Multi-Agent Search with Deadline*

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April 15, 2011

Abstract

This paper studies a finite-horizon search problem in which two or more players are involved. Players can agree upon a proposed object by a unanimous decision. Otherwise, search continues until the deadline is reached, at which players receive predetermined fixed payoffs. If players can benefit from the object of search as soon as they agree, the payoff approximates the Nash bargaining solution in the limit as the realizations of payoffs become frequent, and they reach an agreement almost immediately in the limit. If the benefits are received only at the deadline, the limit payoffs are efficient but sensitive to the distribution of possible payoff profiles. In this case the limit expected duration of search relative to the length of time before the deadline is more than a half, and approximates one in the limit as the number of involved players goes to infinity.

1 Introduction

Search problems that arise in reality often have two common features: The decision to "stop" is made by multiple individuals, and there is a predetermined deadline at which a decision has to be made, whatever it is. For example, legislative committee members may need amend a law proposal by a fixed deadline. A department of a university may need to decide who to hire as a junior faculty. A husband and a wife may need to choose an apartment to live in from September 1st.

Although there is a large body of literature on search problems with a single agent over infinite horizon, there are very few works that diverge from these two assumptions.[1] Recent works by Wilson (2001), Albrecht et al. (2010), Compte and Jehiel (2010), and

*Preliminary draft.

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[1]See Rogerson et al. (2005) for a survey.

Cho and Matsui (2011) analyze situations in which two or more individuals are involved in search, in the context of infinite horizon. As far as our knowledge goes, there is apparently no work that considers a multi-agent search problem in the presence of deadline, despite that in the aforementioned examples and almost all other real-life search problems deadlines seem to be present.

One may argue that infinite horizon is a "proxy for long finite horizon." In this paper we show that this may or may not be a valid argument, depending on the situation that we want to analyze. We also show that, depending on the situations, the duration of search does not shrink to zero even when the frequency of search becomes extreme. This is an insight that we do not expect in an infinite horizon model, as in all the works mentioned above, the search ends immediately when it becomes very frequent.

Specifically, we consider a continuous-time model with finite horizon, in which according to a Poisson process a payoff profile is drawn *iid* from some a priori specified set of possible payoff profiles. Upon each arrival of payoff profiles, involved players choose either "accept" or "reject," and if all players accept, the search ends, while otherwise the search continues until the deadline is reached. At the deadline players obtain a fixed a priori specified payoff. In the apartment-search example, this corresponds to the situation where a broker provides information about apartments to the couple, where the market is a sellers' market so that couples cannot negotiate with the broker about, say the prices. Only when both husband and wife agree, they sign the contract, and otherwise they discard this offer and wait for the next offer from the broker (since the market is a sellers' market a discarded apartment will be taken by some other potential tenant).

We show that (an appropriately defined) trembling-hand equilibrium of this game is (essentially) unique. In the equilibrium, we analyze asymptotic behavior in the limit as the realizations of payoffs become frequent. If players can benefit from the object of search as soon as they agree (in the example this means the couple can rent the apartment as soon as they sign the contract), the payoff approximates a point in the Nash set (Maschler et al. (1988), Herrero (1989)) which generalizes the Nash bargaining solution (Nash (1950)) to nonconvex domains. They reach an agreement almost immediately in the limit. If the benefits are received only at the deadline (which corresponds to the situation in which the couples can only rent an apartment only in September), the limit payoffs are efficient but sensitive to the probability distribution of possible payoff profiles. In this case the limit expected duration of search relative to the length of time before the deadline is more than a half, and approximates one in the limit as the number of involved players goes to infinity. We further investigate the structure of equilibrium and relate the forms of equilibria in these two situations.

The multi-agent search problem is similar to the bargaining problem in that both predict what outcome in a prespecified domain is chosen as a consequence of strategic interaction between agents. On the other hand, the search model is distinguished from bargaining in which players have full control of the proposals, as discussed by Compte and Jehiel (2004, 2010). As opposed to the well-known bargaining models in which a player is chosen as a "proposer" and makes an offer to other players, we assume that there is no "proposer" but rather all players are passive. This assumption captures the feature of situations that we would like to analyze. For example neither the husband nor the wife designs and builds their house for themselves, but looks for an apartment which is already built. The distinction between these "active" and "passive" players is also important when we consider the difference between our work and the standard bargaining literature.[2]

Let us relate our work with the literature. First, some recent papers in economics discuss search models in which a group of multiple decision-makers determine when to stop. Wilson (2001), Compte and Jehiel (2010), and Cho and Matsui (2011) consider a search model in which a unanimous agreement is required to accept an alternative, and show that the equilibrium outcome is close to the Nash bargaining solution when players are patient. Compte and Jehiel (2010) analyze the general majority rule to discuss the size of the set of limit equilibrium outcomes. Albrecht et al. (2010) also consider the general majority rule, and show that cutoffs in their strategies are lower for the decision-makers than for a player in the corresponding single-person search model, and the expected duration search is shorter if they are patient. Alpern and Gal (2009), and Alpern et al. (2010) analyze a search model in which an offer is chosen when one of two decision-makers accepts it, unless one of them cast a veto which can be exercised only finite times in the entire search process.[3] Note that all of the above works consider discrete-time infinite-horizon models.[4]

Second, there is an emerging new field on "revision games," which concerns players' interactions over continuous-time with finite horizon, where opportunities to "revise" actions arise according to a Poisson process (Kamada and Kandori (2009), Kamada and Sugaya (2010a,b), Calcagno and Lovo (2010)). An insight from these works is that when action space is finite (as in our case) the set of equilibria is typically small and the solution can be obtained by (appropriately implemented) backwards induction. Another insight is that a differential equation is useful when characterizing the equilibrium. In our paper we follow and extend these methods to characterize equilibrium.

Third, there are several papers discussing continuous-time bargaining models with

[2]Cho and Matsui (2011) present another view: A drawn payoff profile in the search process may be considered as an outcome in a (unique) equilibrium in a bargaining game which is not explicitly described in the model. From this viewpoint, every player is "active" although the "activeness" is hidden in the model.

[3]Recent papers by Moldovanu and Shi (2010), and Bergemann and Välimäki (2011) analyze search problems where each player receives a private signal in every period.

[4]There is large literature of search models in Operations Research. Fewer works, however, consider multi-person decision problems (See Abdelaziz and Krichen (2007) for a survey). Sakaguchi (1978) proposed a two-player continuous-time infinite-horizon stopping game in which opportunities arrive according to the Poisson process as in our model.

finite horizon, in which players have full control of proposals. Ma and Manove (1993) argue continuous-time bargaining with deadline where two players propose alternately, having options to wait with retaining the right of the proposal. They show that players reach an agreement near the deadline as the delay of transmission of the proposal shrinks. Imai and Salonen (2009) consider a similar setting as ours but in which players are selected as a proposer with equal probability. They analyze two limits of equilibrium payoffs: When the opportunities of proposals tend to be frequent, the payoff profile is near the Raiffa bargaining solution, and as the deadline comes close to the present date, the outcome converges to the Nash bargaining solution. Ambrus and Lu (2010a) consider a model of coalitional bargaining in a similar context with ours.[5] They show general uniqueness of the Markov perfect equilibrium, and characterize the core as the limit equilibrium outcomes in convex games.

Finally, the logic behind our result about the Nash bargaining solution is analogous to that of Wilson (2001), Compte and Jehiel (2010), and Cho and Matsui (2011). Our objective in this paper is not to emphasize this perhaps surprising fact that the Nash bargaining solution arises as the consequence of the dynamic interaction, but rather to note the difference that arises when we consider two different situations about the timing of payoff realization or relative importance of patience vs. frequency of search.

The paper is organized as follows. Section 2 provides a model. In Section 3 we provide basic results. In particular, we show that trembling-hand equilibrium takes the form of cutoff strategies, by which we mean each player at each moment of time has a "cutoff" of payoffs below which they reject offers and otherwise accept. In Section 4 we consider the case in which discounting is not so much important relative to the frequency of search (the case corresponding to the situation where the couple can rent an apartment only in September), and in Section 5 we consider the opposite case, that is, the case in which discounting is important relative to the frequency of search (the couples can rent an apartment as soon as they sign the contract). Section 6 concludes. All proofs are provided in Appendix.

2 Model

There are n players who face a search problem (X, x^d) where $X \subset \mathbb{R}^n$ is a set of possible payoff profiles (which we call allocations), and $x^d \in \mathbb{R}^n$ is a disagreement point assumed to be $x^d = (0, \ldots, 0) \in \mathbb{R}^n$. Let $N = \{1, \ldots, n\}$ be the set of players. As a usual notation, a typical player is denoted by i, and the other players are denoted by $-i$. An allocation $x = (x_1, \ldots, x_n) \in X$ is (strictly) *Pareto efficient* in X if there is no allocation $y = (y_1, \ldots, y_n) \in X$ such that $y_i \geq x_i$ for all $i \in N$ and $y_j > x_j$ for some $j \in N$. An

[5] See Ambrus and Lu (2010b) for an application of their model to legislative process.

allocation $x \in X$ is *weakly Pareto efficient* in X if there is no allocation $y \in X$ such that $y_i > x_i$ for all $i \in N$. A probability measure μ is defined on the Borel set of X. We make mild assumptions about X and μ throughout the paper.

Assumption 1. *1. X is a compact subset of $\mathbb{R}^n$.*

2. There exists a profile $(x_1, \ldots, x_n) \in X$ with $x_i > 0$ for all $i \in N$.

3. μ admits a continuous probability density function f whose support is X.[6]

4. f is bounded away from zero, i.e., $\min_{x \in X} f(x) > 0$.

The first assumption is standard. Note that we do not assume convexity of X. The second is an assumption not to make the problem meaningless. The third and the fourth are standard regularity conditions of the probability measure. Let $\overline{x}_i = \max\{x_i \,|\, (x_i, x_{-i}) \in X \text{ for some } x_{-i}\}$ be the maximum payoff attainable for player i in X, $f_H = \max_{x \in X} f(x) < \infty$ be the upper bound, and $f_L = \min_{x \in X} f(x) > 0$ be the lower bound of f in X. Let $\hat{X} = \{v \in \mathbb{R}^n_+ \,|\, x \geq v \text{ for some } x \in X\}$.

Players search within a finite time interval $[-T, 0]$, on which opportunities of agreement arrive according to the Poisson process with arrival rate $\lambda > 0$. At each opportunity $-t \in [-T, 0]$, nature draws an object which is characterized by an allocation $x = (x_1, \ldots, x_n) \in X$ following an identical and independent probability measure μ. After allocation x is provided, each player simultaneously responds by either accepting or rejecting x without a lapse of time. Let $B = \{\text{accept}, \text{reject}\}$ be the set of responses in this search process. If all players accept x, then they reach the agreement and exit the game, obtaining a payoff profile $e^{-\rho(T-t)}x$ where $\rho \geq 0$ is a common discount rate. If at least one of the players rejects the offer, then they continue search. If players reach no agreement before the deadline at time 0, they obtain the disagreement payoff 0.

Let us define strategies in this game. A history at $-t \in [-T, 0]$ consists of

1. a series of time $(t^1, \ldots, t^k)$ when there was an opportunity of Poisson arrival before $-t$, where $k \geq 0$ and $-T \leq -t^1 < -t^2 < \cdots < -t^k < -t$,

2. allocations $x^1, \ldots, x^k$ drawn at opportunities $t^1, \ldots, t^k$ respectively,

3. allocation $x \in X \cup \{\varnothing\}$ at $-t$ ($x = \varnothing$ if no Poisson opportunity arrives at $-t$),

4. acceptance/rejection decision profiles $(b^1, \ldots, b^k)$, where each decision profile b^l ($l = 1, \ldots, k$) is contained in $B^n \setminus \{(\text{accept}, \ldots, \text{accept})\}$.

We denote a history at time $-t$ by $\big((t^1, x^1, b^1), \ldots, (t^k, x^k, b^k), (t, x)\big)$. Let $\tilde{\mathcal{H}}_t$ be the set of all such histories at time $-t$, and $\tilde{\mathcal{H}} = \bigcup_{-t \in [-T,0]} \tilde{\mathcal{H}}_t$. Let

$$\mathcal{H}_t = \big\{\big((t^1, x^1, b^1), \ldots, (t^k, x^k, b^k), (t, x)\big) \in \tilde{\mathcal{H}}_t \,|\, x \neq \varnothing\big\}$$

[6]In particular, this assumption implies that $X \subset \mathbb{R}^n$ is full-dimensional at all $x \in X$.

be the history at time $-t$ when players have an opportunity, and $\mathcal{H} = \bigcup_{-t\in[-T,0]} \mathcal{H}_t$. A (behavioral) *strategy* σ_i of player i is a function from $\mathcal{H}$ to a probability distribution over the set of responses B.[7] Let Σ_i be the set of all strategies of i, and $\Sigma = \prod_{i\in N} \Sigma_i$. For $\sigma \in \Sigma$, let $u_i(\sigma)$ be the expected payoff when players play σ. A strategy profile $\sigma \in \Sigma$ is a *Nash equilibrium* if $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma'_i, \sigma_{-i})$ for all $\sigma'_i \in \Sigma_i$ and all $i \in N$. Let $u_i(\sigma \,|\, h)$ be the expected payoff of player i given that a history $h \in \tilde{\mathcal{H}}$ realized. A strategy profile $\sigma \in \Sigma$ is a *subgame perfect equilibrium* if $u_i(\sigma_i, \sigma_{-i} \,|\, h) \geq u_i(\sigma'_i, \sigma_{-i} \,|\, h)$ for all $\sigma'_i \in \Sigma_i$, $h \in \mathcal{H}$, and all $i \in N$.

A strategy $\sigma_i \in \Sigma_i$ of player i is a *Markov strategy* if for history $h \in \mathcal{H}_t$ at $-t$, $\sigma_i(h)$ depends only on the time $-t$, and the present allocation x_i^k for player i himself. A strategy profile $\sigma \in \Sigma$ is a *Markov perfect equilibrium* if σ is a subgame perfect equilibrium, and σ_i is a Markov strategy for all $i \in N$. We will later show that players play a Markov perfect equilibrium (except for histories in a zero-measure set) if they follow a trembling-hand equilibrium defined below. For $\varepsilon \in (0, 1/2)$, let Σ^ε be the set of strategy profiles which prescribe probability at least ε for both responses in $\{\text{accept}, \text{reject}\}$ after all histories in $\mathcal{H}$. A strategy profile $\sigma \in \Sigma$ is a *trembling-hand equilibrium* if there exists a sequence $(\varepsilon^k)_{k=1,2,\ldots}$ and a sequence of strategy profiles $(\sigma^k)_{k=1,2,\ldots}$ such that $\varepsilon^k > 0$ for all k, $\lim_{k\to\infty} \varepsilon^k = 0$, $\sigma^k \in \Sigma^{\varepsilon^k}$, σ^k is a Nash equilibrium in the ε^k-constrained game with a restricted set of strategies Σ^{ε^k} for all k, and $\lim_{k\to\infty} \sigma^k(h) = \sigma(h)$.[8]

3 Preliminary Results

In this section, we present preliminary results which may be useful in the subsequent sections. First we show that any trembling-hand equilibria yield the same continuation payoff profile after almost all histories at time $-t \in [-T, 0]$. Therefore the trembling-hand equilibrium is essentially unique and Markov.

Proposition 1. *Suppose that σ, σ' are two trembling-hand equilibria. Then $u_i(\sigma \,|\, h) = u_i(\sigma' \,|\, h')$ for almost all histories $h, h' \in \tilde{\mathcal{H}}_t \setminus \mathcal{H}_t$ and all $i \in N$.*

Note that there exist subgame perfect equilibria in which all players reject any allocations, since they move simultaneously.[9] We introduced trembling-hand equilibrium to rule out such trivial equilibria. In an ε-constrained game, a player will optimally accept a favorable allocation for himself, expecting the others to accept it with a small probability.

[7] This function has to be measurable with respect to an appropriate measure on the set of histories (to be defined).

[8] This equilibrium concept is not the normal-form trembling-hand equilibrium but the extensive-form trembling-hand. Although our extensive-form game involves uncountably many nodes, we call this notion a trembling-hand equilibrium defined as a limit of Nash equilibria in which players are unable to take a pure action at any node.

[9] If players respond sequentially, we can show that any subgame perfect equilibrium consists of cutoff strategies. Therefore our results are essentially independent of the timing of responses of players.

A Markov strategy σ_i of player $i \in N$ is a *cutoff strategy* with cutoff v_i if player i who is to respond at time $-t$ accepts allocation $x \in X$ whenever $x_i \geq v_i$, and rejects it otherwise. For a cutoff profile $v = (v_1, \dots, v_n) \in \hat{X}$, we denote the acceptance set by $A(v) = \{x \in X \,|\, x_i \geq v_i \text{ for all } i \in N\}$. The following argument will show that a trembling-hand equilibrium exists that consists of cutoff strategies. Note that a cutoff strategy profile is a trembling-hand equilibrium if it is a Markov perfect equilibrium.

Suppose that all players play Markov strategies σ, and there is no Poisson arrival at time $-t \in [-T, 0]$. Then player i has an expected payoff $u_i(\sigma \,|\, h)$ at $-t$ independent of history $h \in \tilde{\mathcal{H}}_t \setminus \mathcal{H}_t$ played before time $-t$. We denote the continuation payoff at time $-t$ by $v_i(t, \sigma) = e^{-\rho(T-t)} u_i(\sigma \,|\, h)$.

We hereafter fix a cutoff strategy profile σ, and simply denote by $v_i(t)$ the continuation payoff of player i at time $-t$. Let $A(v(t)) \subset X$ be the set of allocations accepted by the cutoff strategies with cutoff profile $v(t) = (v_1(t), \dots, v_n(t))$. We often denote this set by $A(t)$ with a slight abuse of notation. If σ is a subgame perfect equilibrium, $v_i(t)$ is characterized by the following recursive expression: For $i \in N$,

$$\begin{aligned} v_i(t) &= \int_0^t \left(\int_{X \setminus A(\tau)} v_i(\tau) d\mu + \int_{A(\tau)} x_i d\mu \right) \lambda e^{-(\lambda+\rho)(t-\tau)} d\tau \\ &= \int_0^t \left(v_i(\tau) + \int_{A(\tau)} \big(x_i - v_i(\tau)\big) d\mu \right) \lambda e^{-(\lambda+\rho)(t-\tau)} d\tau. \end{aligned} \tag{1}$$

After time $-t$, players find the first Poisson opportunity at time $-\tau$ with probability density $\lambda e^{-\lambda(t-\tau)}$. If the drawn allocation x falls in $A(\tau)$, they reach agreement with x, or otherwise, they continue search with continuation payoffs $v(\tau)$.

Bellman equality (1) implies that $v_i(t)$ is differentiable in t. Multiplying both sides of (1) by $e^{(\lambda+\rho)t}$ and differentiating both sides yield

$$v_i'(t) = -\rho v_i(t) + \lambda \int_{A(t)} \big(x_i - v_i(t)\big) d\mu$$

for $i \in N$. Therefore we obtain the following ordinary differential equation (ODE) of the continuation payoff profile $v(t) = (v_1(t), \dots, v_n(t))$ defined in $\hat{X}$:

$$v'(t) = -\rho v(t) + \lambda \int_{A(t)} \big(x - v(t)\big) d\mu \tag{2}$$

with an initial condition $v(0) = (0, \dots, 0) \in \mathbb{R}^n$. Let us make a couple of observations about ODE (2). This equality implies that the velocity vector $v'(t)$ is parallel to a convex combination between $v(t)$ and the vector from $v(t)$ to the barycenter of the acceptance set $A(t)$ with respect to the probability measure μ. The absolute value the integral on the right hand side is proportional to the weight $\mu\big(A(t)\big)$. If $\rho = 0$, (2) immediately implies $v_i'(t) \geq 0$ for all t and $i \in N$, and $v_i'(t) = 0$ if and only if $\mu\big(A(t)\big) = 0$.

Now we see that a standard argument of ordinary differential equations shows that the ODE (2) has a solution whenever Assumption 1 holds.[10]

Proposition 2. *A trembling-hand equilibrium exists that consists of Markov cutoff strategies.*

By Proposition 1, the solution of ODE (2) is unique. Therefore the game has essentially a unique trembling-hand equilibrium for given X and μ. Let us denote the unique solution of (2) by $v^*(t;\rho,\lambda)$, the continuation payoff profile in the trembling-hand equilibrium. We simply denote this by $v^*(t)$ as long as there is no room for confusion. Next we observe the asymptotic behavior of $v^*(t)$ when the arrival rate λ becomes large. First we show a useful lemma which is directly derived from the form of ODE (2).

Lemma 3. *For any* $\alpha > 0$, $v^*(t;\rho,\alpha\lambda) = v^*(\alpha t;\rho/\alpha,\lambda)$ *if* $-\alpha t \in [-T, 0]$.

Second we note that, under certain assumptions, $v^*(t;\rho,\lambda)$ converges to an allocation v^* independent of t in the limit of $\lambda \to \infty$. This convergence is obvious if $\rho = 0$. Since $v'(t)$ is always nonnegative and $\hat{X}$ is compact, $\lim_{T\to\infty} v^*(T;0,\lambda)$ clearly exists. Since $v^*(T;0,\lambda) = v^*(\lambda T;0,1)$ by Lemma 3, $v^* = \lim_{\lambda\to\infty} v^*(T;0,\lambda)$ also exists, and is independent of T. If $\rho > 0$, existence of $v^* = \lim_{\lambda\to\infty} v^*(t)$ is not obvious since $v_i'(t)$ may be negative. We postpone a proof of existence of this limit in the case of positive ρ until Proposition 12 in Section 5. An intuition of the proof of independence of t is as follows: By Lemma 3, as α gets larger, the marginal change of ρ/α becomes smaller. Since the right hand side of (2) is continuous in v, $v^*(\alpha t;\rho/\alpha,\lambda)$ does not move very much α is large. Therefore $v^*(\alpha t;\rho/\alpha,\lambda)$ is independent of t in the limit.

In the following two sections, we analyze the limit of the continuation payoffs in the equilibrium and the expected duration that the search process continues, as the frequency of Poisson arrival goes to infinity. This limit is considered in two cases: $\rho = 0$ in Section 4, and $\rho > 0$ in Section 5.

4 Asymptotic Results when the Payoffs Realize at the Deadline

In this section, we consider the case in which players receive benefits of the agreement only at the deadline even when they stop searching earlier. Mathematically, we analyze the limit of the continuation payoff profile $\lim_{\lambda\to\infty} v^*(t)$ in the equilibrium when $\rho = 0$. In this case, $v^*(t)$ is characterized as the unique solution of the following ordinary differential

[10]This is because the right hand side of ODE (2) is continuous in v, and $\hat{X}$ is compact. See a textbook of ODE (e.g., Coddington and Levinson (1955, Chapter 1)) for a general discussion.

equation given by letting $\rho = 0$ in ODE (2):

$$v'(t) = \lambda \int_{A(t)} \big(x - v(t)\big) d\mu \tag{3}$$

with an initial condition $v(0) = (0, \ldots, 0) \in \mathbb{R}^n$. We present asymptotic results of the continuation payoffs $v^*(t)$ as the arrival rate λ tends to infinity.

If λ is very large, it is considered that players have so many opportunities that they can find a good allocation. We discuss efficiency of the limit $v^* = \lim_{\lambda\to\infty} v^*(t; 0, \lambda)$ in X to show that the intuition is basically correct. Let us note that we sometimes consider the limit $\lim_{t\to\infty} v^*(t; 0, \lambda)$ with enlarging the time interval $[-T, 0]$. By Lemma 3, $v^* == \lim_{t\to\infty} v^*(t; 0, \lambda)$ for all λ. This implies that we can consider the two limits interchangeably; the limit of $v^*(t)$ as $t \to \infty$, and the limit as $\lambda \to \infty$ for fixed t.

In general, v^* is not necessarily Pareto efficient in X. There is an example of a probability density function f satisfying Assumption 1 in which $v^*(t)$ converges to an allocation that is not strictly Pareto efficient.

Example 1. Let $n = 2$, $X = \big([0, 1/2] \times [3/4, 1]\big) \cup \big([3/4, 1] \times [0, 1/2]\big)$, and f be the uniform density function on X. By the symmetry with respect to the 45 degree line, we must have $v_1^*(t) = v_2^*(t)$ for all t. Therefore $v^* = (1/2, 1/2)$, which is not Pareto efficient in $\hat{X}$.

Note that v^* is weakly Pareto efficient, and that X is a non-convex set in this example. In fact, we will show that v^* is always weakly efficient for general X, and strictly Pareto efficient if X is convex. Furthermore, even if X is not convex, we may say v^* is "generically" Pareto efficient, that is, v^* is Pareto efficient in X for generic f that satisfy Assumption 1.

First, we show that the limit v^* of continuation payoff profile is *weakly* Pareto efficient in $\hat{X}$.

Lemma 4. *The solution $v^*(t)$ of equation* (3) *converges to a weakly Pareto efficient allocation in $\hat{X}$ as $\lambda \to \infty$.*

Next we show that v^* is (strictly) Pareto efficient for generic probability density function f on X. Let $\mathcal{F}$ be the set of density functions that satisfy Assumption 1. We consider a topology on $\mathcal{F}$ defined by the following distance in $\mathcal{F}$: For $f, \tilde{f} \in \mathcal{F}$,

$$\left| f - \tilde{f} \right| = \sup_{x \in X} \left| f(x) - \tilde{f}(x) \right|.$$

Proposition 5. *The set*

$$\{f \in \mathcal{F} \,|\, v^* \textit{ is Pareto efficient in } X\}$$

is open and dense in $\mathcal{F}$.

This proposition only shows that v^* is efficient for generic f. However, if X is convex, then v^* is efficient for all f.

Proposition 6. *Suppose that X is a convex set. Then v^* is Pareto efficient in X.*

Weak Pareto efficiency leads to an observation that players reach an agreement almost surely if t is very large. Let $p(t)$ be the probability that players reach an agreement in the equilibrium before the deadline given no agreement at time $-t$. Then the continuation payoffs $v^*(t)$ must fall in the set $\{p(t)v \,|\, v \in \hat{X}\}$, which implies $v^*(t)/p(t) \in \hat{X}$. We have $v_i^*(t) > 0$ for all $t > 0$ and $i \in N$ since $v_i^*(t)$ is nondecreasing and $v_i^{*\prime}(0) > 0$ by equation 3. Since there is a positive probability that no opportunity arrives before the deadline, $p(t)$ is smaller than one. Therefore $v^*(t)/p(t) \in \hat{X}$ strictly Pareto dominates $v^*(t)$. This implies $\lim_{t\to\infty} p(t) = 1$ since by Lemma 4 v^* is weakly Pareto efficient in $\hat{X}$. Now we show the following proposition:

Proposition 7. *The probability of agreement before the deadline converges to one as the time interval becomes large.*

In Propositions 5, 6, we showed that $v^*(t)$ almost always converges to the Pareto frontier of X. We consider the inverse problem. For any Pareto efficient allocation w in X which is not at the edge of the Pareto frontier,[11] we show that one may find density f which satisfies Assumption 1 such that the limit of the solution $v^*(t)$ of equation (3) is w.

Proposition 8. *Suppose that w is a Pareto efficient allocation in X such that $w_i > 0$ for all $i \in N$, and w is not located at the edge of the Pareto frontier. Then there exists a probability measure μ satisfying Assumption 1 such that the equilibrium continuation payoff profile $v^*(t)$ converges to w as λ tends to infinity.*

In the proof, we construct a probability density function f to have a large weight near $w \in X$, and show that the limit continuation payoffs is w if there is a sufficiently large weight near w. Note that this claim is not so obvious as it seems. Indeed, we will see in Section 5 that the limit is independent of density f if there is a positive discount rate $\rho > 0$, as long as Assumption 1 holds.

In the rest of this section, we make assumptions on regularity of X around v^* in addition to Assumption 1.

Assumption 2. *1. The limit v^* is Pareto efficient in X.*

2. The Pareto frontier of X is smooth in a neighborhood of v^*.

[11] We formally define this property in the proof given in Appendix A.6.

3. *For the unit normal vector $\alpha \in \mathbb{R}^n_+$ at v^*, $\alpha_i > 0$ for all $i \in N$.*

4. *There exist $\varepsilon > 0$ and $\eta > 0$ such that $\{x \in \mathbb{R}^n_+ \mid |v^* - x| \leq \varepsilon, \alpha \cdot (x - v^*) \leq -\eta\}$ is contained in X, where "$\cdot$" denotes the inner product in $\mathbb{R}^n$.*[12]

The next lemma shows that $v^*(t)$ converges to v^* with a speed of order $(\lambda t)^{-1/n}$. Let $\alpha \in \mathbb{R}^n_+$ be the unit normal vector of the Pareto frontier of X at v^*.

Lemma 9. *Suppose that Assumption 2 holds. As either $t \to \infty$ or $\lambda \to \infty$, $(v^*_i - v^*_i(t))(\lambda t)^{1/n}$ converges to a positive and finite value which is written as*

$$\lim_{t\to\infty}(v^*_i - v^*_i(t))(\lambda t)^{\frac{1}{n}} = \Big(\frac{n+1}{f(v^*)n^{n+1}}\prod_{j\neq i}\frac{\alpha_j}{\alpha_i}\Big)^{\frac{1}{n}}$$

for all $i \in N$.

In the present model with finite λ, it always takes positive time for players to reach an agreement. Then it may be interesting to consider the expected duration of search before the agreement. The next proposition shows that if the Pareto frontier of X is smooth and $v^*(t)$ converges to the Pareto frontier, then the expected duration of the search process in the time interval $[-T, 0]$ is $(n^2T)/(n^2 + n + 1)$.

Proposition 10. *Suppose that Assumption 2 holds. Then the expected duration of search in the equilibrium is $\dfrac{n^2}{n^2+n+1}T$.*

The proposition implies that a positive fraction of time is spent on search, but players do not spend all the time they have. This is a result of a tradeoff between two effects: On one hand, players do not want to wait too much, as doing so would result in disagreement, or the agreement in low payoffs which they would receive if it takes place close to the deadline. On the other hand, players do not want to stop their search immediately, as they are very picky when the deadline is very far away. Being picky is optimal for the players, as there is no discounting. In the next section, we will see that if there is a significant effect of discounting we expect the search to end immediately. Putting it another way, there are two effects of making the arrival rates large. One is that there exist many realizations of payoffs in a given time interval, which makes the possibility of agreement more likely. The other is that as the result of the increase of opportunities, players expect more opportunities in the future, which makes them pickier.

The solution of the expected duration provided in the theorem implies that, if only two players are involved in search, the expected duration is $\frac{4}{7}T$, and it monotonically increases to approach T as n gets larger. We do not think the reason for this is a simple one, which would say that if there are many people it is difficult to all agree on something. The result is rather the consequence of two distinct effects explained above.

[12] Only the second assumption is necessary if X is convex.

5 Asymptotic Results when the Payoffs Realize upon Agreement

In this section, we consider the limit of the continuation payoffs $v^*(t)$ as $\lambda \to \infty$ with discount rate $\rho > 0$ fixed. This is the case in which players receive benefits of the agreement as soon as they agree. Let us revisit the ordinary differential equation (2) that characterizes the equilibrium continuation payoff profile $v^*(t)$ as its unique solution:

$$v'(t) = -\rho v(t) + \lambda \int_{A(t)} \big(x - v(t)\big) d\mu \tag{2}$$

with an initial condition $v(0) = (0, \dots, 0)$.

If λ is large, the right hand side of equation (2) is approximated by the right hand side of equation (3). Therefore, $v^*(t)$ is close to the solution of equation (3) in the case of $\rho = 0$, for λ large relative to ρ. This resemblance of trajectories holds until $v^*(t)$ approaches the boundary of $\hat{X}$. Let v^0 be the limit of the solution of equation (3). Note that v^0 is weakly Pareto efficient by Lemma 4. In the extreme case, we show that $v^*(t)$ approaches v^0 arbitrarily closely.

Proposition 11. *For all $\varepsilon > 0$, there exists $\bar{\lambda} > 0$ such that for all $\lambda \geq \bar{\lambda}$,*

$$\big|v^0 - v^*(t)\big| \leq \varepsilon \quad \textit{for some } t.$$

Before analyzing $v^* = \lim_{\lambda \to \infty} v^*(t; \rho, \lambda)$, let us consider another limit $v^*(\infty; \rho, \lambda) = \lim_{t \to \infty} v^*(t; \rho, \lambda)$. Since the right hand side of equation (2) is not proportional to λ, these two limits do not coincide for positive $\rho > 0$. If the limit $v^*(\infty)$ exists, this must satisfy

$$\rho v^*(\infty) = \lambda \int_A \big(x - v^*(\infty)\big) d\mu \tag{4}$$

where $A = \{x \in X \,|\, x \geq v^*(\infty)\}$. For $\rho > 0$, equality (4) shows $\mu(A) > 0$. Equality (4) also implies that $v^*(\infty)$ is parallel to the vector from $v^*(\infty)$ to the barycenter of A.

We assume simplifying conditions until the end of this section.[13]

Assumption 3. *The boundary of X is smooth, and every component of the normal vector at any boundary point of X is strictly positive.*

Now suppose that λ is very large. Then $\mu(A)$ must be very small, which means that $v^*(\infty)$ is very close to the Pareto frontier of X. By Assumption 1, the density f is approximately uniform in A if A is a very small set. To obtain an intuition, suppose that A is a small n-dimensional pyramid. In such a case, the vector in the right hand side of

[13] We can show basically the same results without this assumption. We avoid complications derived from the indeterminacy of a normal vector on the boundary of X.

equality (4) is parallel to the vector from $v^*(\infty)$ to the barycenter of the Pareto frontier of A.

If v is the barycenter of A, it turns out that the boundary of A at x is tangent to the hypersurface defined by $\prod_{i \in N} x_i$ is constant. Therefore the Nash product is stable at x. We refer to such a point as a *Nash point*, and the set of all Nash points as the *Nash set* of $(X, 0)$ (Maschler et al. (1988), Herrero (1989)). The Nash set contains all local maximizers and all local minimizers of the Nash product. If X is convex, there exists a unique Nash point, which is called the Nash bargaining solution.

The above observation leads to the next proposition.

Proposition 12. *Suppose that any Nash point is isolated in X. Then the limit $v^* = \lim_{\lambda \to \infty} v^*(t)$ exists and belongs to the Nash set of the problem $(X, 0)$. In particular, if X is convex, this limit coincides with the Nash bargaining solution of $(X, 0)$.*

Therefore, the trajectory of $v^*(t)$ for very large λ starts at $v^*(t) = 0$, approaches v^0, and moves along the Pareto frontier until reaching a point close to a Nash point. When X is convex, the result of convergence to the Nash bargaining solution is basically the same as Imai and Salonen (2009) who consider a bargaining model in which players are selected as a proposer with even probability in every opportunity.

Remark 1. Although we consider a finite-horizon model, the threatening power of disagreement at the deadline is quite weak for large λ, since the relative effect of discounting grows as λ becomes large. In fact, the asymptotic result of Proposition 12 is essentially the same as those shown by Wilson (2001), Compte and Jehiel (2010), and Cho and Matsui (2011), all of whom consider the limit as the discount factor goes to one in discrete-time infinite-horizon models.

Nash (1953) himself provided a characterization of the Nash bargaining solution by introducing a static demand game with perturbation.[14] Suppose that X is convex. The basic demand game is a one-shot strategic-form game in which each player i calls a demand $x_i \in \mathbb{R}_+$. Players obtain $x = (x_1, \ldots, x_n)$ if $x \in X$, or 0 otherwise. In the perturbed demand game, players fail to obtain $x \in X$ with a positive probability if x is close to the Pareto frontier. Under certain conditions, he showed that the Nash equilibrium of the perturbed demand game converges to the Nash bargaining solution as the perturbation vanishes.

Let us compare the perturbed Nash demand game with the multi-agent search model with discrete-time and infinite-horizon. Suppose that a probability measure μ on X is given in the infinite-horizon search model with discount factor $0 < \delta < 1$. Let $p(x) = \mu(\{y \in X \mid y \geq x\})$ be the probability that players come across an allocation which Pareto dominates or equals $x \in X$ in a period. Let $x \in X$ be the cutoff profile in the stationary

[14] We here follow a slightly modified game considered by Osborne and Rubinstein (1990, Section 4.3). Despite the difference, the model conveys the same insight as the original.

subgame perfect equilibrium. Then player i's continuation payoff is x_i, and he loses $(1-\delta)x_i$ if players cannot agree in a period. Therefore his expected loss from discounting is $(1-p(x))(1-\delta)x_i$. If we take a probability $P(x)$ to satisfy $1-P(x) = (1-p(x))(1-\delta)$, player i loses the same amount when $x \in X$ is demanded in the perturbed demand game where probability of successful agreement is $P(x)$.

The key tradeoff in this game, the attraction to larger demands or the fear of failure, is parallel to that in the multi-agent search, to be pickier or to avoid loss from discounting.

Now we consider the duration of search in the equilibrium. In contrast to Proposition 10 in the case of $\rho = 0$, we show that players reach an agreement almost immediately if λ is very large.

Proposition 13. *For all $-t \in (-T, 0]$ and all $\varepsilon > 0$, there exists $\bar{\lambda} > 0$ such that the probability that players reach an agreement before time $-t$ in the equilibrium is larger than $1 - \varepsilon$ for all $\lambda \geq \bar{\lambda}$.*

6 Conclusion

We investigated an n-person search problem with deadline in which an agreement opportunity arrives according to the Poisson process, and the drawn object is adopted by a unanimous acceptance. If players cannot reach any agreement before the deadline, they obtain a predetermined payoff profile. We analyze the limit of the equilibrium continuation payoffs as objects are drawn more and more frequently. First, if players receive payoffs at the deadline, the continuation payoffs are efficient but sensitive to the distribution of objects in search. The limit expected duration of search is longer than a half of the length of the give time interval, increases in n, and converges to one as n goes to infinity. Second, if players receive payoffs immediately after they agree, the continuation payoffs converges to a Nash point, and the duration of search is almost zero in the limit.

Appendix

A.1 Proof of Proposition 1

Suppose that there exists at least one trembling-hand equilibrium. We show that the continuation payoff of player i at time $-t$ is unique in any trembling-hand equilibrium.

Let $\overline{v}_i^{\varepsilon}(t)$ and $\underline{v}_i^{\varepsilon}(t)$ be the supremum and the infimum of the set of continuation payoffs $u_i(\sigma \,|\, h)$ of player i after all histories $h \in \tilde{\mathcal{H}}_t \setminus \mathcal{H}_t$ at time $-t$ in all Nash equilibria σ in the ε-constrained game. Let $w_i^{\varepsilon}(t) = \overline{v}_i(t) - \underline{v}_i(t)$, $\bar{w}^{\varepsilon}(t) = \max_{i \in N} w_i^{\varepsilon}(t)$. We will show that $\bar{w}^{\varepsilon}(t) = 0$ for all $\varepsilon > 0$ for any time $-t \in [-T, 0]$. Note that $\bar{w}^{\varepsilon}(0) = 0$ for all ε.

Let us consider the ε-constrained game. If player i accepts an allocation $x \in X$ at time $-t$, he will obtain x_i with probability at least ε^{n-1}. Accepting x is a dominant action of player i if the following inequality holds:

$$\varepsilon^{n-1} x_i + (1 - \varepsilon^{n-1}) \underline{v}_i^\varepsilon(t) > \overline{v}_i^\varepsilon(t),$$

which implies,

$$x_i > \overline{v}_i^\varepsilon(t) + \frac{1 - \varepsilon^{n-1}}{\varepsilon^{n-1}} w_i^\varepsilon(t).$$

Let $\tilde{v}_i^\varepsilon(t) = \overline{v}_i^\varepsilon(t) + \frac{1-\varepsilon^{n-1}}{\varepsilon^{n-1}} w_i^\varepsilon(t)$. Then $\tilde{v}_i^\varepsilon(t) - \underline{v}_i^\varepsilon(t) = \frac{1}{\varepsilon^{n-1}} w_i^\varepsilon(t)$.

Let $X_i^1(t) = \{x \in X \,|\, x_i \geq \tilde{v}_i^\varepsilon(t)\}$, $X_i^m(t) = \{x \in X \,|\, \underline{v}_i^\varepsilon(t) \leq x_i \leq \tilde{v}_i^\varepsilon(t)\}$, and $X_i^0(t) = \{x \in X \,|\, x_i \leq \underline{v}_i^\varepsilon(t)\}$. Any player i accepts $x \in X_i^1(t)$ and rejects $x \in X_i^0(t)$ with probability $1 - \varepsilon$ after almost all histories at time $-t$. Note that $X = \big(\bigcup_{j \in N} X_j^m(t)\big) \cup \big(\bigcup_{(s_1,\ldots,s_n) \in \{0,1\}^n} \bigcap_{j \in N} X_j^{s_j}(t)\big)$ (although not disjoint). Then

$$\begin{aligned}
\overline{v}_i^\varepsilon(t) \leq \int_0^t \Big(&\sum_{j \in N} \int_{X_j^m(\tau)} \overline{x}_i d\mu \\
&+ \sum_{(s_1,\ldots,s_n) \in \{0,1\}^n} \int_{\bigcap_{j \in N} X_j^{s_j}(\tau)} \big((1-\varepsilon)^{\sum_{j \in N} s_j} \varepsilon^{\sum_{j \in N}(1-s_j)} x_i \\
&\quad + (1 - (1-\varepsilon)^{\sum_{j \in N} s_j} \varepsilon^{\sum_{j \in N}(1-s_j)}) \overline{v}_i^\varepsilon(\tau)\big) d\mu \Big) \lambda e^{-(\lambda+\rho)(t-\tau)} d\tau,
\end{aligned}$$

and

$$\begin{aligned}
\underline{v}_i^\varepsilon(t) \geq \int_0^t \Big(&\sum_{(s_1,\ldots,s_n) \in \{0,1\}^n} \int_{\bigcap_{j \in N} X_j^{s_j}(\tau)} \big((1-\varepsilon)^{\sum_{j \in N} s_j} \varepsilon^{\sum_{j \in N}(1-s_j)} x_i \\
&\quad + (1 - (1-\varepsilon)^{\sum_{j \in N} s_j} \varepsilon^{\sum_{j \in N}(1-s_j)}) \underline{v}_i^\varepsilon(\tau)\big) d\mu \Big) \lambda e^{-(\lambda+\rho)(t-\tau)} d\tau.
\end{aligned}$$

Therefore $w_i^\varepsilon(t) = \overline{v}_i(t) - \underline{v}_i(t)$ is estimated as follows:

$$
\begin{aligned}
w_i^\varepsilon(t) \le &\int_0^t \Big(\sum_{j\in N} \int_{X_j^m(\tau)} \overline{x}_i d\mu \\
&+ \sum_{(s_1,\dots,s_n)\in\{0,1\}^n} \int_{\bigcap_{j\in N} X_j^{s_j}(\tau)} \big(1-(1-\varepsilon)^{\sum_{j\in N} s_j} \varepsilon^{\sum_{j\in N}(1-s_j)}\big) w_i^\varepsilon(\tau) d\mu \Big) \lambda e^{-(\lambda+\rho)(t-\tau)} d\tau \\
\le &\int_0^t \Big(\sum_{j\in N} f_H \overline{x}_i \frac{1}{\varepsilon^{n-1}} w_j^\varepsilon(\tau) \prod_{k\ne j} \overline{x}_k \\
&+ \sum_{(s_1,\dots,s_n)\in\{0,1\}^n} \int_X \big(1-(1-\varepsilon)^{\sum_{j\in N} s_j} \varepsilon^{\sum_{j\in N}(1-s_j)}\big) w_i^\varepsilon(\tau) d\mu \Big) \lambda e^{-(\lambda+\rho)(t-\tau)} d\tau \\
\le &\int_0^t \Big(\sum_{j\in N} f_H \max_{k\in N}\{\overline{x}_k\} \frac{1}{\varepsilon^{n-1}} \prod_{k\ne j} \overline{x}_k \\
&+ \sum_{(s_1,\dots,s_n)\in\{0,1\}^n} \big(1-(1-\varepsilon)^{\sum_{j\in N} s_j} \varepsilon^{\sum_{j\in N}(1-s_j)}\big) \Big) \overline{w}^\varepsilon(\tau) \lambda e^{-(\lambda+\rho)(t-\tau)} d\tau.
\end{aligned}
$$

Since the above inequality holds for all $i \in N$, there exists a constant $L > 0$ such that the following inequality holds:

$$\overline{w}^\varepsilon(t) \le \int_0^t L\overline{w}^\varepsilon(\tau) e^{-(\lambda+\rho)(t-\tau)} d\tau.$$

Let $W^\varepsilon(t) = \int_0^t \overline{w}^\varepsilon(\tau) e^{(\lambda+\rho)\tau} d\tau$. Then

$$
\begin{aligned}
W^{\varepsilon\prime}(t) &= \overline{w}^\varepsilon(t) e^{(\lambda+\rho)t} \\
&\le W^\varepsilon(t).
\end{aligned}
$$

Therefore we have $\frac{d}{dt}\big(W^\varepsilon(t)e^{-t}\big) \le 0$, which implies $\overline{w}^\varepsilon(t) \le W^\varepsilon(t) \le 0$ since $\overline{w}^\varepsilon(0) = 0$. Hence, $\overline{w}^\varepsilon(t) = 0$ for all t and all $\varepsilon > 0$. Any trembling-hand equilibria yield the same continuation payoffs after almost all histories at time $-t \in [-T, 0]$.

A.2 Proof of Lemma 3

Let $w^*(t;\rho,\lambda) = v^*(\alpha t; \rho/\alpha, \lambda)$. By equation (3), $w^*(t;\rho,\lambda)$ is the solution of

$$w'(t/\alpha) = -\frac{\rho}{\alpha} w(t/\alpha) + \lambda \int_{A(w(t/\alpha))} \big(x - w(t/\alpha)\big) d\mu,$$

which is equivalent to

$$\frac{d}{d\tau} w(\tau) = -\rho w(\tau) + \alpha\lambda \int_{A(w(\tau))} \big(x - w(\tau)\big) d\mu$$

where $\tau = t/\alpha$, $w(t) = v(\alpha t)$. The solution of the second equation is $w^*(t;\lambda) = v^*(t;\alpha\lambda)$. Therefore we have $v^*(t;\rho,\alpha\lambda) = v^*(\alpha t;\rho/\alpha,\lambda)$ as desired.

A.3 Proof of Lemma 4

By Lemma 3, it suffices to show that $v^* = \lim_{t\to\infty} v^*(t)$ is weak Pareto efficient.

Let $A = \{x \in X \mid x \geq v^*\}$. Suppose that there exists $x = (x_1, \dots, x_n) \in A$ such that $x_1 > v_1^*, \dots, x_n > v_n^*$. By Assumption 1, there exists a closed subset $Y \subset A$ such $\mu(Y) > 0$, and $y_i = \inf\{x_i \mid (x_1, \dots, x_n) \in Y\} > x_i$. By ODE (3), we have

$$\begin{aligned} v_i^{*\prime}(t) &= \lambda \int_{A(t)} \big(x_i - v_i^*(t)\big) d\mu \\ &\geq \lambda \int_W (y_i - v_i^*) d\mu \\ &= \lambda (y_i - v_i^*) \mu(Y) > 0. \end{aligned}$$

This inequality implies that $v_i^*(t) \geq \lambda(y_i - v_i^*)\mu(Y)t + v_i^*(0)$, which tends to infinity as $t \to \infty$. This contradicts the fact that $v_i^*(t)$ is convergent. Hence x is weakly Pareto efficient in $\hat{X}$.

A.4 Proof of Proposition 5

Let $v^*(t; f)$ be the solution of ODE (3) for density $f \in \mathcal{F}$, and $v^*(f) = \lim_{\lambda\to\infty} v^*(t; f) = \lim_{t\to\infty} v^*(t; f)$.

First we show that the set is open, i.e., for all $f \in \mathcal{F}$ with $v^*(f)$ Pareto efficient, $\varepsilon > 0$, and a sequence $f_k \in \mathcal{F}$ ($k = 1, 2, \dots$) with $|f_k - f| \to 0$ ($k \to \infty$), there exist $\delta > 0$ and $\bar{k}$ such that

$$|v^*(f_k) - v^*(f)| \leq \varepsilon$$

for all $k \geq \bar{k}$.

Since $\lim_{t\to\infty} v^*(t; f) = v^*(f)$, for all $\delta > 0$ there exists $\bar{t} > 0$ such that $|v^*(f) - v^*(t; f)| \leq \delta$ for all $t \geq \bar{t}$. By Pareto efficiency of $v^*(f)$, let $\delta > 0$ be sufficiently small so that $A\big(v^*(\bar{t}; f) - (\delta, \delta, \dots, \delta)\big)$ is contained in the ε-ball centered at $v^*(f)$. Since the right hand side of ODE (3) is continuous in v, the unique solution of (3) is continuous with respect to parameters in (3). Therefore, for a finite time interval $[0, T]$ including $\bar{t}$, there exists $\bar{k}$ such that $|v^*(t; f_k) - v^*(t; f)| \leq \delta$ for all $t \in [0, T]$ and all $k \geq \bar{k}$. This implies that $v^*(t; f_k) \in A\big(v^*(\bar{t}; f) - (\delta, \delta, \dots, \delta)\big)$, thereby $v^*(f_k) \in A\big(v^*(\bar{t}; f) - (\delta, \delta, \dots, \delta)\big)$. Hence we have $|v^*(f_k) - v^*(f)| \leq \varepsilon$.

Second we show that the set is dense, i.e., for all $f \in \mathcal{F}$ with $v^*(f)$ not strictly Pareto efficient in X and all $\varepsilon > 0$, there exists $\tilde{f} \in \mathcal{F}$ such that $|f - \tilde{f}| \leq \varepsilon$ and $v^*(\tilde{f})$ is Pareto efficient. Since $v^*(f)$ is only weakly Pareto efficient in $\hat{X}$, there exists Pareto efficient $y \in X$ which Pareto dominates $v^*(f)$. Let $I = \{i \in N \mid y_i = v_i^*(f)\}$ and $J = N \setminus I$. Since y is Pareto efficient, there is $\delta > 0$ such that if $x \in X$ is weakly Pareto efficient, satisfies $|y - x| \leq \delta$, and $y_i = x_i$ for some $i \in N$, then there is no $\tilde{x} \in X$ such that $\tilde{x}_i > y_i$ and $|y - \tilde{x}| \leq \delta$.

By Assumption 1, for any small $\delta/2 > \eta > 0$, there is a small ball contained in X centered at $\tilde{y}$ with $|y - \tilde{y}| \leq \eta$. Let g be a continuous density function whose support is the above small ball, takes zero on the boundary of the ball, and the expectation of g is exactly $\tilde{y}$. Let $\tilde{f} = (1 - \frac{\varepsilon}{|f|+|g|})f + \frac{\varepsilon}{|f|+|g|}g \in \mathcal{F}$. Since f and g are bounded from above, $|f - \tilde{f}| \leq \varepsilon$.

Since $v^*(f)$ is weakly Pareto efficient, if $v^*(f) \in A(v)$, then $A(v) \subset \bigcup_{i \in N}\big([v_i, v_i^*(f)] \times \prod_{j \neq i}[0, \overline{x}_j]\big)$. If $|v^*(f) - v| \leq \xi$ where $\xi > 0$ is very small,

$$\begin{aligned}\int_{A(v)} (x_i - v_i) f(x) dx &\leq f_H \sum_{j \in N} (v_j^*(f) - v_j) \prod_{k \in N} \overline{x}_k \\ &\leq \xi n f_H \prod_{k \in N} \overline{x}_k\end{aligned}$$

If $v^*(f) \in A(v)$, $\min_{j \in N}(y_j - v_j) \geq 2\eta$ and $|v^*(f) - v| \leq \xi$, we have

$$\begin{aligned}\int_{A(v)} (x_i - v_i)\tilde{f}(x)dx - \int_{A(v)} (x_i - v_i) f(x) dx &= \int_{A(v)} (x_i - v_i)(\tilde{f}(x) - f(x))dx \\ &= \frac{\varepsilon}{|f| + |g|}\int_{A(v)} (x_i - v_i)(g(x) - f(x))dx \\ &\geq \frac{\varepsilon}{|f| + |g|}\Big((\tilde{y}_i - v_i) - \Big(\xi n f_H \prod_{k \in N} \overline{x}_k\Big)\Big).\end{aligned}$$

If $j \in J$ and $|v^*(f) - v| \leq \xi$ where $\xi > 0$ is very small, then

$$\int_{A(v)} (x_j - v_j)\tilde{f}(x)dx - \int_{A(v)} (x_j - v_j) f(x) dx \geq \frac{\varepsilon}{2(|f| + |g|)}(\tilde{y}_j - v_j^*(f)).$$

Let $w(t) = v^*(t; \tilde{f}) - v^*(t; f)$. Since ODE (3) is continuous in the parameters, for all $\zeta > 0$, there exists $\varepsilon > 0$ such that $|w(t)| \leq \zeta$ for all $t \in [0, T]$. Suppose that T and t are very large so that $|v^*(f) - v^*(t; f)| \leq \xi$. For $j \in J$, $w'_j(t)$ is estimated as follows:

$$
\begin{aligned}
w'_j(t) &= \lambda \int_{A(v^*(t;\tilde{f}))} (x_j - v^*_j(t;\tilde{f}))\tilde{f}(x)dx - \lambda \int_{A(v^*(t;f))} (x_j - v^*_j(t;f))f(x)dx \\
&= \lambda \int_{A(v^*(t;\tilde{f}))} (x_j - v^*_j(t;\tilde{f}))\tilde{f}(x)dx - \lambda \int_{A(v^*_j(t;\tilde{f})))} (x_j - v^*_j(t;\tilde{f})))f(x)dx \\
&\quad + \lambda \int_{A(v^*(t;\tilde{f}))} (x_j - v^*_j(t;\tilde{f}))f(x)dx - \lambda \int_{A(v^*(t;f))} (x_j - v^*_j(t;f))f(x)dx \\
&\geq \frac{\lambda\varepsilon}{2(|f|+|g|)}(\tilde{y}_j - v^*_j(f)) - \lambda \int_{A(v^*(t;f))\cap A(v^*(t;\tilde{f}))} w_j(t)f(x)dx \\
&\quad - \lambda \int_{A(v^*(t;f))\setminus(A(v^*(t;f))\cap A(v^*(t;\tilde{f})))} (x_j - v^*_j(t;f))f(x)dx \\
&\geq \frac{\lambda\varepsilon}{2(|f|+|g|)}(\tilde{y}_j - v^*_j(f) - \zeta) - \lambda\zeta\xi \sum_{k\in N}\prod_{l\neq k}\overline{x}_l - \lambda\xi n f_H \prod_{k\in N}\overline{x}_k.
\end{aligned}
$$

Therefore when $\xi > 0$ is sufficiently small, $w'_j(t)$ is bounded away from zero:

$$
w'_j(t) \geq \frac{\lambda\varepsilon}{4(|f|+|g|)}(\tilde{y}_j - v^*_j(f) - \zeta).
$$

This implies that for small $\varepsilon > 0$ and large t, $v^*_j(t;\tilde{f}) > v^*(f)$ for all $j \in J$. Then the similar method to Step 3 in the proof of Proposition 6 shows that $v^*(t;\tilde{f})$ converges to a Pareto efficient allocation in X.

A.5 Proof of Proposition 6

Let $A = \{x \in X \,|\, x \geq v^*\}$. Let $I = \{i \in N \,|\, x_i = v_i \text{ for all } x \in A\} \subset N$, and $J = N \setminus I$. Suppose that there exists $x \in X$ which Pareto dominates v^*, thereby $J \neq \emptyset$.

Step 1: We show that I is nonempty. If there is no such player, there exist $y(1), \ldots, y(n)$ such that $y(j) \in A$ and $y_j(j) > v^*_j$ for all $j \in N$. This implies that $y = \frac{1}{n}\sum_{j\in N} y(j)$ strictly Pareto dominates v^*. Since X is convex, y also belongs to A. This contradicts the weak Pareto efficiency of v^* shown in Lemma 4.

Step 2: Next we show that if v^* is not Pareto efficient in X, and $i \in I$, then $x_i \leq v^*_i$ for all $x \in X$.

Let i be the player in I. Suppose that there exists $y \in X$ with $y_i > v^*_i$. Since X is convex, $\alpha y + (1-\alpha)x \in X$ for all $0 \leq \alpha \leq 1$ and $x \in X$. Since we assumed that there exists $x \in X$ which Pareto dominates v^*, $x_j > v^*_j$ for $j \in J$. Then there exists $\alpha > 0$ such that $\alpha y + (1-\alpha)x \geq v^*$, and $\alpha y_j + (1-\alpha)x_j > v^*_j$ for some j. By Step 1, we must have $x_i = v^*_i$. Therefore, $\alpha y_i + (1-\alpha)x_i > v^*_i$, which contradicts the fact that $i \in I$.

Step 3: Finally we show that $v^*(t)$ converges to a Pareto efficient allocation in X as $t \to \infty$.

By convexity of X, we may find $y_j, \bar{y}_j$ $(j \in J)$ such that $v_j^* < y_j < \bar{y}_j$, and $\prod_{i\in I}[v_i^* - \varepsilon, v_i^*] \times \prod_{j\in J}[y_j, \bar{y}_j]$ is contained in X for small $\varepsilon > 0$. Let $\varepsilon \in (0, 1/2)$ be sufficiently small such that $\varepsilon \leq \dfrac{2f_L \prod_{j\in J}(\bar{y}_j - y_j)}{f_H \prod_{j\in J} \overline{x}_j}$. Since $v^*(t)$ converges to v^* as $t \to \infty$, there exists $\bar{t}$ such that $\max_{i\in N}\{v_i^* - v_i^*(t)\} \leq \varepsilon$ whenever $t \geq \bar{t}$. Note that by Proposition **??**, $v_i^* - v_i^*(t) > 0$ for all t and $i \in N$. Let $Y(t) = \prod_{i\in I}[v_i^*(t), v_i^*] \times \prod_{j\in J}[y_j, \bar{y}_j] \subset A(t)$.

We have $A(t) \subset \prod_{i\in I}[v_i^*(t), v_i^*] \times \prod_{j\in J}[0, \overline{x}_j]$ since there is no $x \in A(t)$ with $x_i > v_i^*$. By equation (3), for $i \in I$,

$$\begin{aligned}
v_i^{*\prime}(t) &= \lambda \int_{A(t)} \big(x_i - v_i^*(t)\big) d\mu \\
&\leq \lambda \int_{\prod_{i'\in I}[v_{i'}^*(t), v_{i'}^*]} \big(x_i - v_i^*(t)\big) \int_{\prod_{j\in J}[0,\overline{x}_j]} f_H \prod_{j\in J} dv_j \prod_{i'\in I} dv_{i'} \\
&\leq \frac{1}{2}\lambda f_H (v_i^* - v_i^*(\bar{t})) \prod_{i'\in I}(v_{i'}^* - v_{i'}^*(t)) \prod_{j\in J} \overline{x}_j
\end{aligned}$$

for all $t \geq \bar{t}$. On the other hand, for $j \in J$,

$$\begin{aligned}
v_j^{*\prime}(t) &= \lambda \int_{A(t)} \big(x_j - v_j^*(t)\big) d\mu \\
&\geq \lambda \int_{Y(t)} (y_j - v_j^*) d\mu \\
&= \lambda (y_j - v_j^*) \mu(Y(t)) \\
&\geq \lambda f_L (y_j - v_j^*) \prod_{i\in I}(v_i^* - v_i^*(t)) \prod_{j'\in J}(\bar{y}_{j'} - y_{j'}).
\end{aligned}$$

Then for $i \in I$ and $j \in J$,

$$\begin{aligned}
\frac{v_i^{*\prime}(t)}{v_j^{*\prime}(t)} \cdot \frac{v_j^* - v_j^*(\bar{t})}{v_i^* - v_i^*(\bar{t})} &\leq \frac{f_H (v_i^* - v_i^*(\bar{t}))(v_j^* - v_j^*(\bar{t})) \prod_{j\in J} \overline{x}_j}{2 f_L \prod_{j'\in J}(\bar{y}_{j'} - y_{j'})} \\
&\leq \frac{(v_i^* - v_i^*(\bar{t}))(v_j^* - v_j^*(\bar{t}))}{\varepsilon} \\
&\leq \varepsilon \leq \frac{1}{2}
\end{aligned}$$

for all $t \geq \bar{t}$. Therefore,

$$\frac{v_i^{*\prime}(\bar{t})}{v_j^{*\prime}(\bar{t})} \leq \frac{v_i^* - v_i^*(\bar{t})}{2\big(v_j^* - v_j^*(\bar{t})\big)}$$

holds for all $t \geq \bar{t}$. This inequality implies

$$v_i^*(t) - v_i^*(\bar{t}) \leq \frac{v_i^* - v_i^*(\bar{t})}{2\big(v_j^* - v_j^*(\bar{t})\big)} \big(v_j^*(t) - v_j^*(\bar{t})\big)$$

for all $t \geq \bar{t}$. By letting $t \to \infty$ in the above inequality, we have $0 < v_i^* - v_i^*(\bar{t}) \leq$

$\big(v_i^* - v_i^*(\bar{t})\big)/2$, a contradiction. Hence v^* is strictly Pareto efficient in X.

A.6 Proof of Proposition 8

First, we define the notion of the edge of the Pareto frontier. Suppose that w is Pareto efficient in X, and $w_i > 0$ for all $i \in X$. Let us denote an $(n-1)$-dimensional subspace orthogonal to w by $D = \{z \in \mathbb{R}^n \mid w \cdot z = 0\}$. For $\xi > 0$, let D_ξ be an $(n-1)$-dimensional disk defined as

$$D_\xi = \{z \in D \mid |z| \leq \xi\},$$

and let S_ξ be its boundary. We say that a Pareto efficient allocation w in X is *not* located at the Pareto frontier of X if there is $\xi > 0$ such that for all vector $z \in D_\xi$ there is a scalar $\alpha > 0$ such that $\alpha(w+z)$ is Pareto efficient in X. We denote this allocation by $w_z \in X$.

Let $B_\varepsilon(x) = \{x \in X \mid |w - x| \leq \varepsilon\}$ for $y \in X$ and $\varepsilon > 0$. We denote the volume of $B_\varepsilon(y)$ by $V_\varepsilon(y)$, and the volume of the n-dimensional ball with radius ε by V_ε. Note that $\min_{y \in X} V_\varepsilon(y) > 0$ by Assumption 1. Let g be a continuous density function on n-dimensional ball centered at $0 \in \mathbb{R}^n$ with radius ε, assumed to take zero on the boundary of the ball. Let $\tilde{f}$ be the uniform density function on X. For a Pareto efficient allocation y, we define a probability density function f_y on X by

$$f_y(x) = \eta \tilde{f}(x) + (1-\eta) g(y-x) \frac{V_\varepsilon}{V_\varepsilon(y)}$$

where $\eta > 0$ is small. Note that $f_y(x)$ is uniformly bounded above and away from zero in x and y.

For $z \in D_\xi$, let $\tilde{\varphi}(z)$ be the limit of the solution of ODE (3) with density f_{w_z}, and $\varphi(z) = \tilde{\varphi}(z) + \delta w \in D$ for some $\delta \in \mathbb{R}$. By the form of ODE (3), the solution of (3) with density f_{w_z} is continuously deformed if z changes continuously. Since w is not at the edge of the Pareto frontier, $\tilde{f}_{w_z}$ is also Pareto efficient in X and comes close to w if ξ, ε, and η are small. Therefore $\varphi(z)$ is a continuous function from D_ξ to D. The rest of the proof consists of two steps.

Step 1: We show that $|\varphi(z) - z| \leq \xi/2$ if $\varepsilon > 0$ and $\eta > 0$ are small. If a density function has a positive value only in $B_\varepsilon(y)$, then the barycenter of $A(t)$ is always contained in $B_\varepsilon(y)$. In such a case, the limit allocation belongs to $B_\varepsilon(y)$. As $\eta \to 0$, f_y approaches the above situation. Therefore, for sufficiently small $\eta > 0$, the distance between the limit allocation and y is smaller than 2ε. For ε very small, we have $|\varphi(z) - z| \leq \xi/2$.

Step 2: We show that there is $z \in D_\xi$ such that $\varphi(z) = 0$. If not, $\psi(z) = \dfrac{\xi\varphi(z)}{|\varphi(z)|}$ is a continuous function from D_ξ to S_ξ. Let $\Phi : D_\xi \times [0,1] \to S_\xi$ be a homotopy defined by $\Phi(z,s) = \frac{\xi(sz+(1-s)\psi(z))}{|sz+(1-s)\psi(z)|}$. (Note that the denominator is positive by Step 1.) This means that S_ξ is a deformation retract of D_ξ, which is not true. A contradiction.

Hence for f_{w_z} such that $\varphi(z) = 0$, the limit allocation coincides with w.

A.7 Proof of Lemma 9

Let $f_H(t) = \max_{x\in A(t)} f(x)$, and $f_L(t) = \min_{x\in A(t)} f(x)$. Since f is continuous, both $f_H(t)$ and $f_L(t)$ are continuous and converge to $f(v^*)$ as $t \to \infty$. For $\varepsilon > 0$, there is $\bar{t}$ such that $|v^* - v^*(t)| \leq \varepsilon$ for all $t \geq \bar{t}$. For $\eta > 0$, let

$$\underline{A}(t) = \{x \in \mathbb{R}^n_+ \mid x \geq v(t), \alpha \cdot (x - v^*) \leq -\eta\}, \text{ and}$$
$$\overline{A}(t) = \{x \in \mathbb{R}^n_+ \mid x \geq v(t), \alpha \cdot (x - v^*) \leq \eta\}.$$

Then the volume of $\underline{A}(t)$ (with respect to the Lebesgue measure on $\mathbb{R}^n$) is

$$V(\underline{A}(t)) = \frac{1}{n} \prod_{j\in N} \Big(\frac{\alpha \cdot (v^* - v^*(t))}{\alpha_j} - \eta\Big),$$

and the volume of $\overline{A}(t)$ is

$$V(\overline{A}(t)) = \frac{1}{n} \prod_{j\in N} \Big(\frac{\alpha \cdot (v^* - v^*(t))}{\alpha_j} + \eta\Big).$$

By Assumption 2, there exists $\eta > 0$ such that $\underline{A}(t) \subset A(t) \subset \overline{A}(t)$ for all $t \geq \bar{t}$. The rest of the proof consists of two steps.

Step 1: We show that for any two players $i, j \in N$, $\lim_{t\to\infty} v_j^{*\prime}(t)/v_i^{*\prime}(t) = \alpha_i/\alpha_j$. The ith coordinate of the right hand side of equation (3) is estimated as

$$f_L(t) \int_{\underline{A}(t)} \big(x_i - v_i^*(t)\big) dx$$
$$\leq \int_{A(t)} \big(x_i - v_i^*(t)\big) f(x) dx \leq f_H(t) \int_{\overline{A}(t)} \big(x_i - v_i^*(t)\big) dx.$$

Therefore,

$$\frac{\lambda f_L(t) V(\underline{A}(t))}{n+1} \Big(\frac{\alpha \cdot (v^* - v^*(t))}{\alpha_i} - \eta\Big)$$
$$\leq v_i^{*\prime}(t) \leq \frac{\lambda f_H(t) V(\overline{A}(t))}{n+1} \Big(\frac{\alpha \cdot (v^* - v^*(t))}{\alpha_i} + \eta\Big) \qquad \text{(A.1)}$$

for all $t \geq \bar{t}$ and $i \in N$. By letting $\varepsilon \to 0$, $\eta \to 0$, and $t \to \infty$, we have $\lim_{t\to\infty} v_j^{*\prime}(t)/v_i^{*\prime}(t) = \alpha_i/\alpha_j$.

Step 2: By Step 1, for i and small $\delta > 0$, there exist $\tilde{t}$ such that such that

$$(1-\delta)\frac{\alpha_i}{\alpha_j} \le \frac{v_j^* - v_j^*(t)}{v_i^* - v_i^*(t)} \le (1+\delta)\frac{\alpha_i}{\alpha_j}$$

for all $t \ge \tilde{t}$ and $j \in N$. Therefore,

$$n(1-\delta)(v_i^* - v_i^*(t)) \le \frac{\alpha \cdot (v^* - v^*(t))}{\alpha_i} \le n(1+\delta)(v_i^* - v_i^*(t)).$$

By inequality (A.1), we have

$$\begin{aligned}
&\frac{\lambda f_L(t)}{n(n+1)}\big(n(1-\delta)(v_i^* - v_i^*(t)) - \eta\big)^2 \prod_{j\neq i}\Big(n(1-\delta)\frac{\alpha_i}{\alpha_j}(v_i^* - v_i^*(t)) - \eta\Big) \\
&\quad \le v_i^{*\prime}(t) \le \frac{\lambda f_H(t)}{n(n+1)}\big(n(1+\delta)(v_i^* - v_i^*(t)) + \eta\big)^2 \prod_{j\neq i}\Big(n(1+\delta)\frac{\alpha_i}{\alpha_j}(v_i^* - v_i^*(t)) + \eta\Big)
\end{aligned}$$

for all $t \ge \tilde{t}$ and $j \in N$. Thus for large t, $v_i^*(t)$ is approximated by the solution of the following ordinary differential equation:

$$v_i'(t) = \frac{\lambda f(v^*) n^n}{n+1}(n(v_i^* - v_i^*(t)))^2 \prod_{j\neq i}\frac{\alpha_i}{\alpha_j}(v_i^* - v_i(t)).$$

By solving this, we have an approximation

$$v^* - v_i^*(t) = \Big(C + \frac{\lambda f(v^*) n^{n+1} t}{n+1}\prod_{j\neq i}\frac{\alpha_i}{\alpha_j}\Big)^{-\frac{1}{n}}$$

where C is a constant. Hence,

$$\lim_{t\to\infty}(v_i^* - v_i^*(t))(\lambda t)^{\frac{1}{n}} = \Big(\frac{n+1}{f(v^*) n^{n+1}}\prod_{j\neq i}\frac{\alpha_j}{\alpha_i}\Big)^{\frac{1}{n}},$$

which is a positive constant.

A.8 Proof of Proposition 10

By Assumption 2, $A(t)$ is approximated as

$$\{x \in \mathbb{R}_+^n \,|\, x \ge v^*(t), \alpha \cdot (x - v^*) \le 0\}.$$

By Lemma 9, $\mu(A(t))$ is approximated as

$$\begin{aligned}\mu(A(t)) &= f(v^*)n^{n-1}\prod_{i\in N}(v_i^* - v_i^*(t))\\ &= f(v^*)n^{n-1}\prod_{i\in N}\left(\left(\frac{n+1}{f(v^*)n^{n+1}}\prod_{j\neq i}\frac{\alpha_j}{\alpha_i}\right)^{\frac{1}{n}}(\lambda t)^{-\frac{1}{n}}\right)\\ &= \frac{n+1}{n^2\lambda t}\end{aligned}$$

if t is large. For $s \in [0,T]$, the probability that players reach an agreement before time $-(T-s)$ is

$$1 - e^{\int_{T-s}^{T}\mu(A(t))\lambda dt} = 1 - \left(\frac{\lambda n^2(T-s)+n+1}{\lambda n^2 T + n + 1}\right)^{\frac{n+1}{n^2}}.$$

This probability is approximated by $1 - \left(\frac{T-s}{T}\right)^{\frac{n+1}{n^2}}$. Therefore the expected duration of the search process is

$$\int_0^T s\frac{d}{ds}\left[1 - \left(\frac{T-s}{T}\right)^{\frac{n+1}{n^2}}\right]ds = \frac{n^2}{n^2+n+1}T.$$

A.9 Proof of Proposition 11

Let $v^0(t;\lambda)$ be the solution of (3) for $\rho = 0$. Fix any $t \in [0,T]$. Recall that $v^0(t;\alpha\lambda) = v^0(\alpha t;\lambda)$ for all $\alpha > 0$. Since $\lim_{\lambda\to\infty} v^0(t;\lambda) = v^0$, there exists $\bar{\lambda}^1 > 0$ such that

$$\begin{aligned}\left|v^0 - v^0(t;\lambda)\right| &= \left|v^0 - v^0(\lambda t;1)\right|\\ &\leq \varepsilon/2 \end{aligned} \tag{A.2}$$

for all $\lambda \geq \bar{\lambda}^1$.

Since the right hand side of ODE (2) is continuous in ρ, λ, and uniformly Lipschitz continuous in v, the unique solution $v^*(t;\rho,\lambda)$ is continuous in ρ, λ for all $t \in [0,T]$.[15] Recall that $v^*(t;\rho,\alpha\lambda) = v^*(\alpha t;\rho/\alpha,\lambda)$ for all $\alpha > 0$. Therefore by continuity in ρ, there exists $\bar{\lambda}^2 > 0$ such that

$$\begin{aligned}\left|v^*(t;\rho,\lambda) - v^0(t;\lambda)\right| &= \left|v^*(\lambda t;\rho/\lambda,1) - v^0(\lambda t;1)\right|\\ &\leq \varepsilon/2 \end{aligned} \tag{A.3}$$

for all $\lambda \geq \bar{\lambda}^2$. By adding (A.2) and (A.3), we obtain the desired inequality for $\bar{\lambda} = \max\{\bar{\lambda}^1, \bar{\lambda}^2\}$.

[15] See, e.g., Coddington and Levinson (1955, Theorem 7.4 in Chapter 1).

A.10 Proof of Proposition 12

Let $v(t)$ be the solution of ODE (2). The proof consists of five steps.

Step 1: We show that for any $t > 0$, $\mu(A(t)) \to 0$ as $\lambda \to \infty$. If not, there exist a positive value $\varepsilon > 0$ and an increasing sequence $(\bar{\lambda}_k)_{k=1,2,\ldots}$ such that $\mu(A(t)) \geq \varepsilon$ for all $\bar{\lambda}_k$. Since X is compact and f is bounded from above, there exists $\eta > 0$ such that $\mu\big(A(v(t) + (\eta, \ldots, \eta))\big) \geq \varepsilon/2$. In fact, since

$$\begin{aligned}\mu\big(A(v(t)) \setminus A(v(t) + (\eta, \ldots, \eta))\big) &\leq \sum_{i \in N} \mu\Big([v_i(t), v_i(t) + \eta] \times \prod_{j \neq i} [0, \overline{x}_j]\Big) \\ &\leq f_H \sum_{i \in N} \eta \prod_{j \neq i} \overline{x}_j,\end{aligned}$$

we have $\mu\big(A(v(t) + (\eta, \ldots, \eta))\big) \geq \varepsilon/2$ for $\eta = \dfrac{\varepsilon}{2 f_H \sum_{i \in N} \prod_{j \neq i} \overline{x}_j}$. For this η, the integral in ODE (2) is estimated as

$$\begin{aligned}\int_{A(t)} \big(x_i - v_i(t)\big) d\mu &\geq \int_{A(v(t)+(\eta,\ldots,\eta))} \big(x_i - v_i(t)\big) d\mu \\ &\geq \int_{A(v(t)+(\eta,\ldots,\eta))} \eta d\mu \\ &\geq \eta\varepsilon/2.\end{aligned}$$

By ODE (2),

$$v_i'(t) \geq -\rho \overline{x}_i + \bar{\lambda}_k \eta \varepsilon/2,$$

which obviously grows infinitely as $\bar{\lambda}_k$ becomes large. This contradicts compactness of X.

Step 2: We compute the direction of $\int_{A(t)} \big(x_i - v_i(t)\big) d\mu$ in the limit as $\lambda \to \infty$. By Step 1, the boundary of X contains all accumulation points of $\{v_i(t) \mid \lambda > 0\}$ for fixed $t > 0$. Fix an accumulation point $v^*(t)$. There exists an increasing sequence $(\lambda_k)_{k=1,2,\ldots}$ with $v^*(t) = \lim_{k \to \infty} v(t)$. By Assumption 3, there exists a unit normal vector of X at $v^*(t)$, which we denote by $\alpha \in \mathbb{R}_{++}$.

Step 1 implies that $v(t)$ is very close to the boundary of X when λ_k is very large. By smoothness of the boundary of X, $A(t)$ looks like a polyhedron defined by convex hull of $\{v(t), v(t) + (z_1(t), 0, \ldots, 0), v(t) + (0, z_2(t), 0, \ldots, 0), \ldots, v(t) + (0, \ldots, 0, z_n(t))\}$ where $z_i(t)$'s are positive length of edges such that the last n vertices are on the boundary of X. This vector $z(t)$ is parallel to $(1/\alpha_1, \ldots, 1/\alpha_n)$. Let $r(t)$ be the ratio between the length of $z(t)$ and $(1/\alpha_1, \ldots, 1/\alpha_n)$, i.e., $r(t) = z_1(t)\alpha_1 = \cdots = z_n(t)\alpha_n$.

Since density f is bounded from above and away from zero, distribution μ looks almost uniform on $A(t)$ if λ_k is large. Then the integral $\int_{A(t)} \big(x_i - v_i(t)\big) d\mu$ is almost parallel to the vector from $v(t)$ to the barycenter of the polyhedron, namely, $z(t)/(n+1)$. Therefore $\int_{A(t)}(x_i - v_i(t))d\mu$ is approximately parallel to $(1/\alpha_1, \dots, 1/\alpha_n)$ when λ_k is large.

Step 3: We show that $\sum_{i \in N} \alpha_i v_i'(t) \geq 0$ for large λ. Let $(\lambda_k)_{k=1,2,\dots}$ be the sequence defined in Step 2. For large λ_k, $A(t)$ again looks like a polyhedron with the uniform distribution. By Step 2, the ODE near $v_i(t)$ is written as

$$v_i'(t) = -\rho v_i(t) + \lambda_k \frac{z_i(t)}{n+1} \cdot \mu(A(t)). \tag{A.4}$$

Note that $v_i(t)$ is close to $v_i^*(t)$ and $\mu(A(t))$ is order n of the length of $z(t)$. By replacing the above equation by $r(t)$, ODE (A.4) approximates

$$r'(t) = a - \lambda_k b r(t)^{n+1} \tag{A.5}$$

for some constants $a, b > 0$. Since $r(t)$ is large when t is small, the above ODE shows that $r(t)$ is decreasing in t. Therefore $\mu(A(t))$ is also decreasing in t. For large λ_k, this implies that

$$\alpha \cdot v'(t) = \sum_{i \in N} \alpha_i v_i'(t) \geq 0.$$

Step 4: We show that the Nash product is nondecreasing if λ is large. By ODE (A.4), we have

$$\alpha_i v_i'(t) = -\rho \alpha_i v_i(t) + \beta \tag{A.6}$$

where $\beta = \lambda_k \mu(A(t))/(n+1)$ independent of i. Let us assume without loss of generality that $\alpha_1 v_1'(t) \geq \cdots \geq \alpha_n v_n'(t)$. Then we must have $1/\alpha_1 v_1(t) \geq \cdots \geq 1/\alpha_n v_n(t)$.

Let $L(t) = \sum_{i \in N} \log v_i(t)$ be log of the Nash product. Then $L'(t) = \sum_{i \in N} v_i'(t)/v_i(t)$. By Chebyshev's sum inequality,

$$\begin{aligned} L'(t) &= \sum_{i \in N} \frac{v_i'(t)}{v_i(t)} \\ &\geq \frac{1}{n}\Big(\sum_{i \in N} \alpha_i v_i'(t)\Big)\Big(\sum_{i \in N} \frac{1}{\alpha_i v_i(t)}\Big) \geq 0. \end{aligned}$$

Hence, $L(t)$ is nondecreasing if λ_k is large. Moreover, equality holds if and only if $\alpha_1 v_1'(t) = \cdots = \alpha_n v_n'(t)$ or $\alpha_1 v_1(t) = \cdots = \alpha_n v_n(t)$.

Step 5: We show that $v(t)$ converges to a point in the Nash set as $\lambda \to \infty$. Step 4 shows that $L'(t)$ converges to zero as $\lambda \to \infty$. Then $\alpha_1 v_1'(t) = \cdots = \alpha_n v_n'(t)$ or $\alpha_1 v_1(t) = \cdots = \alpha_n v_n(t)$ in the limit of $\lambda \to \infty$. The former case implies $v_i'(t) = 0$ for all $i \in N$ by Step 3. Then ODE (A.6) shows that the latter case holds. Therefore the latter case always holds in the limit of $\lambda \to \infty$. This implies that the boundary of X at $v^*(t)$ is tangent to the hypersurface defined by "Nash product $= \prod_{i\in N} v_i^*(t)$." Hence any accumulation point $v^*(t)$ belongs to the Nash set.

Since we assumed that the Nash set consists of isolated points, $v^*(t)$ is isolated. If $v(t)$ does not converge to $v^*(t)$, there is $\delta > 0$ such that for any $\bar{\lambda}$ there exists $v(t)$ with $\lambda \geq \bar{\lambda}$. Let $\delta > 0$ be small such that there is no point in the Nash set in $\{x \in X \mid |v^*(t) - x| \leq \delta\}$. Since $v(t)$ is continuous with respect to λ, for any $\bar{\lambda}$, there exists $\lambda > \bar{\lambda}$ such that $\delta/2 \leq |v^*(t) - v(t)| \leq \delta$. Since $\{x \in X \mid \delta/2 \leq |v^*(t) - x| \leq \delta\}$ is compact, $v(t)$ must have an accumulation point in this set. This contradicts the fact that any accumulation point is contained in the Nash set. Furthermore, $v^*(t)$ does not depend on t since $v^*(t)$ is continuous in t.

A.11 Proof of Proposition 13

(Sketch of proof): The ODE (A.5) is approximated by a linear ODE, which has a solution converging to v^* with an exponential speed. Therefore for large λ, $r(t)$ is approximated by $r(t) = \left(\frac{a}{\lambda b}\right)^{\frac{1}{n+1}}$. Since $\mu(A(t))$ is proportional to $r(t)^n$, $\mu(A(t)) = c\lambda^{-\frac{n}{n+1}}$ for a constant $c > 0$. the probability that players reach an agreement before time $-(T-s)$ is

$$1 - e^{-\int_{T-s}^{T} \mu(A(t))\lambda dt} = 1 - e^{-sc\lambda^{\frac{1}{n+1}}},$$

which converges to one as $\lambda \to \infty$.

References

Abdelaziz, F. and S. Krichen (2007): "Optimal stopping problems by two or more decision makers: a survey," *Computational Management Science*, 4, 89–111.

Albrecht, J., A. Anderson, and S. Vroman (2010): "Search by committee," *Journal of Economic Theory*, 145, 1386–1407.

Alpern, S. and S. Gal (2009): "Analysis and design of selection committees: a game theoretic secretary problem," *International Journal of Game Theory*, 38, 377–394.

Alpern, S., S. Gal, and E. Solan (2010): "A sequential selection game with vetoes," *Games and Economic Behavior*, 38, 1–14.

Ambrus, A. and S.-E. Lu (2010a): "A continuous-time model of multilateral bargaining," Mimeo.

——— (2010b): "Legislative bargaining with long finite horizons," Mimeo.

Bergemann, D. and J. Välimäki (2011): "Efficient Search by Committee," Mimeo.

Calcagno, R. and S. Lovo (2010): "Preopening and Equilibrium Selection," Mimeo.

Cho, I.-K. and A. Matsui (2011): "A Search Theoretic Foundation of Nash Bargaining Soluiton," Mimeo.

Coddington, E. A. and N. Levinson (1955): *Theory of Ordinary Differential Equations*, McGraw-Hill.

Compte, O. and P. Jehiel (2004): "Bargaining over Randomly Generated Offers: A new perspective on multi-party bargaining," Mimeo.

——— (2010): "Bargaining and Majority Rules: A Collective Search Perspective," *Journal of Political Economy*, 118, 189–221.

Herrero, M. J. (1989): "The Nash Program: Non-Convex Bargaining Problems," *Journal of Economic Theory*, 49, 266–277.

Imai, H. and H. Salonen (2009): "Limit Solutions for Finite Horizon Bargaining Problems," Mimeo.

Kamada, Y. and M. Kandori (2009): "Revision Games," Mimeo.

Kamada, Y. and T. Sugaya (2010a): "Asynchronous Revision Games with Deadline: Unique Equilibrium in Coordination Games," Mimeo.

——— (2010b): "Policy Announcement Game: Valence Candidates and Ambiguous Policies," Mimeo.

Ma, C.-T. A. and M. Manove (1993): "Bargaining with Deadlines and Imperfect Player Control," *Econometrica*, 61, 1313–1339.

Maschler, M., G. Owen, and B. Peleg (1988): "Paths leading to the Nash set," in *The Shapley value: Essays in Honor of Lloyd S. Shapley*, ed. by A. E. Roth, Cambridge University Press, 321–330.

Moldovanu, B. and X. Shi (2010): "Search Committees," Mimeo.

Nash, J. F. (1950): "The Bargaining Problem," *Econometrica*, 18, 155–162.

——— (1953): "Two-Person Cooperative Games," *Econometrica*, 21, 128–140.

Osborne, M. J. and A. Rubinstein (1990): *Bargaining and Markets*, Academic Press.

Rogerson, R., R. Shimer, and R. Wright (2005): "Search-Theoretic Models of the Labor Market: A Survey," *Journal of Economic Literature*, 43, 959–988.

Sakaguchi, M. (1978): "When to Stop: Randomly Appearing Bivariate Target Values," *Journal of the Operations Research Society of Japan*, 21, 45–58.

Wilson, C. A. (2001): "Mediation and the Nash bargaining solution," *Review of Economic Design*, 6, 353–370.