rate $1/n$. This implies that $\sqrt{n}(P_n(\widehat{\beta}_{n,\Delta}) - v(\theta_0, x(\theta_0))) \to 0$ as $n \to \infty$. Therefore, this term goes to 0 as $n \to \infty$. From Lemma A.6, the second term also goes to 0. *Q.E.D.*

A.6. *A Central Limit Theorem for Order Statistics from $m-$Dependent Heterogeneous Distributions*

In Appendix A.4, we apply Theorem 2.1 of Sen (1968), which we reproduce here:

LEMMA A.6: *Let $X_1, X_2, ...$ be $m-$dependent random variables. Let $F_i(\cdot)$ be the marginal distribution function of X_i and let $F_{i,h}(\cdot,\cdot)$ be the joint distribution function of X_i and X_{i+h}. Let $\overline{F}_n(\cdot) = \frac{1}{n}\sum_{i=1}^{n} F_i(\cdot)$ be the average distribution with the density function $\overline{f}_n$. Let Y_n be the pth sample quantile of $(X_1, ..., X_n)$ and let ξ_n be the pth population quantile of $\overline{F}_n(x)$. Suppose that $f_i(x)$ is continuous in the neighborhood of ξ_n and*

$$0 < inf_{1\leq i\leq n} f_i(\xi_n) \leq sup_{1\leq i\leq n} f_i(\xi_n) < \infty. \tag{27}$$

Let

$$v_{n,m}^2 = \frac{1}{n}\sum_{i=1}^{n} F_i(\xi_n)(1-F_i(\xi_n)) + \frac{2}{n}\sum_{h=1}^{m}\sum_{i=1}^{n-h}\left\{F_{i,h}(\xi_n,\xi_n) - F_i(\xi_n)F_{i+h}(\xi_n)\right\}.$$

Suppose $inf_n v_{n,m} > 0$. Then,

$$\sqrt{n}\frac{\overline{f}_n(\xi_n)}{v_{n,m}(\xi_n)}(Y_n - \xi_n) \to_d N(0,1).$$

Lemma A.6 allows dependent heterogeneous distributions of signals and is more powerful than the central limit theorems used in the previous literature:

- If $X_1, X_2, ...$ are iid from the common distribution F with the density f, Lemma A.6 becomes $\sqrt{n}\frac{f(\xi_n)}{F(\xi_n)(1-F(\xi_n))}(Y_n - \xi_n) \to_d N(0,1)$, which is Theorem 10.3 of David and Nagaraja (2003).
- The result of Balkema and de Haan (1978) used in Hong and Shum (2004) is a special case where the distribution of signals is independent uniform.

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