A SEARCHER VERSUS HIDER GAME
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Abstract

We introduce a search game in which the hider has only partial information about the searcher’s resource. Say, the hider can be a terrorist trying to hide and the searcher can be special forces trying to catch him, terrorist does not know the number of forces involved in the search but just its distribution. This plot is modeled by a non-zero-sum game and extends previous works on zero-sum allocation games. The other plot for the considered game can be inspired by wireless networks application. Namely, here a hider is, say, a terrorist, who wants to allocate a malicious node in network in attempt to reduce network connectivity (and thereby undermine the network’s security), and the searcher is a network operator who applies appropriate measures to detect malicious nodes to maintain network performance. We investigate how the information about the total search resources available to the hider can influence the behaviour of both players. For the case where the distribution has two mass points we developed geometrical approach allowing us to track down properties of the equilibrium, constructed the equilibrium strategies in closed form and proved its uniqueness. The two mass points distribution is a crucial important one in this plot since it presents the most often happened case, namely, large and small resources and how just the thread that large resources can be involved impacts on the hider. The large resources could represent the total searching force including the reserves while the small resources represents the special forces that are on call for search. Also, we discuss what happen if the distribution consists from more than two mass points.

1 Introduction

The theory of how to search for lost, missing, hidden and even evasive objects has been a subject of serious scientific research for more than sixty years and it is called Search Theory. Koopman did the initial work on search theory during World War II for the US Navy ([8]). The Navy’s primary search objects were enemy ships and submarines, and its own downed fliers adrift on the ocean. Koopman developed the general fundamental principles of search theory before he could get down to the specifics of naval problems. Bayesian search theory is a branch of search theory where Bayesian statistics is applied
to ([6, 11]). If the agent who hides something (say, a treasure, or a gun) knows that he
is being pursued or searched, and he does his best not to be detected, then the search
problem can be considered as game theoretical one with two agents involved: a searcher
and a hider ([2, 3, 5, 4, 7, 10, 12]). And then we deal with hider versus searcher (or hider-
searcher) game. Most hider-searcher games that have been considered in the literature
are zero-sum and have complete information. This paper introduces a new type of hider-
searcher game with incomplete information about the searcher’s resource, extending the
complete information games considered by [2, 7, 10] which originally was suggested as a
model for construction of antiballistic missile defence.

Namely, we study the following plot where the hider can be a terrorist trying to hide
and the searcher can be special forces trying to catch him, terrorist does not know the
number of forces involved in the search but just its distribution. This plot is modelled
by a non-zero-sum game and extends previous works on zero-sum allocation games. The
other plot for the considered game can be inspired by wireless networks applications.
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in attempt to reduce network connectivity (and thereby undermine the network’s secu-
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about the total search resources available to the hider can influence the behaviour of both
players. For the case where the distribution has two mass points we developed geomet-
rical approach allowing us to track down properties of the equilibrium, constructed the
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tribution is a crucial important one in this plot since it presents the most often happened
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be involved impacts on the hider. The large resources could represent the total searching
force including the reserves while the small resources represents the special forces that are
on call for search. Also, we discuss what happen if the distribution consists from more
than two mass points.

2 The game

We first describe a standard hide-and-search game on an integer interval [1, n]. The hider
selects one of the n points and hides there and the searcher tries to find him. The game is
zero-sum. For pure strategies, the payoff to the searcher is 1 if he finds the hider, otherwise
it is 0. A mixed hider strategy is a vector \( p = (p_1, \ldots, p_n) \) where \( p_i \) is the probability that
hider hides at point \( i \). The searcher seeks the hider by dividing the given total continuous
search effort \( \bar{x} \), allocating the effort \( x_i \) to each point \( i \). A mixed searcher strategy therefore
is a vector \( x = (x_1, \ldots, x_n) \) such that \( \sum_{i=1}^{n} x_i = \bar{x} \) where \( \bar{x} \) is the total resource that the
searcher has and \( x_i \) is the resource allocated for the search at point \( i \). The probability
that the searcher discovers the hider depends on two detection parameters \( \lambda_i > 0 \) and
\( \alpha_i \in (0, 1] \), and it is given by \( \alpha_i(1 - \exp(-\lambda_i x_i)) \). This is an extension of Koopman’s
formulation ([9]), in which \( \alpha_i \) is taken equal to one. The parameter \( \lambda_i \) represents the
difficulty of searching the location. The difficulty is inversely proportional to \( \lambda_i \). The
parameter \( \alpha_i \) is the probability of catching the hider, which in our game may be less than
The payoff to the searcher is the probability of finding the hider:

\[ v_S(p, x) = \sum_{i=1}^{n} \alpha_ip_i(1 - \exp(-\lambda_ix_i)). \]

Here we assume that the search parameters \( \alpha_i, \lambda_i \) are known to both players. If also the total search resource \( \bar{x} \) is known to both players then in the zero-sum scenario, in which the hider wants to minimize the probability of detection, the payoff to the hider is \(-v_S(p, x)\). Hide-and-search games like this have been considered by [2], [10] and [7] and have a unique Nash equilibrium.

In this paper we assume that hider does not know with certainty the total resource searcher has in his disposition but only its distribution, namely, he knows that this resource is \( \bar{x}^k \) with probability \( q^k \) where \( k = 1, 2 \) and \( q^1 + q^2 = 1 \). The searcher knows what resource he has in his disposition. To deal with this situation we introduce two types of searcher, namely, we will say that the searcher has type \( k \) \( (k = 1, 2) \) if his total resource is \( \bar{x}^k \). A searcher strategy of type \( k \) is given by \( x_k = (x^k_1, \ldots, x^k_n) \) with \( \sum_{i=1}^{n} x^k_i = \bar{x}^k \) and his payoff is given by:

\[ v^k_S(p, x^k) = \sum_{i=1}^{n} \alpha_ip_i(1 - \exp(-\lambda_ix^k_i)). \]

The payoff to the hider is:

\[ v_H(p, (x^1, x^2)) = -\sum_{k=1}^{2} q^k \sum_{i=1}^{n} \alpha_ip_i(1 - \exp(-\lambda_ix^k_i)). \]

We want to find an equilibrium strategies, that is, we want to find \( p^* \) and \( (x^1_*, x^2_*) \) such that for any strategies \( p, (x^1, x^2) \) the following inequalities hold

\[ v_H(p, (x^1, x^2)) \leq v_H(p^*, (x^1_*, x^2_*)), \]
\[ v^k_S(p^*, x^k) \leq v^k_S(p^*, x^k_*), \]
\[ v^2_S(p^*, x^2) \leq v^2_S(p^*, x^2_*). \]

Our goal is to find the equilibrium in closed form to trace down what impact the extra information produces on players behavior. Also, we would like to get sure whether the equilibrium is unique. Just existence and uniqueness of equilibrium without any tips what it is going on with solution follows from [1]. In this paper we develop a geometrical technic which allows us to solve the stated purposes not only construct the solution in closed form but also to prove. Also, in discussion section we show what difficulties arise if one would like to apply this technique for the case where more than two sizes of searching team could be on.

3 Auxiliary results

Since \( v_H \) is linear in \( p \) and \( v^k_S \) is concave in \( x^k \) applying a mixed linear and non-linear optimization approaches implies the following result connecting equilibrium with Lagrange multipliers and the minimal induced probability of detection.
Theorem 1 \((p, (x^1, x^2))\) is an equilibrium if and only if there are non-negative \(\omega\) (the minimal induced probability of detection), \(\nu^1\) and \(\nu^2\) (the Lagrange multipliers) such that

\[
\begin{align*}
\alpha_i \lambda_i p_i \exp(-\lambda_i x^1_i) &= \nu^1 \quad \text{for } x^1_i > 0 \\ &\leq \nu^1 \quad \text{for } x^1_i = 0,
\end{align*}
\]

(1)

\[
\begin{align*}
\alpha_i \lambda_i p_i \exp(-\lambda_i x^2_i) &= \nu^2 \quad \text{for } x^2_i > 0 \\ &\leq \nu^2 \quad \text{for } x^2_i = 0
\end{align*}
\]

(2)

and

\[
q^1 \alpha_i (1 - \exp(-\lambda_i x^1_i)) + q^2 \alpha_i (1 - \exp(-\lambda_i x^2_i)) \begin{cases} \omega & \text{for } p_i > 0 \\ \geq \omega & \text{for } p_i = 0. \end{cases}
\]

(3)

In the following theorem the equilibrium is specified in closed form as functions on \(\omega, \nu^1\) and \(\nu^2\).

Theorem 2 Let \((p, (x^1, x^2))\) be an equilibrium. Then \(p_i > 0\) for any \(i\), and

(a) if \(x^1_i > 0\) and \(x^2_i = 0\) then

\[
p_i = \frac{q^1 \nu^1}{\lambda_i (q^1 \alpha_i - \omega)}
\]

(4)

\[
x_i = \frac{1}{\lambda_i} \ln \left( \frac{q^1 \alpha_i}{q^1 \alpha_i - \omega} \right)
\]

(5)

and

\[
\omega \leq q^1 \alpha_i \left( 1 - \frac{\nu^1}{\nu^2} \right),
\]

(6)

(b) if \(x^2_i > 0\) and \(x^1_i = 0\) then

\[
p_i = \frac{q^2 \nu^2}{\lambda_i (q^2 \alpha_i - \omega)}
\]

(7)

\[
x_i = \frac{1}{\lambda_i} \ln \left( \frac{q^2 \alpha_i}{q^2 \alpha_i - \omega} \right)
\]

(8)

and

\[
\omega \leq \max q^2 \alpha_i \left( 1 - \frac{\nu^2}{\nu^1} \right),
\]

(9)

(c) if \(x^1_i > 0\) and \(x^2_i > 0\) then

\[
p_i = \frac{q^1 \nu^1 + q^2 \nu^2}{\lambda_i (\alpha_i - \omega)}
\]

(10)

\[
x^1_i = \frac{1}{\lambda_i} \ln \left( \frac{\alpha_i (q^1 \nu^1 + q^2 \nu^2)}{\nu^1 (\alpha_i - \omega)} \right),
\]

(11)

\[
x^2_i = \frac{1}{\lambda_i} \ln \left( \frac{\alpha_i (q^1 \nu^1 + q^2 \nu^2)}{\nu^2 (\alpha_i - \omega)} \right)
\]

(12)

and

\[
\max \left\{ q^1 \alpha_i \left( 1 - \frac{\nu^1}{\nu^2} \right), q^2 \alpha_i \left( 1 - \frac{\nu^2}{\nu^1} \right) \right\} < \omega \leq \alpha_i.
\]

(13)
Proof. First note that since \( p_i > 0 \) at least for one point \( i \) the Lagrange multipliers by (1) and (2) as well as \( \omega \) have to be positive.

The next note that \( p_i > 0 \) for any \( i \). Since if we assume that there is an \( i \) such that \( p_i = 0 \), then, by (1) and (2), \( x_i^1 = 0, x_i^2 = 0 \). Thus, by (3), \( \omega = 0 \). This contradiction proves that \( p_i > 0 \) for any \( i \).

(a) Since \( p_i > 0 \), \( x_i^1 > 0 \) and \( x_i^2 = 0 \), (1) and (3) imply that
\[
\alpha_i \lambda_i p_i \exp(-\lambda_i x_i^1) = \nu^1, \tag{14}
\]
and (3) yields that
\[
\alpha_i \lambda_i p_i \leq \nu^2. \tag{16}
\]
By (15) we have that
\[
\omega \leq q^1 \alpha_i. \tag{17}
\]
By (14),
\[
x_i^1 = \frac{1}{\lambda_i} \ln \left( \frac{\alpha_i p_i \lambda_i}{\nu^1} \right). \tag{18}
\]
Substituting this \( x_i \) into (15) yields the following equation for \( p_i \):
\[
\frac{q^1 (\alpha_i p_i \lambda_i - \nu^1)}{p_i \lambda_i} = \omega
\]
solving which implies (4). Substituting (4) into (15) yields (5). By (16) and (4) we have that
\[
\frac{\nu^1 q^1 \alpha_i}{q^1 \alpha_i - \omega} \leq \nu^2.
\]
Thus, taking into account (17) we obtain (6).

(b) follows from (a) by symmetry.

(c) Since \( p_i > 0 \), \( x_i^1 > 0 \) and \( x_i^2 > 0 \), (1) – (3) imply that
\[
\alpha_i \lambda_i p_i \exp(-\lambda_i x_i^1) = \nu^1, \tag{19}
\]
\[
\alpha_i \lambda_i p_i \exp(-\lambda_i x_i^2) = \nu^2 \tag{20}
\]
and
\[
q^1 \alpha_i (1 - \exp(-\lambda_i x_i^1)) + q^2 \alpha_i (1 - \exp(-\lambda_i x_i^2)) = \omega. \tag{21}
\]
By (19) and (20) we have that
\[
x_i^1 = \frac{1}{\lambda_i} \ln \left( \frac{\alpha_i p_i \lambda_i}{\nu^1} \right), \quad x_i^2 = \frac{1}{\lambda_i} \ln \left( \frac{\alpha_i p_i \lambda_i}{\nu^2} \right). \tag{22}
\]
Substituting (22) into (21) implies
\[ \frac{\alpha_i \lambda_i p_i - q_1 \nu_1 - q_2 \nu_2}{p_i \lambda_i} = \omega \]
solving which we obtain (10). Substituting (10) into (22) yields (11) and (6).
We can rewrite result of the previous theorem in more compact form introducing notation for subsets where either both type of searcher perform search or only one of them as follows

**Theorem 3** Each equilibrium has to be of the form
\[ (p, (x^1, x^2)) = (p(\omega, \nu^1, \nu^2), (x^1(\omega, \nu^1, \nu^2), x^2(\omega, \nu^1, \nu^2))) \]
where
\[ p_i(\omega, \nu^1, \nu^2) = \begin{cases} \frac{q^1 \nu^1}{\lambda_i (q^1 \alpha_i - \omega)}, & i \in I_{10}(\omega, \nu^1, \nu^2), \\ \frac{q^2 \nu^2}{\lambda_i (q^2 \alpha_i - \omega)}, & i \in I_{01}(\omega, \nu^1, \nu^2), \\ \frac{q^1 \nu^1 + q^2 \nu^2}{\lambda_i (\alpha_i - \omega)}, & i \in I_{11}(\omega, \nu^1, \nu^2), \end{cases} \] (23)
\[ x_i^1(\omega, \nu^1, \nu^2) = \begin{cases} 0, & i \in I_{01}(\omega, \nu^1, \nu^2), \\ \frac{1}{\lambda_i} \ln \left( \frac{q^1 \alpha_i}{q^1 \alpha_i - \omega} \right), & i \in I_{10}(\omega, \nu^1, \nu^2), \\ \frac{1}{\lambda_i} \ln \left( \frac{\alpha_i (q^1 \nu^1 + q^2 \nu^2)}{\nu^1 (\alpha_i - \omega)} \right), & i \in I_{11}(\omega, \nu^1, \nu^2), \end{cases} \] (24)
\[ x_i^2(\omega, \nu^1, \nu^2) = \begin{cases} 0, & i \in I_{01}(\omega, \nu^1, \nu^2), \\ \frac{1}{\lambda_i} \ln \left( \frac{q^2 \alpha_i}{q^2 \alpha_i - \omega} \right), & i \in I_{10}(\omega, \nu^1, \nu^2), \\ \frac{1}{\lambda_i} \ln \left( \frac{\alpha_i (q^1 \nu^1 + q^2 \nu^2)}{\nu^2 (\alpha_i - \omega)} \right), & i \in I_{11}(\omega, \nu^1, \nu^2) \end{cases} \] (25)
for some positive \( \omega, \nu^1 \) and \( \nu^2 \), where
\[ I_{10} = \left\{ i \in [1, n] : \omega \leq q^1 \alpha_i \left( 1 - \frac{\nu^1}{\nu^2} \right) \right\}, \]
\[ I_{01} = \left\{ i \in [1, n] : \omega \leq q^2 \alpha_i \left( 1 - \frac{\nu^2}{\nu^1} \right) \right\}, \]
\[ I_{11} = \left\{ i \in [1, n] : \max\left\{ q^1 \alpha_i \left( 1 - \frac{\nu^1}{\nu^2} \right), q^2 \alpha_i \left( 1 - \frac{\nu^2}{\nu^1} \right) \right\} < \omega \leq \alpha_i \right\}. \] (26)
It is clear that for a fixed \( \nu^1 \) and \( \nu^2 \) either \( I_{10}(\omega, \nu^1, \nu^2) \) or \( I_{01}(\omega, \nu^1, \nu^2) \) is empty, namely, the following result holds.

**Lemma 1** (a) If \( \nu_1 > \nu_2 \) then
\[ I_{10}(\omega, \nu^1, \nu^2) = \emptyset, \]
\[ I_{11}(\omega, \nu^1, \nu^2) = \left\{ i \in [1, n] : q^1 \alpha_i \left( 1 - \frac{\nu^1}{\nu^2} \right) < \omega \leq \alpha_i \right\}. \]
(b) If \( \nu_1 < \nu_2 \) then
\[ I_{01}(\omega, \nu^1, \nu^2) = \emptyset, \]
\[ I_{11}(\omega, \nu^1, \nu^2) = \left\{ i \in [1, n] : q^1 \alpha_i \left( 1 - \frac{\nu^1}{\nu^2} \right) < \omega \leq \alpha_i \right\}. \]
It turns out that there is a straight correspondence between the total search resources and the Lagrangian multipliers.

**Theorem 4** If $\bar{x}^1 > \bar{x}^2$ then $\bar{\nu}^1 < \bar{\nu}^2$. If $\bar{x}^1 < \bar{x}^2$ then $\bar{\nu}^1 > \bar{\nu}^2$.

**Proof.** Suppose, contrary to our claim, that $\bar{x}^1 > \bar{x}^2$ and $\bar{\nu}^1 > \bar{\nu}^2$. Then for $i \in I_{11}(\omega, \nu^1, \nu^2)$

$$x_i^1(\omega, \nu^1, \nu^2) = \frac{1}{\lambda_i} \ln \left( \frac{\alpha_i (q^1 \nu^1 + q^2 \nu^2)}{\nu^1 (\alpha_i - \omega)} \right)$$

$$< \frac{1}{\lambda_i} \ln \left( \frac{\alpha_i (q^1 \nu^1 + q^2 \nu^2)}{\nu^2 (\alpha_i - \omega)} \right) = x_i^2(\omega, \nu^1, \nu^2)$$

Thus, since by the above lemma, $I_{10}(\omega, \nu^1, \nu^2) = \emptyset$, we have that

$$\bar{x}^1 = \sum_{i=1}^n x_i^1(\omega, \nu^1, \nu^2) < \sum_{i=1}^n x_i^2(\omega, \nu^1, \nu^2) = \bar{x}^2.$$ 

This contradiction completes the proof of the theorem.

4 The main result

The optimal $\omega, \nu^1$ and $\nu^2$ have to be found from the condition that all the search resources has to be applied and vector $p$ has to be a probability one, namely, from the following conditions:

$$\sum_{i=1}^n x_i^1(\omega, \nu^1, \nu^2) = \bar{x}^1,$$

$$\sum_{i=1}^n x_i^2(\omega, \nu^1, \nu^2) = \bar{x}^2,$$

$$\sum_{i=1}^n p_i(\omega, \nu^1, \nu^2) = 1$$

which are equivalent to the following ones:

$$H_S^1(\omega, \nu^2/\nu^1) = \bar{x}^1; \quad (27)$$

$$H_S^2(\omega, \nu^1/\nu^2) = \bar{x}^2; \quad (28)$$

$$H_H(\omega, \nu^1, \nu^2) = 1; \quad (29)$$

where

$$H_S^1(\omega, \tau) = \sum_{i: \omega \leq q^1 \alpha_i (1-1/\tau)} \frac{1}{\lambda_i} \ln \left( \frac{q^1 \alpha_i}{q \alpha_i - \omega} \right)$$

$$+ \sum_{i: \max(q^1 \alpha_i (1-1/\tau), q^2 \alpha_i (1-1/\tau)) < \omega \leq \alpha_i} \frac{1}{\lambda_i} \ln \left( \frac{\alpha_i (q^1 + q^2 \tau)}{\alpha_i - \omega} \right); \quad (30)$$

$$H_S^2(\omega, \tau) = \sum_{i: \omega \leq q^2 \alpha_i (1-1/\tau)} \frac{1}{\lambda_i} \ln \left( \frac{q^2 \alpha_i}{q \alpha_i - \omega} \right)$$

$$+ \sum_{i: \max(q^1 \alpha_i (1-1/\tau), q^2 \alpha_i (1-1/\tau)) < \omega \leq \alpha_i} \frac{1}{\lambda_i} \ln \left( \frac{\alpha_i (q^1/\tau + q^2)}{\alpha_i - \omega} \right); \quad (31)$$
and
\[
H_H(\omega, \nu^1, \nu^2) = \sum_{i \in I_1} \frac{q^1 \nu^1}{\lambda_i (q^1 \alpha_i - \omega)} + \sum_{i \in I_2} \frac{q^2 \nu^2}{\lambda_i (q^2 \alpha_i - \omega)} + \sum_{i \in I_1} \frac{q^1 \nu^1 + q^2 \nu^2}{\lambda_i (\alpha_i - \omega)}.
\]

(32)

Without lost of generality we can assume for a while that
\[
\bar{x}^1 < \bar{x}^2.
\]

(33)

Then, by Theorem 3.4,
\[
\nu^1 > \nu^2, \text{ so } \tau = \nu^2 / \nu^1 < 1.
\]

(34)

Thus, by (30) and (31),
\[
H_1^S(\omega, \tau) = \sum_{i \in q^1 a_i} \frac{1}{\lambda_i} \ln \left( \frac{\alpha_i (q^1 + q^2 \tau)}{\alpha_i - \omega} \right),
\]

(35)

\[
H_2^S(\omega, 1/\tau) = \sum_{i \in \omega \leq q^2 a_i} \frac{1}{\lambda_i} \ln \left( \frac{q^2 \alpha_i}{q^2 \alpha_i - \omega} \right) + \sum_{i \in q^2 a_i < \omega \leq \alpha_i} \frac{1}{\lambda_i} \ln \left( \frac{\alpha_i (q^1 / \tau + q^2)}{\alpha_i - \omega} \right).
\]

(36)

Function $H_2^S(\omega, \tau)$ has the following properties. It is continuous on $\omega$ and $\tau$, it is increasing on $\omega$ and $\tau$. For a fixed $\tau$, $H_2^S(\omega_{\min} - 0, \tau) = \infty$ where $\omega_{\min} = \min_k \alpha_k$ and $H_2^S(0+, \tau) = 0$.

Thus, for a fixed $\tau > 0$ there is $\omega(\tau)$ such that
\[
H_2^S(\omega(\tau), \tau) = \bar{x}^1.
\]

This function $\omega(\tau)$ is decreasing on $\tau$ and
\[
\omega(0+) = \omega_0, \quad \omega(1) = \omega_1,
\]

where $\omega_0$ is the root in $(0, \omega_{\min})$ of the equation:
\[
\sum_{i : q^2 a_i < \omega \leq \alpha_i} \frac{1}{\lambda_i} \ln \left( \frac{q^2 \alpha_i}{\alpha_i - \omega} \right) = \bar{x}^1
\]

(37)

and $\omega_1$ is the root in $(0, \omega_{\min})$ of the equation:
\[
\sum_{i : \omega \leq \alpha_i} \frac{1}{\lambda_i} \ln \left( \frac{\alpha_i}{\alpha_i - \omega} \right) = \bar{x}^1.
\]

Similarly, $H_2^S(\omega, 1/\tau)$ has the following properties. It is continuous on $\omega > 0$ and $\tau \in (0, 1]$, it is increasing on $\omega$ and decreasing on $\tau$. Then function $H_S(\tau) := H_2^S(\omega^1(\tau), 1/\tau)$ is continuous and decreasing on $\tau$. Also, by (37) and (33)
\[
H_1(1) = \sum_{i : \omega \leq \alpha_i} \frac{1}{\lambda_i} \ln \left( \frac{\alpha_i}{\alpha_i - \omega^1(\tau)} \right) = \bar{x}^1 < \bar{x}^2.
\]
Also,

\[ H_1(0+) = \infty. \]

Thus, there is the unique \( \tau^* \) such that

\[ H_1(\tau^*) = \bar{x}^2. \]

Finally, note that

\[ H_H(\omega, \nu^1, \nu^2) = \nu^2 \bar{H}_H(\omega, \tau) \]

where

\[
\bar{H}_H(\omega, \tau) = \sum_{i \in I_{10}} \frac{q^1 \tau}{\lambda_i(q^1 \alpha_i - \omega)} + \sum_{i \in I_{01}} \frac{q^2}{\lambda_i(q^2 \alpha_i - \omega)} + \sum_{i \in I_{11}} \frac{q^1/\tau + q^2}{\lambda_i(\alpha_i - \omega)}.
\]

Thus, the optimal \( \nu^2 = \nu^*_2 \) are given as follows:

\[ \nu^*_2 = \frac{1}{\bar{H}_H(\omega^*, \tau^*)}. \]

Thus, we have proved the following result.

**Theorem 5** The game has the unique equilibrium and it is given by Theorem 3 where the optimal \( \omega, \nu_1, \nu_2 \) are given as follows

(a) If \( \bar{x}^1 < \bar{x}^2 \) then \( \omega = \omega^1(\tau^*) \) where \( \tau^* \) is the unique root of the equation \( H_1(\tau^*) = \bar{x}^2 \), \( \nu^2 = 1/\bar{H}_H(\omega^*, \tau^*) \) and \( \nu^1 = \nu^2/\tau^* \).

(b) The case \( \bar{x}^1 > \bar{x}^2 \) can be defined by symmetry.

## 5 Numerical results

Our game generalizes a previous search game with resource allocation [2], [7] in which the resource \( \bar{x} \) is known to both players. In our game, hider does not know the exact value of \( \bar{x} \) but just its distribution, namely, he knows that the search team has size \( \bar{x}^1 \) with probability \( q^1 \) and size \( \bar{x}^2 \) with probability \( q^2 \). Without loss of generality we can say that \( \bar{x}^1 \leq \bar{x}^2 \). The expected values \( q^1 \bar{x}^1 \) and \( q^2 \bar{x}^2 \) signify the cost of the operation and it is reasonable to assume that they are the same for this numerical modelling. Hider can consider the big size search resources as a threat, that forces the hider to spread out more evenly over the locations so that the small thread, arose from small size search resources, has a bigger chance of catching the hider. In real life, the large resources could represent the total force including the reserves while the small resources represents the special forces that are on call.

The parameters in our numerical simulation are as follows.

\[
\alpha = (0.8, 0.8, 0.8, 1, 1, 1, 1, 1, 1, 1)
\]
\[
\lambda = (0.8, 1, 1.2, 0.8, 0.9, 1, 1.1, 1.2, 1.3, 1.4)
\]
\[
\bar{x}^1 = 0.2, \quad \bar{x}^2 = 10, \quad q^1 = \frac{45}{49}, q^2 = \frac{4}{49}
\]

(38)
So the bigger searcher resource arises a small team with a large probability. The parameters are chosen in such a way that the expected value of the resources is equal to one. The first location is the most attractive to the hider: he has a chance of twenty percent of never being found and the location requires a maximal effort of the searcher, since $\lambda_1$ is minimal.

The bars in the left-hand figure (Fig.1) represent the allocation of the searcher resources as a percentage of the total resource, per location. Blue bars represent the small resources and red bars represent the big resources. The figure shows that the small resources concentrates on the first three locations, that are most attractive to the hider. The big resources allocates relatively more capacity to the unattractive locations. Our numerical experiments show that this is a general phenomenon: the small operation should always concentrate on locations in which the hider is hard to find. The right-hand figure shows that the hider has a slight preference for the first location. The probability of catching the hider in this example is equal to 6.6%.

If the searcher resources would be just one size and equals to one, then the probability of catching the hider under the given parameters is equal to 9.1%. Our numerical experiments show that if the expected resource $q^1 \bar{x}^1 + q^2 \bar{x}^2$ is constant, then the probability of catching the hider is maximized when $\bar{x}^1 = \bar{x}^2$. If the cost of maintaining a medium scale force is equal to the cost of a small force combined with a large backup of reservists, then the medium scale force is certainly preferable.

6 Distribution has more than two mass points

In this paper we consider the plot where hider does not know the value of search resource, just he knows it could be one of two values (typically, either a small or a large one) and he knows its distribution. It is natural to wonder whether our result remains valid if this distribution consists from more than two mass points. Much of the analysis remains valid, but there is one point that gets more complicated, as we will show it now.
Say, hider knows that searcher has in his disposition the total search resource $\bar{x}^r (r \in [1, R])$ with probability $q^r$, where $\sum_{r=1}^R q^r = 1$. In analogy of equations (1)–(3) we find

$$\alpha_i \lambda_i p_i \exp(-\lambda_i \bar{x}^r_i) \begin{cases} \nu^r & \text{for } x^r_i > 0 \\ \leq \nu^r & \text{for } x^r_i = 0 \end{cases}$$

for $r \in [1, R]$, and

$$\sum_{r=1}^R \nu^r \alpha_i (1 - \exp(-\lambda_i \bar{x}^r_i)) = \omega. \quad (39)$$

Let $I_r = \{i \in [1, n] : x^r_i > 0\}$ be the subset of locations that are being searched by searcher of type $r$ and $B_i = \{r \in [1, R] : x^r_i > 0\}$ be the set of searcher’s type which looks at point $i$. Then, by (39) and (40), the equilibrium $(p, (x^1, \ldots, x^R))$ has to be given as follows:

$$x^r_i = x^r_i(\omega, \nu^1, \ldots, \nu^R)$$

$$= \frac{1}{\lambda_i} \ln \left( \frac{\alpha_i \sum_{t \in B_i} q^r t' \nu^t}{\alpha_i \sum_{t \in B_i} q^r t' - \omega} \right), \quad i \in [1, n], \ r \in [1, R] \quad (41)$$

and

$$p_i = p_i(\omega, \nu^1, \ldots, \nu^R) = \frac{1}{\lambda_i} \frac{\sum_{t \in B_i} q^r t' \nu^t}{\sum_{t \in B_i} q^r t' - \omega}, \quad i \in [1, n]. \quad (42)$$

These $\omega, \nu^1, \ldots, \nu^R$ can be found as solution of the system of equations:

$$H^S_5(\omega, \nu^1, \ldots, \nu^R) := \sum_{i=1}^n x^r_i(\omega, \nu^1, \ldots, \nu^R) = \bar{x}^r, \quad r \in [1, R], \quad (43)$$

$$H_H(\omega, \nu^1, \ldots, \nu^R) := \sum_{i=1}^n p_i(\omega, \nu^1, \ldots, \nu^R) = 1. \quad (44)$$

Now we focus on the case of $R = 3$ to investigate how to find a solution of (43) and (44) and on its way also to prove its uniqueness. Without loss of generality we can assume that $\bar{x}^1 < \bar{x}^2 < \bar{x}^3$. Then $\nu^1 > \nu^2 > \nu^3$ and $I_1 \subseteq I_2 \subseteq I_3 = [1, n]$.

Following the approach suggested into this paper we have to look at the relations:

$$H^S_1 = \sum_{i} \frac{1}{\lambda_i} \ln \left( \frac{\alpha_i \nu^1 q^1 + q^2 \nu^2 + q^3 \nu^3}{\nu^1} \right),$$

$$H^S_2 = \sum_{i} \frac{1}{\lambda_i} \ln \left( \frac{\alpha_i \nu^1 q^1 + q^2 \nu^2 + q^3 \nu^3}{\nu^2} \right),$$

$$+ \sum_{i \neq 1} \frac{1}{\lambda_i} \ln \left( \frac{\alpha_i \nu^1 q^1 + q^2 \nu^2 + q^3 \nu^3}{\nu^3} \right),$$

$$H^S_3 = \sum_{i} \frac{1}{\lambda_i} \ln \left( \frac{\alpha_i \nu^1 q^1 + q^2 \nu^2 + q^3 \nu^3}{\nu^1} \right),$$

$$+ \sum_{i \neq 1} \frac{1}{\lambda_i} \ln \left( \frac{\alpha_i \nu^1 q^1 + q^2 \nu^2 + q^3 \nu^3}{\nu^2} \right),$$

$$+ \sum_{i \neq 1} \frac{1}{\lambda_i} \ln \left( \frac{\alpha_i \nu^1 q^1 + q^2 \nu^2 + q^3 \nu^3}{\nu^3} \right).$$
The sums in these functions depend on the parameters $\tau = \nu^2/\nu^1$ and $\sigma = \nu^3/\nu^1$ and as before we may use notation $\bar{H}^j(\omega, \tau, \sigma) := H^j(\omega, \nu^1, \nu^2, \nu^3)$ for $j \in [1,3]$. Up to here the analysis is completely analogous, but now we come to the point where we have to establish that there is a unique equilibrium point, i.e., that the three surfaces $H^j = \bar{x}^j$ intersect in exactly one point. As before the three sums $\bar{H}^j$ are increasing with $\omega$. The sum $\bar{H}^1$ increases with $\sigma$ and $\tau$ while $\bar{H}^3$ decreases with these parameters. The sum $\bar{H}^2$ increases with $\sigma$ but decreases with $\tau$. Now a topological argument can be used again to show that the surfaces have a common point, so an equilibrium exists - and there is an algorithm to find it - but to establish uniqueness, we need more.

The difficulty of these equations is that $I_1$ and $I_2$ can be proper subsets, and indeed in general they are. If $I_1$ is equal to $I_3$, then the equations reduce to

$$\sum_{I_3} \frac{1}{\lambda_i} \ln \left( \frac{\alpha_i(q^1\nu^1 + q^2\nu^2 + q^3\nu^3)}{\alpha_i - \omega} \right) - \sum_{I_3} \frac{\ln \nu_j}{\lambda_i} = \bar{x}^j.$$

The first sum in this equation is independent of $j$. One quickly verifies that the ratios $\tau$ and $\sigma$ now are independent of the value of that sum. To solve $H^j(\omega, \sigma, \tau) = \bar{x}^j$ we need to adjust $\omega$ only, having preset the parameters $\tau$ and $\sigma$, as it were, in such a way that they are in tune once $\omega$ solves one of the equations $H^j = \bar{x}^j$. It immediately follows that the equilibrium is unique. Now if $I_1$ is a proper subset, then the analysis becomes more intricate. Since the equilibrium varies continuously with the parameters, it does seem likely that it remains unique. However, a solution to this problem seems difficult and requires a more sophisticated way of handling the equations.

References


