Can Anyone Learn To Play Mixed Strategies Equilibrium? Evidence From Chess Competitions

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Abstract

I examine how different kinds of chess players prepare for the next game in competitions, a strategic situation that has a Nash equilibrium in mixed strategies, to determine whether anyone could learn to play the equilibrium. The data show that the professional players play mixed strategies equilibrium, but the inexperienced future professional chess players do not. The amateurs do not use the equilibrium strategies when they are young and also after many years of experience. The data suggest that being able to learn how to play mixed strategies is an innate ability. Some of the inexperienced amateurs, but none of the experienced amateurs or inexperienced future professionals seem to use simpler second-best behavioral responses as substitutes for computing complicated Nash equilibria.

1 Introduction

A few recent empirical studies have examined whether people play mixed strategies equilibrium. Walker and Wooders (2001), Chiappori et al. (2002), and Palacios-Huerta (2003) find that experienced athletes use the equilibrium mix of strategies in the field; Palacios-Huerta and Volij (2006) finds that professional soccer players do so in the laboratory as well. The subjects of laboratory studies by Rapoport and Boebel (1992), Mookherjee and Sopher (1994), Ochs (1995), and McCabe et al. (2000) generally do not play the equilibrium. These findings seem to suggest that the more familiar one is with situations in which being unpredictable is rewarded, the more likely one is to play mixed strategies equilibrium. Indeed, Cheung and Friedman (1997) and Erev and Roth (1998) show that most subjects of laboratory experiments learn how to behave closer to the game theoretical predictions. Few of them, however, eventually use the precise equilibrium mix of strategies. The difference between the findings of laboratory and field studies could be alternatively explained by the sorting of people into occupations. Those who can play mixed strategies equilibrium self select in activities where the trait is rewarded and gain experience using it. Very few of the subjects of laboratory studies understand mixed strategies equilibrium and thus have little experience with situations in which this ability would be useful.

In this paper, I exploit the panel structure of a large and very heterogenous group of subjects observed in their natural environment to separate the effect of learning from that of sorting. The panel allows me to observe those who become professionals as well as those who self select to different occupations over time at various levels of experience. This paper brings new evidence showing that some people fail to play the mixed strategies equilibrium even after many years of experience. These are the people who eventually choose different occupations. Learning has therefore a limited role in explaining the equilibrium behavior generally observed in the field among professionals. The very selective sorting of people into the occupation examined is partially responsible for the observed equilibrium behavior. The paper also brings new evidence showing that some of the people who do not play mixed strategies equilibrium use simpler second-best behavioral responses as substitutes for computing complicated Nash equilibria.

The subjects examined are chess players preparing the next game of chess in competitions. A well known result in game theory states that chess has a solution in pure strategies, so chess might seem as an unlikely area for testing mixed strategies equilibrium. Due to its extraordinary complexity, however, chess players cannot use backward induction to solve the game. Deciding the first several moves before the game starts and preparing for the openings that might stem from opponent's choices of moves is an important part of the game of chess played in competitions. If a chess player always chooses the same move to start a game, or if he always uses the same responses to his opponent's first moves, he puts himself to a great disadvantage by allowing his opponent to prepare only the openings that start with or contain these moves. Although the context and the required skills are different, the strategic situation in which the chess player finds himself before the start of a game in competitions is very similar to that of a tennis player who serves or a soccer player kicking a penalty kick.

Like in the previous studies, inexperienced amateurs do not play equilibrium strategies. Experienced players who later drop out of chess competitions fail to play the equilibrium, even after many years of experience in a field in which strategic thinking is of paramount importance. One interpretation of the result is that these players are simply not able to learn how to play equilibrium in mixed strategies. This is certainly not a unique interpretation, but others are much less convincing. One could instead argue that the future drop outs might simply not try hard enough. However, these experienced amateurs are training to become professional chess players, so the lack of motivation is probably not a convincing explanation for their failure to play the equilibrium. Another possible explanation is that these players might decide not to learn the mixed strategies equilibrium because they expect to drop out of chess competitions in the future. The data prepared for a companion paper show that before dropping out of chess competitions these players improve considerably their chess playing skills. Facing a likely change in occupation these players should invest less in a very field-specific skill such as playing chess than in a general one that is more transferable across professions such as using mixed strategies.

The data rejects the hypothesis of equilibrium play for all groups of players except experienced professionals. Young and inexperienced future professionals do not play mixed strategies equilibrium. This suggests that playing mixed strategies is not an innate skill. Gaining experience therefore plays a role in generating the equilibrium behavior observed in the field for these players, but its role is limited. Some players are able to learn how to play mixed strategies, and they are the ones who self select to this occupation, while others are not able to learn and they choose different occupations. The paper adds to the growing literature suggesting that the selection of subjects for tests matters, such as Harrison and List (2003), who show that the experienced professional baseball-card dealers do not fall prey to ”winner's curse” in auctions, Benjamin et al. (2001), who show that nursing students estimate health-related risks with more accuracy than randomly selected students, and Lazear et al. (2005), who show that almost two-thirds of the altruistic behavior in a dictator game experiment disappears when the subjects are allowed to sort themselves into a different game.

The skills examined in preparing for the game of chess include anticipation of opponent's responses and preparation for each of the situations that may ensue. These skills are more useful in many other occupations, such as lawyer, teacher, or manager, than the skills of athletes usually examined in the literature. Another difference between athletes and chess players is the object of their decision. Tennis and soccer players make split-second decisions about the very immediate, near-present future. In contrast, the chess players examined in this paper prepare the game with at least one day in advance. Owing to nature of the strategic situation and the skills examined, this finding has a somewhat broader interpretation, contributing to the “nurture versus nature” debate in the labor literature. Accumulation of human capital through “nurture” is important, but it would be inefficient without people sorting to the occupations in which they have a “natural” comparative advantage.

The second contribution of the paper answers the question: How exactly do the people who seem to be unable to learn to play the equilibrium behave in situations that would require them to play mixed strategies? The data show that a subset of the people who do not play mixed strategies seem to address only one of the strategies in the opponent's equilibrium mix of strategies. These behavioral responses are cheaper to compute than the full equilibrium in mixed strategies and they are optimal when the opponent chooses that particular strategy in the mix. The data show that some of the inexperienced amateurs seem to over-prepare for the move White made in the previous game. Choosing an opening move different than the one used in previous games improves White's expected outcome by 17.2 percent when playing against an 8 year old who eventually drops out of competitions. Some of the inexperienced amateurs also seem to underestimate the probability with which the opponent chooses less frequently used moves. Choosing a move used less frequently in the past games increases one's expected outcome in the current game by 12.7 percent. The proportion of players deviating from the equilibrium strategies in this particular way decreases over time and becomes insignificant after the age of 20, when the competitions are no longer segregated by age.

All these findings in the data are robust to alternative specifications and tests, inclusion of player fixed effects, pooling the data, and coding the opening moves in various different ways. The results do not seem to be generated by omitted variables or by measurement errors. Compared to previous tests of mixed strategies equilibrium, using chess data has some disadvantages: chess players may not use mixed strategies, aggregation of games across different players might cause biased results, and the outcome of the game of chess depends on more variables than the first moves. A test using all the games played for the World Championship title between 1886 and 2002 in a context in which the payoff matrix is identical across games for the same pair of players confirms the hypothesis of mixed strategies equilibrium play among top players.

The advantages of using chess outweigh the few disadvantages. Chess is a game rather than a sport, so it is a more natural choice for this kind of research. The strategies are known in advance and fully observable when played. By contrast, the strategies in tennis and soccer penalty kicks include direction, spin, strength, and other items, of which only direction is observable. In sports, the strategy set might also vary across players; no one can bend it like Beckham. However, in chess the strategy set consists of moves available to all players that are known in advance and fully observable by the researcher.

2 The Role Of Openings In Modern Chess

Chess is a two-person, zero-sum, finite, sequential, extensive-form game with perfect information; chess therefore has a sequential equilibrium in pure strategies which could be found by backward induction. This result has little importance to chess players because spanning the game of chess in its extensive form and reaching the terminal nodes is a task beyond the capability of the human mind, even with the help of extremely fast computers. Hardy (1999) estimates the number of the nodes in the extensive-form game of chess at $10^{10^{50}}$. A conservative estimate of the nodes after the first 40 moves is 10^{43}; it would take Deep Blue, the fastest chess computer in the world, capable of spanning 200 million nodes per second, 10^{21} years to span the game in its extensive form thus far.

The players cannot use backward induction to solve chess in pure strategies so the actual game of chess is played in conditions of extremely bounded rationality. Chess players span a subset of the tree of the game by starting in the current position and evaluating the positions resulting after several moves according to certain rules, choosing then the move that leads to the most advantageous position. Chess is played against the clock: the players have a limited time to make a number of moves. They face a trade-off between width and depth in spanning the tree, and at the same time between spending more time analyzing the current position and preserving time for future use. In order to preserve game time and to improve their analysis of the tree, chess players spend considerable time and effort preparing for the games: they learn general principles that allow them to eliminate moves that are likely to be dominated from each position or to discover dominant moves that were probably overlooked by their opponent.

A considerable fraction of preparation time is dedicated to the openings moves. Chess always starts in the same initial position in which pieces have limited mobility, so the tree generated by the first several moves is relatively less complex and easier to analyze. Players could analyze the positions generated by the first moves as they play the game, but that would be a poor allocation of their time. They are better off deciding before the game starts on a contingency plan of moves to be played to each move their opponent could make in the beginning of the game. This allows them to substitute valuable time during the game with relatively less valuable preparation time before the game starts and to use additional expertise. Chess players have been able to analyze the beginning of the tree almost completely and to eliminate the sequences of moves leading to disadvantageous positions. The remaining set of sequences of moves, called "openings," consist of sequences of a few best responses to the few best moves the opponent may play. Players memorize openings, rehearse them, and virtually always play them. They also study the weaknesses of deviating from the known openings. Any deviation from the recommended sequence of moves gives the opponent a considerable advantage.

Figure 1 shows a very simplified version of the game of chess in the extensive form, starting with White's opening moves from the initial position. Suppose White can move either "e4" or "d4." If White moves "e4," then Black can respond with either "e5" or with another move, among which "c5" is the most common. To White's "d4," Black can respond with either "d5" or with a set of moves, most commonly "Nf6." If White chooses to play "e4" as his first move, and Black chooses "e5" as the response to "e4," the players end up playing one of the openings called "open games"; if Black chooses to play anything else, they end up playing one of the openings called "semi-open games." If Black answers "d5" to White's "d4", the players will play one of the openings labeled "closed games," and if Black answers with anything else, they play one of the openings known as "Indian systems."

In a game of chess, White can choose between 20 moves and Black has 20 responses. The top players examined in this paper, however, use only 4 out of the 20 possible opening moves: "e4," "d4," "c4," and "Nf3," and the last three are considered equivalent by many chess theorists. Lower ranked players used these 4 opening moves in more than 99 percent of the games played. The theoretical model in the next section uses two moves and two responses, but the results could be generalized to multiple moves and responses. What is important here is not necessarily the number of moves, but the tree structure of the game of chess: once a player makes a move, a subset of openings can no longer be played and thus the time spent by players on preparing them does not influence the outcome of the game. Another imprecision of Figure 1 is that in chess distinct sequences of moves could lead to the same position. For example, White playing "d4," Black responding "d5," followed by White's "e4" would lead to the same position as the sequence "e4," "d5," and "d4." In practice, however, this almost never happens: "d5" is one of the optimal responses chosen by Black to White's "d4," but it is not optimal when White plays "e4," and it most likely would not be followed by "d4." Therefore, the game of chess players play in the opening stage of a game of chess is more appropriately characterized as a tree and not a lattice.

By vastly simplifying the tree of the game of chess in its extensive form, Figure 1 obscures the sheer number of openings and variants of openings in chess. The Oxford Companion to Chess has more than 1300 named openings and variants, many of them containing traps, very good moves, and moves to be avoided. Five volumes of Encyclopedia of Chess Openings merely classify and describe, but not analyze all the played openings. Amazon.com lists 2,357 books analyzing chess openings; more than 300 of them are written on Sicilian Defence alone, which is only one of the few "semi-open" games in Figure 1. For comparison, Amazon.com lists only 674 books on chess endgame, and even fewer, 232, on the middle-game. Learning openings and studying them is a very important part of a chess player's preparation. In chess competitions, the players find out the identity of their opponent at least one day before the game. A fraction of the preparation time for the next game is spent on choosing the opening move or the response to the opening move and on preparing for the possible ensuing openings. The players allocate preparation time among openings before the beginning of the chess game and thus before observing the opponent's choice of moves.

Figure 1 illustrates the strategic decisions to be made before the game starts. It shows that one does not need to prepare all the possible openings for a given game. Once White has decided to play "e4" in the next game he does not have to prepare any of the closed games and Indian systems; he can allocate all the preparation time to studying the open and semi-open games. Similarly, once Black decides on a few responses to each of White's possible moves he can allocate the preparation time only to the openings that would be played with positive probability. On the other hand, a player may want to be unpredictable in his choice of opening moves and responses. If Black knew that White always chooses "e4," for example, he could either allocate all his preparation time to open games, or he could allocate all his time to semi-open games, while White would still have to allocate some time to both. The strategic choice of opening move and the allocation of preparation time are modeled as a game with an equilibrium in mixed strategies in the next section. For various payoff functions, the game could have an equilibrium in pure strategies, but in practice that outcome is unlikely: a player would have to be extraordinarily skilled at playing openings that start with a certain move to overcome the advantage he gives his opponent by always choosing that move. The strategic situation applies to experienced top players as well as to beginners. Because of their experience and stock of knowledge of openings acquired through preparing the previous games, the experienced players may need less time to prepare the openings in the current game. They may prepare openings in more detail and may surprise their opponent later in the game with a variant of a given opening, but they face the same strategic considerations: they will not be able to play the variants prepared for a certain move if the opponent chooses to play another move.

3 A Model Of Choice Of Opening Move and Time Allocation

This section formalizes the discussion about the importance of opening moves in chess. I describe and analyze a game of choice of opening moves and allocation of preparation time. I prove the existence of an equilibrium, I discuss the conditions for the equilibrium to take place in mixed strategies, and I derive the theoretical propositions tested in the empirical section.

Two players, White and Black, decide before the game of chess between them starts what opening moves to make in the game and how much time to allocate to prepare each opening that might result from their moves. A strategy for White in this game is to choose an opening move and to allocate the preparation time to each openings that could result from his opening move. A strategy for Black is to choose a response to each opening moves that White may choose and to allocate the preparation time for openings that may result from her responses. The payoffs depend on each player's ability at playing the selected opening and on the preparation time devoted to that opening. Although complicated by the choice of preparation time, the game has the same basic structure as the game of matching pennies: White gains if he chooses to prepare a different opening than what Black prepares.

White's strategies have the form $(i, t_w^i) \in \{1, 2\} \times [0, 1]$, where $i = 1(2)$ corresponds to playing move $e4(d4)$ in the game and t_w^1 corresponds to allocating that fraction of time to preparing open games and the remaining to the semi-open games; similarly t_w^2 is allocated to closed games and $1 - t_w^2$ to Indian systems. Black's strategies have the form $(j, t_w^j) \in \{1, 2, 3, 4\} \times [0, 1]$; strategy $(1, t_b^1)$ corresponds to preparing to play $e5$ to White's $e4$ and $d5$ to White's $d4$ and allocating a fraction of t_b^1 to preparing the open games and $1 - t_b^1$ to closed games; $(2, t_b^2)$ corresponds to preparing to play $e5$ to White's $e4$ and $Nf6$ to White's $d4$ and allocating a fraction of t_b^2 to preparing open games and $1 - t_b^2$ to Indian systems, and so forth.

The payoffs depend on the opening that will ensue from their choices of opening moves and responses and the time allocated to that opening: $U_w((i, t_w^i), (j, t_b^j)) = P_{i,j}(t_w^i, t_b^j) = -U_b((i, t_w^i), (j, t_b^j))$. The functions $P_{i,j}$ depend on the choices i and j and are monotonic in their arguments: given that a certain opening has been selected, a player will get a larger payoff if more preparation time is allocated to that opening and less to the other opening. For example, if White chooses $(2, 0.25)$ (to play $d4$ and to allocate 0.25 of his time to preparing closed games) and Black chooses $(1, 0.7)$ (to play $e5$ to $e4$ and $d5$ to $d4$ and to allocate 0.7 of his preparation time to open games) they will end up playing a closed game, to which White allocated 0.25 and Black allocated 0.3 of their time: $U_w((2, 0.25), (1, 0.7)) = P_{2,1}(0.25, 0.7) = f_c(0.25, 0.3)$. Given Black's strategy, White's payoff would have been improved if he played $(2, 1)$ instead: $P_{2,1}(1, 0.7) = f_c(1, 0.3) > f_c(0.25, 0.3) = P_{2,1}(0.25, 0.7)$; given White's strategy, Black would have been better off if he played $(1, 0)$: $-P_{2,1}(0.25, 0) = -f_c(0.25, 1) > -f_c(0.25, 0.3) = -P_{2,1}(0.25, 0.7)$.

Games with infinite strategy sets do not always have Nash equilibria. The particularities of the payoff function together with a few assumptions about their shape guarantee that in this setup a Nash equilibrium exists; a few mild assumptions about the payoff functions are needed to show that the Nash equilibrium takes place in mixed strategies with a very limited support. The latter assumptions need not be satisfied for all players: in practice most players use more than one opening move, but very few use only one opening move.

Consider any pure strategy that White could use (i, t_w^i). The properties of the payoff function described above assure that the Black's best response to it is given by $\arg\max_j U_b((i, t_w^i), (j, t_b^j))$ where $t_b^j \in \{0, 1\}$. Therefore if White always uses an opening move, Black can best respond by always preparing only one of the openings that start with that move. A mild assumption assures that this cannot be an equilibrium: if Black always prepares a single opening, White could get an advantage by playing the other opening move. Let $BR((i, t_w^i)) = \text{argmax}_j\{U_b((i, t_w^i), (j, 0)), U_b((i, t_w^i), (j, 1))\}$ be Black's best response to White's (i, t_w^i)

Assumption 1: $U_w((k, t_w^k), BR((i, t_w^i))) > U_w((i, t_w^i), BR((i, t_w^i)))$, for a $t_w^k \in [0, 1], k \neq i$ for any $i \in \{1, 2\}$.

Assumption 1 rules out equilibria in which White uses pure strategies: if an Nash equilibrium exists, it must take place in mixed strategies. If Assumption 1 does not hold, the Nash equilibrium exist in pure strategies by the definition of best response. Let $(\pi, f_1(t_w^1), f_2(t_w^2))$ with $\pi \in [0,1]$ and $\int_0^1 f_i(t_w^i)\,dt_w^i = 1$ be a White's mixed strategy that uses potentially the entire strategy set: White plays strategies $(1, t_w^1)$ with probability $\pi \cdot f_1(t_w^1)$ and strategies $(2, t_w^2)$ with probability $(1-\pi) \cdot f_2(t_w^2)$. It is straightforward to show that for any Black's pure strategy, the White's mixed strategy $(\pi, f_1(t_w^1), f_2(t_w^2))$ is equivalent to White playing the mixed strategy $(1, a_1)$ with probability π and $(2, a_2)$ with probability $1-\pi$, where $a_i = \int_0^1 t_w^i \cdot f_i(t_w^i)\,dt_w^i$ is the average time allocated to the first opening that could ensue after playing move i. So if an equilibrium in mixed strategies takes place its support has at most two of White's strategies. Similarly, the support of Black's equilibrium strategy has at most four strategies.

Let $p_w^1 \in [0,1]$ be the probability with which White uses $(1, t_w^1)$ and $p_w^2 = 1 - p_w^1$ be the probability with which White uses $(2, t_w^2)$. Let $p_b^1, p_b^2, p_b^3 \in [0,1]$ be the probability with which Black uses $(1, t_b^1)$, $(2, t_b^2)$, and $(3, t_b^3)$, respectively; $p_b^4 = 1 - p_b^1 - p_b^2 - p_b^3$ is the probability with which Black uses $(4, t_b^4)$. White's expected payoff is:

$$\begin{aligned} EU_w = -EU_b = p_w \cdot [p_b^1 \cdot f_O(t_w^1, t_b^1) + p_b^2 \cdot f_O(t_w^1, t_b^2) + p_b^3 \cdot f_S(1 - t_w^1, t_b^3) + p_b^4 \cdot f_S(1 - t_w^1, t_b^4)] \\ + p_w^2 \cdot [p_b^1 \cdot f_C(t_w^2, 1 - t_b^1) + p_b^2 \cdot f_I(1 - t_w^2, 1 - t_b^2) + p_b^3 \cdot f_C(t_w^2, 1 - t_b^3) + p_b^4 \cdot f_I(1 - t_w^2, 1 - t_b^4)] \end{aligned}$$

Assuming that the functions f_i are continuous in their parameters, for any $(p_w^1, p_b^1, p_b^2, p_b^3) \in [0,1]^4$, White's expected payoff (and the negative of Black's expected payoff) is a continuous and differentiable function in $(t_w^1, t_w^2, t_b^1, t_b^2, t_b^3, t_b^4) \in [0,1]^6$. The first order conditions $\frac{\partial EU_w}{\partial t_w^i} = 0$, for any $i \in \{1,2\}$ and $\frac{\partial -EU_w}{\partial t_b^j} = 0$, for any $j \in \{1,2,3,4\}$ are continuous and the best responses defined by them have a fixed point by Kakutami's Fixed Point Theorem $({t_w^1}^*, {t_w^2}^*, t_b^{1*}, t_b^{2*}, t_b^{3*}, t_b^{4*}) \in [0,1]^6$ that can be interpreted as the equilibrium time allocations if White chooses $e4$ with probability p_w^1 and $d4$ with probability p_w^2, and Black chooses strategies 1 through 4 with probability p_b^1 through p_b^4.

Using ${t_w^i}^*(p_w^1, p_b^1, p_b^2, p_b^3)$ and $t_b^{j*}(p_w^1, p_b^1, p_b^2, p_b^3)$, White's problem reduces at choosing p_w^1 that maximizes $EU_w(p_w^1, p_b^1, p_b^2, p_b^3) = EU_w(p_w^1, p_b^1, p_b^2, p_b^3, {t_w^i}^*(p_w^1, p_b^1, p_b^2, p_b^3), t_b^{j*}(p_w^1, p_b^1, p_b^2, p_b^3))$, given Black's choice of (p_b^1, p_b^2, p_b^3); similarly for Black. Because ${t_w^i}^*$ and t_b^{j*} are continuous in $(p_w^1, p_b^1, p_b^2, p_b^3)$, $EU_w(p_w^1, p_b^1, p_b^2, p_b^3)$ is also continuous in $(p_w^1, p_b^1, p_b^2, p_b^3)$, so the first order conditions with respect to p_w^1 and p_b^j are continuous and the best response functions defined by them have a fixed point by Kakutami's Fixed Point Theorem. This fixed point $({p_w^1}^*, p_b^{1*}, p_b^{2*}, p_b^{3*})$ together with functions ${t_w^i}^*(p_w^1, p_b^1, p_b^2, p_b^3)$ and ${t_b^j}^*(p_w^1, p_b^1, p_b^2, p_b^3)$ define the Nash equilibrium of this game. Assumption 1 guarantees that the equilibrium is in mixed strategies; if Assumption 1 does not hold, equilibrium still exists, but in pure strategies.

If the game has a Nash equilibrium in mixed strategies (as opposed to one in pure strategies) the expected payoff to each player should be the same for each move in the support of the equilibrium mixed strategy. One does not fully observe the strategies used by each player: although the moves are observed, the preparation time is not. The characteristics of the equilibrium strategies derived above guarantee that the opening move is enough to fully characterize the strategy used by White.

Proposition 1. For a given pair of players, if both players play the mixed strategies equilibrium, then White's expected payoff of the chess game is independent of the opening move played in the game.

Proof: The proposition could be proved by examining the first order condition of $EU_w(p_w^1, p_b^1, p_b^2, p_b^3) = EU_w(p_w^1, p_b^1, p_b^2, p_b^3, {t_w^i}^*(p_w^1, p_b^1, p_b^2, p_b^3), {t_b^j}^*(p_w^1, p_b^1, p_b^2, p_b^3))$ with respect to p_w^1. All the marginal effects of p_w^1 on ${t_w^i}^*$ and ${t_b^j}^*$ are zero by the envelope theorem and the definition of optimal times given probabilities of using certain moves, so $p_b^1 \cdot f_O({t_w^1}^*, t_b^{1*}) + p_b^2 \cdot f_O({t_w^1}^*, t_b^{2*}) + p_b^3 \cdot f_S(1 - {t_w^1}^*, t_b^{3*}) + p_b^4 \cdot f_S(1 - {t_w^1}^*, t_b^{4*}) = p_b^1 \cdot f_C({t_w^2}^*, 1 - t_b^{1*}) + p_b^2 \cdot f_I(1 - {t_w^2}^*, 1 - t_b^{2*}) + p_b^3 \cdot f_C({t_w^2}^*, 1 - t_b^{3*}) + p_b^4 \cdot f_I(1 - {t_w^2}^*, t_b^{4*})$

The similar proposition for Black is not empirically useful because Black's strategy cannot be observed: although the pairs of openings uniquely identify the strategy used by Black, one cannot observe what response Black has prepared to White's unused move.

If chess players select the opening moves and responses and allocate the preparation time according to the prescription of the mixed strategies equilibrium, their choice of strategies should be determined exclusively by the payoff matrix. In a sequence of games, for example, extraneous variables such as the past choices of strategies by each player should not affect the way the current game is played.

Proposition 2. If both players play the mixed strategies equilibrium, then

a) White's expected payoff when he chooses a different move than the one chosen in the previous game should be the same as his expected payoff when he chooses the move chosen in the previous game;

b) White's expected payoff when he chooses a move used less frequently in the past should be the same as his expected payoff when he chooses a move used more frequently in the past.

Proof: The proof is trivial and follows from Proposition 1. Let $i = 1$ be the move used in the previous game (or the move used more frequently in the past.) Then Proposition 1 states that the expected payoff from moving $i = 1$ should be the same as the expected payoff from moving $i = 2$.

4 Data and Empirical Strategy

The data come from ChessBase 8.0, a collection of games played in competitions used by professional chess players. The database contains roughly 2,300,000 games of chess starting with what is probably the first recorded game in 1560 and continuing until 2002. About 1,900,000 of them are played in competitions after 1950 and are reported by the chess players themselves. Competition rules require players to record their and opponent's name, their rankings, the date at which the game took place, all the moves, and the outcome of the game, and then to file the recording with the referees, from whom ChessBase collects them.

For a large subsample of games, the competition in which the game took place and the strength of the players are also recorded. Almost all competitions are round-robin tournaments, in which a player meets all the other players, so players would not consider losing in the current game to meet a weaker opponent in the future. The strength of a player is estimated through the ELO coefficient, an estimator discovered by the Hungarian mathematician Arpad Elo and implemented by the International Chess Federation, FIDE, in the 1970s. ELO is a maximum likelihood type of estimator, derived from the observed outcome of the games previously played by a player. Winning improves the ELO coefficient and winning against a player with a high ELO coefficient improves a player's ELO coefficient more. As expected, the players' ELO coefficients are very good predictors of the outcome of the game.

The set of players in the database is very heterogeneous. The database contains all the World Championship matches between 1886 and 2002. At the other end of the spectrum, the database contains games played by less experienced players. Some of these less experienced players later become professional chess players with many games recorded in the database over time. Some other players drop out of the database when they choose a different occupation.

Proposition 1, the equality of payoff across all the opening moves used by White, is the standard test of equilibrium play used in the literature, but when examining multiple, heterogeneous subjects, this test alone is insufficient. The equality of payoff across White's moves might be generated by alternative, out of equilibrium behavior of some of the players, so a failure to reject the null hypothesis may not be enough to reject non equilibrium behavior. An example using the simpler game of matching pennies in which White loses 1 when Black matches his choice of Heads or Tails, and wins 1 otherwise illustrates this point. Suppose White plays the equilibrium strategy of Heads with probability 0.5 and Tails with probability 0.5. Most Black players use the equilibrium strategy, but a fraction π of them simply choose Heads if and only if White chose Heads in the previous game. They behave as if they believed that White's strategy is simply to repeat the move used last. In this case, Proposition 1 alone would not be sufficient to reject the out-of-equilibrium play because the expected payoff is equal across all actions. The expected payoff to White from playing Heads at time t is $EU_w(H_t) = (1-\pi)\cdot[0.5\cdot(-1)+0.5\cdot 1]+\pi\cdot[P(H_{t-1})\cdot EU_{w,n}(H_t|H_{t-1})+P(T_{t-1})\cdot EU_{w,n}(H_t|T_{t-1})]$, where $P(H_{t-1}) = 0.5$ is the probability with which White chose heads in the previous game and $EU_{w,n}(H_t|H_{t-1}) = -1$ is the payoff to White from choosing heads again when meeting a player who does not play equilibrium strategy. Similarly, $P(T_{t-1}) = 0.5$ and $EU_{w,n}(T_t|H_{t-1}) = 1$, so $EU_w(H_t) = 0 = EU_w(T_t)$ and testing Proposition 1 fails to reject out of equilibrium behavior.

A solution is complementing the tests of Proposition 1 with testing Proposition 2. I compare the payoff of the games in which White uses a different choice than that used in the previous game to the payoff of the games in which White makes the same choice would highlight Blacks' non-equilibrium behavior. White's expected payoff from making a different move is: $EU_w(M_t|M_t \neq M_{t-1}) = 0.5\cdot EU_w(H_t|T_{t-1})+0.5\cdot EU_w(T_t|H_{t-1}) = 0.5\cdot[(1-\pi)\cdot 0+\pi\cdot 1]+0.5\cdot[(1-\pi)\cdot 0+\pi\cdot 1] = \pi$. White's expected payoff from choosing the same move as in the previous game is: $EU_w(M_t|M_t = M_{t-1}) = 0.5 \cdot EU_w(H_t|H_{t-1}) + 0.5 \cdot EU_w(T_t|T_{t-1}) = 0.5 \cdot [(1-\pi)\cdot 0 + \pi \cdot (-1)] + 0.5 \cdot [(1-\pi)\cdot 0 + \pi \cdot (-1)] = -\pi$, so $EU_w(M_t|M_t \neq M_{t-1}) \neq EU_w(M_t|M_t = M_{t-1})$ which contradicts Proposition 2.

Testing both propositions represents an improvement over the usual test of Proposition 1 alone. Rejecting Proposition 1 is sufficient to reject the Nash equilibrium, but it sheds little light into how precisely the out-of-equilibrium players behave. Rejecting Proposition 2 suggests that some players behave as if they believed that their opponent would always repeat the same move, or as if they believed that he would always choose the move most frequently used in the past. Furthermore, Proposition 2 could also be used to measure π, the fraction of players who do not play equilibrium strategies because $\pi = \frac{EU_w(M_t|M_t \neq M_{t-1}) - EU_w(M_t|M_t = M_{t-1})}{2}$.

The data set has a couple of drawbacks that affect the sample selection and empirical strategy. Most importantly, very few professional players, and virtually no amateurs play many games against the same opponent. Chiappori et al. (2002) show that aggregating across various pairs of players could result in selection bias if the payoff matrix of the game described in the previous section is pair-of-players specific. In the next section, I assume that the payoff matrix is White player-specific, but it does not depend on Black. This assumption is first tested and the data fails to rejected it. The assumption allows to test Proposition 1 player by player using all the games in which a player used White. It also warrants the inclusion of White player fixed effects in tests of Proposition 2.

One question routinely asked in the literature is whether players play mixed strategies equilibrium more often as they grow older and become more accustomed with the strategic environment. Unfortunately, the data set does not contain information about the age of the players or their birth year. I obtained the birth year of 160 professional players with long careers and many games recorded in the data set from the ChessBase's website.[1] These players play 42,449 games against 7,216 Black players. It is virtually impossible to obtain the birth year or the age of all these Black players. Chess players participate in competitions segregated by age until they are 20. Therefore White's age is a very good indicator for

[1]I select the top 40 male and the top 40 female players. To correct for possible sample-selection biases, I also select 40 lower ranked male and female professional players (with an ELO around 2300 in 2002.) The list of the selected players is presented in Table A1.

Black's age while White is younger than 20. Once White is older than 20, his age is not a good indicator of Black's age, so the effect of experience could be identified only for the group of players between the ages 8 and 20.

5 Empirical Results

5.1 Testing The Assumption Of Identical Black Players

The assumption of identical black players is tested in a regression framework. The results are reported in Table 1. If the payoff matrix does not depend on the Black player, then the Black-fixed effects should be jointly statistically insignificant in all regressions. The dependent variables examined are: whether White uses the moves "e4," "d4," "c4," or "Nf3," the moves used in more than 99 percent of the games in the sample; as well as whether White uses an opening move not used in the previous game, or one used less frequently in the previous games. Chess theorists consider moves "d4," "c4," and "Nf3" to be effectively identical: Black's responses to each are the same, and White follows any move with the other two, so each move leads to the same position and set of available openings after two additional moves. Therefore, whether White uses any of these moves is also tested. Each regression contains the following control variables: the ELO coefficients of both players, White's age and age squared, and Black's experience in chess competition and its square. They also contain year fixed effects and White player-fixed effects.

The data fail to reject the hypothesis of identical Black players in six out of seven tests at very large significance levels. The identity of player using Black does not explain White choosing to move "e4," "d4," or "c4," nor does it explain White deciding to use an opening move different than the one used in the previous game or an opening move used less frequently in the past. The exception is whether White uses "Nf3" as the opening move. That move is one of the less frequently used moves among the four tested. When "Nf3" is pooled with "d4" and "c4" under the assumption that they represent the same move, the data fail to reject the hypothesis of identical Black players at a very large significance level. All the subsequent tests are repeated under the assumption that "Nf3," "d4," and "c4" represent the same move. The test results are robust to this alternative specification.

Table 1 also contains a test of the joint significance of the Black-fixed effects when the dependent variable is the outcome of the game of chess. This is not a test of the hypothesis of homogeneous Black players because the identity of the player using Black may affect the outcome of the game of chess after the opening game was played out. The null hypothesis of joint significance of the Black-fixed effects is rejected at 5 percent significance level. After controlling for the identity of White, the player's productivity and their age and experience, the outcome of the game of chess depends on the identity of Black. This suggests that the control variables fail to capture all aspects of Black's productivity. Therefore the subsequent tests should include Black-fixed effects in samples in which the data fail to reject the assumption of their joint insignificance.

5.2 Testing Equilibrium Play

I test Proposition 1 by running models of the form:

$$O_{i,j,t} = X_{i,j,t}\,\alpha + \sum_{i=1}^{N_W} \beta_i \cdot DW_i + \sum_{i=1}^{N_W} \sum_{k=1}^{N_{M_i}-1} \delta_{i,k} \cdot DW_i \cdot DM_k + \epsilon_{i,j,t}$$

$O_{i,j,t}$ is the outcome of the game of chess at time t between player i as White and player j as Black: 1 if White wins, 0.5 if it is a draw and 0 if Black wins. $X_{i,j,t}$ is a set of control variables: the players' ELO coefficient, White's age and age squared and Black's experience and experience squared. DM_k are dummy variables for move k. Including DW_i, a dummy variable for White player i, allows the payoff matrix, as well as the unobserved component of White's chess playing skill to vary across players using White. According to Proposition 1, White does not have an advantage in the opening selection game when choosing a certain move over another. Thus, controlling for the players' chess skills and other variables, the outcome of the game of chess should be independent of the opening move played. Under the null hypothesis of equilibrium play the coefficients $\delta_{i,k}$ on the dummy variables for move k played by White i should be jointly insignificant. The results are presented in Panel A of Table 2. The top three rows show the p-value of the joint significance test of the $\delta_{i,k}$ coefficients, the F statistic and the number of degrees of freedom. The next rows show whether White and Black player fixed effects are included in the specification based on a test of their joint significance.[2]

Column 1 of Table 2 shows the results of the tests using the full sample of games. The group of players using Black is very heterogeneous, so I repeat this test and subsequent tests in subsets of the full sample segmented along two dimensions. One dimension is the players' age. Chess competitions are segregated by age when the players are younger than 20, so I distinguish between younger and older players. The other dimension is whether Black becomes a professional chess player. I observe the career length and the total number of games recorded in the data set for each Black player and I determine whether a player became a professional chess player. A professional player has a career longer than 10 years and has more than 100 games recorded in the database. The criterion is arbitrary but using different arbitrary criteria for determining whether a player became professional does not significantly alter the results. Columns 2 - 5 present the tests of equilibrium play using games played by amateur and professional Black players when they are younger or older than 20.

Proposition 2 is tested by running several regressions of the form:

$$O_{i,j,t} = X_{i,j,t}\,\alpha + \sum_{i=1}^{N_W} \beta_i \cdot DW_i + \gamma_1 \cdot \text{COM}_{i,j,t} + \gamma_2 \cdot \text{COM}_{i,j,t} \cdot \text{White's Age}_{i,t} + \epsilon_{i,j,t}$$

$O_{i,j,t}$, $X_{i,j,t}$, and DW_i are the outcome of the game of chess, control variables and White fixed effects; Black and opening move fixed effects are also included when the data do not reject their joint significance. Each regression tests whether a characteristic of the opening move (COM) affects the outcome of the game.

Each row of Panel B of Table 2 reports the coefficients γ_1 and γ_2 from regressions of the outcome of the game on various control variables and a characteristic of the opening move. The standard errors are

[2] I include Black fixed effects if the data reject their joint insignificance to capture the unobserved component of Black's chess productivity. I do not include them if they are jointly insignificant because including irrelevant variables would increase the variance of the estimates, making it more difficult to reject the hypothesis of equilibrium play. Consistently including Black player fixed effects does not change the analysis. These alternative specifications are available upon request.

in parentheses. The characteristic of the opening move in the top four rows is White using a different move than the one chosen in the previous game. The next four rows show the coefficients and standard errors from regressions in which the characteristic of the opening move is the frequency of usage in the past games. In the next four rows the variable of interest is whether White uses his most frequently used move and in the last four rows the tested variable is a whether White changes the opening move to a less frequently used move.

The data fail to reject the hypothesis of equilibrium play in the full sample of games and in the games played by experienced professional players, who constitute more than half of the full sample. In columns 1 and 5 of Panel A in Table 2, the p value of the F test of joint significance of $\delta_{i,k}$ is 0.129 and 0.367, respectively, so the data fail to reject Proposition 1. Proposition 2 also cannot be rejected, as none of the γ_1 coefficients in the columns 1 and 5 of Panel B are significant.[3]

The hypothesis of equilibrium play is rejected for all other groups of players. Among experienced amateurs, Proposition 2 is not rejected by the data, but Proposition 1 is rejected at conventional significance levels. The p value of the joint significance test of opening move dummy variables in column 4 is 0.004, so each White player has an advantage when using certain opening moves against this group of Black players. The evidence against young players who later become professional chess players is somewhat weaker. In column 3, the data fail to reject Proposition 1 at 5 percent significance level, but they reject it at 10 percent significance level; Proposition 2 cannot be rejected by the data.

Inexperienced players who later drop out of chess competitions do not play the mixed strategies equilibrium. The data fail to reject Proposition 1 in column 2, but they do reject Proposition 2. The estimates of the coefficients are statistically and economically significant. When both players are age 8, White improves the outcome of the game in his favor by 17.2 percent when he uses an opening move different than the one used in the previous game. White also improves the outcome of the game if he uses a different move than the one used most frequently in the past by an average of 12.7 percent. The less frequently

[3]I exclude the intersection between the characteristic of the move and White's age in samples where White is older than 20. I am interested in how the proportion of Black players incorrectly playing the game changes with Black's age. When White is younger than 20, White's age is the same as Black's age, so White's age could be used as a proxy. When White is older than 20, she meets players who are also older than 20, but not necessarily her age. Estimating both γ_1 and γ_2 does not change the results.

used the move is in the past, the greater is White's advantage. For example, if White uses one move 60 percent of the time and another 40 percent of the time, using the least frequently used move instead of the other gives him an advantage of 5.5 percent. White's advantage is larger if White uses an opening move 80 percent of the time and another opening move 20 percent of the time: White improves the outcome of the game in his favor by 16.7 percent when he uses the less frequently used opening move. The largest advantage to White is when he changes the opening move from the one used in the previous game, to an opening move used less frequently in the past: White wins one in five games he would otherwise have lost. This suggests that not all Black players in this group play the mixed strategies equilibrium: some of them deviate from the optimal strategy by preparing too much for the move White used in the previous game or for the move White uses most frequently. By the time both players reach age 20, the advantage White has from changing the opening move and choosing less frequently used opening moves diminishes and becomes statistically and economically insignificant.

The empirical results reveal the deviations from equilibrium made by some of the inexperienced players that lead to out of equilibrium play. Young amateurs seem to overestimate the probability with which a player chooses an opening move if the player used that move in the previous game. They also seem to underestimate the probability with which a player uses a strategy from his mix of strategies, if that strategy is used less frequently in the past. At the same age, those who later become professional chess players do not make these deviations.

5.3 Robustness Chess

The results of testing Propositions 1 and 2 are very robust. Including dummy variables for the year in which the game took place does not change the results. Consistently including Black player dummy variables, even in the specifications in which they are statistically insignificant, causes one to fail to reject the hypothesis of equilibrium play among experienced amateurs and young future professionals, but that is the expected outcome of including irrelevant independent variables and does not affect the main argument of the paper. The results are also very robust to using a likelihood ratio test instead of the F test.

Pooling the data, or using a ordered logit specification instead of the linear probability model also does not change the test results of the theoretical propositions. Most importantly, the results are very robust to various alternative codings of the opening move variables. They do not change if one groups the moves "d4," "c4," and "Nf3" as representing the same move as some chess opening textbooks suggest, or if one bundles together all the moves used less frequently by the players. All the tests using these alternative specifications are available upon request.

5.3.1 Testing Proposition 1 Using Games Played By The Same Pair Of Players

Between 1886 and 2002, 25 pairs of players played in 40 matches for the Male World Championship.[4] The matches consist of a large series of games in which the opponents alternate playing White and Black. I compile two samples of games, consisting of all the games played by each player of the pair as White against the other as Black. I use these 50 samples of games to test whether Proposition 1 holds.

Using these games addresses many of the potential drawbacks of the testing procedure used in the paper. The games take place between the same pair of players, so the payoff matrix is identical and the testing does not depend on the hypothesis of identical Black players. Players know each other, have ample preparation time, and are able to study past opponent's games. The rules of the match are such that the players always prefer a win to a draw and that in turn to a loss, so strategic losing or quick drawing never happens. These samples of games are an ideal test of the robustness of the hypothesis of equilibrium mixed strategies play among professional players.

Under the null hypothesis of equilibrium play, the outcome of the game should be independent of the opening move chosen by White. I consider the outcome of a game (win, draw, or loss) as a random draw from a multinomial probability distribution. The probability distribution function of outcomes should not vary within each subsample corresponding to an opening move. I test whether the probability distribution functions corresponding to each opening move are identical using a generalization of the Pearson's

[4] A total of 28 distinct players played in these matches. Karpov played the largest number of games: 223 games against 4 different players in 9 distinct matches. Kasparov against Karpov is the pair playing the largest number of games: 144 games in 5 distinct matches. In all the games but two, White starts with one of the following moves: e4, d4, c4, or Nf3. In 1978 Kortschnoj starts with g3 against Karpov and wins the game. In 1972 Fisher doesn't start at all against Spassky, losing by forfeit. These two games were omitted from the analysis.

nonparametric chi-square test for equality of distributions, as shown in Mood et al. (1974).[5]

Table 3 reports the results of the Pearson test for each of the 50 samples. The first three columns have information about each sample: the year in which the pair of players first met in a match and the identity of players. The next four columns show the mix of strategies chosen by White. Some players do not use all four moves, and in that case the zeroes corresponding to an unused move are omitted. Columns 8 through 11 show the average payoff in the games that started with a specific move. Finally, the last three columns report the value of the Pearson statistic, the number of degrees of freedom, and the p-value of the statistic under the null hypothesis. A very large difference in payoffs following different opening moves yields a large Pearson statistic and a low p-value, which would indicate a rejection of the null hypothesis of equilibrium play.

Some players use pure strategies. Steinitz, for example, plays the same move in all 10 games against Zukertort. In 9 out of the 50 samples, White always chooses the same move, so these 9 samples cannot be used to test Proposition 1. Among the 41 samples of the games in which White used mixed strategies, the null hypothesis is rejected only once (which represents 2.4 percent of the tests) at the 5 percent level of significance, in 1910, for a match in which there are only five observations. The null hypothesis is rejected four times (which represents 9.7 percent of the tests) at a 10-percent level of significance, for Capablanca against Alekhine in 1927, Spassky against Petrosian, starting in 1966, and for the matches in which Karpov plays against Kasparov, starting in 1984. Overall, the data support the hypothesis of mixed strategies equilibrium play.[6]

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[5] Under the null hypothesis, the Pearson statistic

$$Q = \sum_{i=1}^{N_m} \sum_{j=1}^{N_o} \frac{[N_{i,j} - n_i \cdot \frac{N_{1,j}+N_{2,j}+N_{3,j}+N_{4,j}}{n_1+n_2+n_3+n_4}]^2}{n_i \cdot \frac{N_{1,j}+N_{2,j}+N_{3,j}+N_{4,j}}{n_1+n_2+n_3+n_4}}$$

has a limiting chi-square distribution with $(N_m - 1) \cdot (N_o - 1)$ degrees of freedom, where N_m represents the number of distinct opening moves used and N_o represents the number of outcomes. $j = 1, ..., N_o$ indexes the game outcomes ($j = 1$ for win, $j = 2$ for draw, and $j = 3$ for loss), whereas $i = 1, ..., N_m$ indexes the first moves ($i = 1$ for e4, $i = 2$ for d4, $i = 3$ for c4, and $i = 4$ for Nf3). n_i is the number of games that start with move i and $N_{i,j}$ is the number of games that start with move i and have the outcome j.

[6] Additional tests give support to the hypothesis of equilibrium in mixed strategies play. Under the null hypothesis, each of the Pearson statistics is distributed chi-squared. Therefore the sum of these Pearson statistics is distributed chi-squared with a number of degrees of freedom equal to the sum of the individual degrees of freedom. The value of the cumulative statistic is 104.77, which is distributed with 107 degrees of freedom; the p-value associated with this test is 0.542. The equilibrium play

5.3.2 Omitted Variables

The outcome of a game of chess depends not only on the variables included in the regressions, but also on variables that are not recorded in the data base. If these omitted variables are observed by the players, they may influence both the outcome of the game and their choice of opening moves, biasing the estimates γ_1 and γ_2 in tests of Proposition 2. One could envision the following scenario: a good player usually plays "d4," but when he meets a considerably weaker player who does not seem to be well prepared against "e4," he changes the opening move to "e4." Because he changes the opening move only when he meets weaker opponents, the change in opening move and a better outcome for White are correlated but the former does not cause the latter.

Without actually observing the omitted variables it is difficult to strongly reject the scenario above, but several pieces of evidence suggest that the relation between changing the opening move and wins among inexperienced amateurs is not spurious. Table 1 shows that the propensity to change the opening move or to use a less frequently used opening move is not correlated with the identity of Black. The observed measures of Black's ability also do not seem to be consistently correlated with changing the opening move or using a more obscure move, as Table 4 shows. Consistent with the scenario above, meeting a more experienced and better Black player decreases White's propensity to change the opening move or to play a less frequently used move. Although these coefficients are statistically significant, their magnitude is extremely small: for example, 5 years of experience for the Black player would decrease White's propensity to change the opening move by 1 percent. Other variables that are correlated with Black's ability, such as the length of Black's career and the number of Black's games recorded in the data base, as well as whether Black becomes a professional player are uncorrelated with the probability of White changing the opening move or playing an unusual move. Also, if meeting weaker opponents induces White to change the opening move or play less frequently used moves, as White gets older and meets better sorted players, he should do it less often. The results in Table 4 show that White changes the opening move and uses less frequently used moves more often as he gets older.[7]

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across all samples cannot to be rejected by the data at any reasonable level of significance.

[7]Furthermore, if the above scenario were true, White should have better results when he switches from his favored move

5.3.3 Measurement Error

The issue of measurement error may arise in when testing Proposition 2 from treating the ChessBase database as if it were comprehensive, containing all the games ever played in competitions by a given player. If this assumption is incorrect one may have doubts about the accuracy of the variables describing the characteristics of the opening move and about whether Black was correctly identified as a professional player. I address these concerns in increasing order of importance.

Determining whether the Black player became professional would not be considerably affected by the fact that some games are missing. The database contains all the major international tournaments as well as many major national tournaments in many countries, at many levels of strength and degrees of professionalism in the past 60 years, so if a player is not shown to be playing in any of these tournaments it is safe to consider him or her not a professional player. The variables measuring the frequency of a move would also not be affected considerably if the games recorded in the database are randomly selected from the larger population of all the games played by a certain player. Under this assumption the frequency of a move in the sample of the games recorded in the database matches the frequency of the move in the population of all the games played.

If a player plays other games that are not recorded, however, one cannot be sure whether the player actually changed the opening move from the previous, unobserved game. The error measurement in this limited dependent variable could bias the estimators toward zero and even change their sign. Although this possible bias should be kept in mind when examining the estimates in the top row of Panel B in Table 4, I do not consider it to be important for several reasons. What is important here is the sign of the estimates and the Monte Carlo simulations show that it requires more than half of the observations to be incorrectly coded for the sign of the estimates to change. Also, if the bias affects all the groups proportionally, the comparison of the estimates among groups can still be performed even with the biased estimates, and that

to another move, because that's when the opponent is weaker, compared to the results when he changes back to the preferred move. The effects of a change in the opening move from and to the preferred move are actually equal. The largest effect on the game outcome is when White changes an unusual move with another unusual move, which suggest that when some of the inexperienced Black observe that White used an unusual move, they expect him to use his preferred move in the next game. These models are available upon request.

comparison shows that inexperienced amateurs make some mistakes that other groups of players do not.

6 Conclusions

The empirical analysis of a large and heterogenous sample of chess players conducted in this paper suggests that playing mixed strategies is neither an innate trait nor a skill learnable by everyone. Instead, the data suggest that being able to learn how to play mixed strategies is an innate ability. The future professional chess players do not play the equilibrium strategies when they are young, but the experienced professionals do. The players who drop out of chess competitions to choose a different occupation fail to play the equilibrium strategies when they are young but also after many years of experience. To the extent to which the skills analyzed in this strategic situation, such as anticipation and preparation, apply to other occupations, these findings provide an insight into the more general "nurture versus nature" debate in the labor economics literature. Learning alone cannot generate the kind of behavior rewarded by the markets; the endogenous sorting of people to jobs in which they have a comparative advantage is essential for observing equilibrium behavior in the field.

The data show that chess players need a long time to learn to play the mixed strategies equilibrium in a relatively complex game and some of them never do so. These findings help explaining why is it so difficult to observe equilibrium behavior in the laboratory: researchers cannot afford to give subjects the time to become familiar with the task and are usually not able to mimic the market selection mechanism. The failure of even experienced chess players, a group of people with above average analytic abilities that are familiar with strategic thinking to play equilibrium mixed strategies puts the results of laboratory experimenters in perspective: equilibrium behavior will likely remain the exception rather than the norm in the laboratory. Therefore the researchers ought to exercise caution when extending the results of laboratory experiments in which the assignment of subjects to tasks is done exogenously, by the researchers, to the behavior observed in actual markets, in which the people sort themselves endogenously to occupations and tasks.

Finally, the data reveal that some players substitute cheaper, second-best behavioral responses for playing the more complex mixed strategies equilibrium. These players seem to be using the best response to only one of the strategies in the opponent's equilibrium mix. They prepare too much for the games that start with the move White used in the last game, or with the move White used most frequently, and therefore they are relatively unprepared when White chooses a different move. The best response to only one of the strategies used by the opponent in the past could serve as a substitute for computing the equilibrium mix because these responses would be optimal at least some of the time. Understanding these second-best behavioral responses should be part of future empirical research of mixed strategies play. Although they shed little light into how professionals behave, they could explain the behavior of people in situations in which computing the equilibrium is beyond their ability.

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FIGURE 1 - A VERY SIMPLIFIED VERSION OF THE GAME OF CHESS IN EXTENSIVE FORM:
FEW OPENING MOVES, RESPONSES AND OPENINGS, AS WELL AS THE BOARD POSITION AFTER EACH MOVE OR SEQUENCE OF MOVES

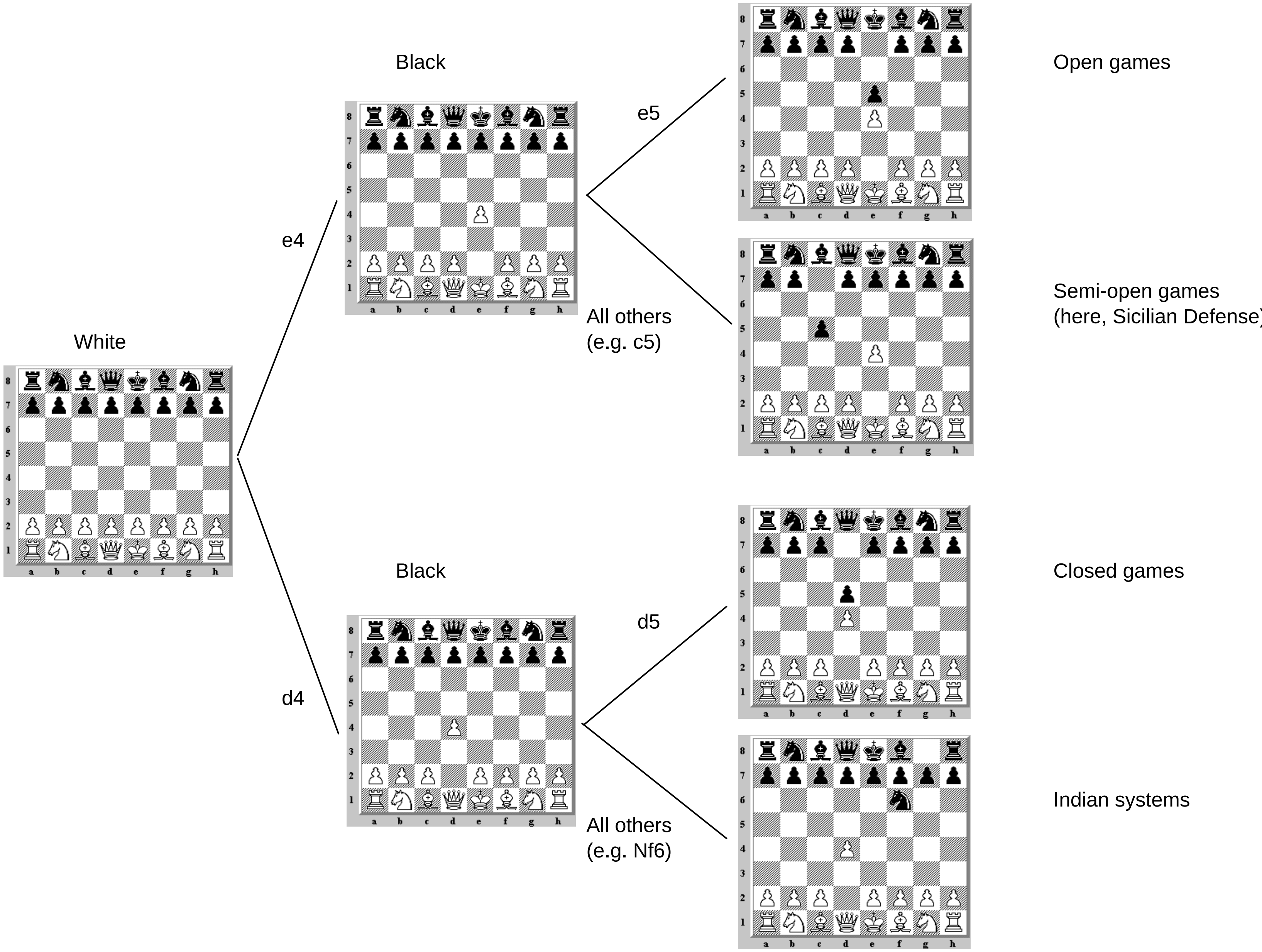

TABLE 1: TESTING WHETHER BLACK PLAYERS ARE HOMOGENEOUS

	Dependent Variable			
Independent variables	White moves e4	White moves d4	White moves c4	White moves Nf3
F statistic: joint significance of Black-fixed effects [p value listed below]	1.00 [p = 0.48]	0.92 [p = 1.00]	0.94 [p = 1.00]	1.04 [p = 0.03]
Year fixed effects included?	Yes	Yes	Yes	Yes
White-fixed effects included?	Yes	Yes	Yes	Yes
Black-fixed effects included?	Yes	Yes	Yes	Yes
R squared:	0.648	0.488	0.174	0.258

	Dependent Variable			
Independent variables	White moves d4, c4 or Nf3	White changes the opening move	White uses a less frequently used move	Outcome of the game
F statistic: joint significance of Black-fixed effects [p value listed below]	0.99 [p = 0.70]	0.98 [p = 0.85]	0.97 [p = 0.97]	1.18 [p = 0.00]
Year fixed effects included?	Yes	Yes	Yes	Yes
White-fixed effects included?	Yes	Yes	Yes	Yes
Black-fixed effects included?	Yes	Yes	Yes	Yes
R squared:	0.642	0.187	0.239	0.121

TABLE 2: TESTING MIXED STRATEGIES EQUILIBRIUM PLAY

	Ful Sample (1)	Amateurs under 20 (2)	Professionals under 20 (3)	Amateurs over 20 (4)	Professionals over 20 (5)
A. Null hypothesis: For a given player using White, the expected outcome of the game is the same regardless of the opening move used					
p value of the joint test	0.129	0.477	0.090	0.004	0.367
F statistic	1.08	1.00	1.15	1.27	1.03
Degrees of freedom	(426, 34642)	(120, 2588)	(171, 8570)	(249, 5543)	(317, 21869)
Covariates included?	Yes	Yes	Yes	Yes	Yes
White player-fixed effects included?	Yes	Yes	Yes	Yes	Yes
p value of the joint test	0.000	0.026	0.000	0.000	0.000
F statistic	2.45	1.29	1.95	1.53	2.25
Degrees of freedom	(159, 41841)	(108, 2588)	(112, 8570)	(145, 5543)	(147, 24347)
Black player-fixed effects included?	Yes	No	No	No	Yes
p value of the joint test	0.000	0.695	0.500	0.711	0.000
F statistic	1.17	0.97	1.00	0.98	1.22
Degrees of freedom	(7215, 34642)	(1772, 2588)	(1850, 6725)	(3487, 2113)	(2484, 21869)
Number of observations:	42,449	2,823	8,860	5,944	24,823
R squared:	0.131	0.205	0.158	0.188	0.143

Notes: Statistics in the table are based on ordinary least squares models in which the dependent variable is the outcome of the game (1 for win, 0.5 for draw, 0 for loss.) The table assumes homogeneity of players using Black and allows for heterogeneity of players using White.The hypothesis tested is whether for a given player using White the outcome of the game is the same regardless of the opening move played. The covariates are the strength of the players measured through the ELO coefficient, White's age and age squared and Black's experience in competitions and its square. All models include White player fixed effects to control for differences in the payoff matrix and for White's unobserved productivity. Black fixed effects are included only when testing in samples in which their joint insignificance is rejected.

TABLE 2: continued

	Ful Sample (1)	Amateurs under 20 (2)	Professionals under 20 (3)	Amateurs over 20 (4)	Professionals over 20 (5)
B. Null hypothesis: For a given player using White, the previous history of play has no effect on the current expected outcome of the game					
Change in opening move (COM)	0.001	0.292	-0.074	0.004	0.001
	(0.005)	(0.152)*	(0.101)	(0.014)	(0.005)
COM x White's Age		-0.015	0.005		
		(0.009)*	(0.005)		
The past frequency of use (F)	0.010	-0.519	0.114	-0.135	0.009
	(0.008)	(0.234)**	(0.170)	(0.074)*	(0.010)
F x White's Age		0.030	-0.008		
		(0.013)**	(0.010)		
Using a less frequently used move (LFM)	-0.003	0.239	-0.053	0.031	-0.005
	(0.005)	(0.133)*	(0.135)	(0.031)	(0.006)
LFM x White's Age		-0.014	0.002		
		(0.008)*	(0.008)		
COM x LFM	-0.001	0.346	-0.007	0.012	-0.002
	(0.005)	(0.179)*	(0.146)	(0.023)	(0.006)
COM x LFM x White's Age		-0.019	0.002		
		(0.010)*	(0.008)		
Number of observations:	42,449	2,823	8,860	5,944	24,823
Covariates included?	Yes	Yes	Yes	Yes	Yes
White player-fixed effects included?	Yes	Yes	Yes	Yes	Yes
Black player-fixed effects included?	Yes	No	No	No	Yes
White x Move fixed effects included?	No	No	Yes	Yes	No

Notes: The coefficients in the table are based on ordinary least squares models in which the dependent variable is the outcome of the game. The covariates are the players' strength, White's age and age squared and Black's competition experience and its square. White player fixed effects are included in all models. Black player and White x Move fixed effects are included only when testing in samples in which their joint insignificance is rejected.

* - indicates rejection at 10 percent significance level

** - indicates rejection at 5 percent significance level

TABLE 3: TESTING FOR EQUALITY OF PAYOFF ACROSS OPENING MOVES
CHESS GAMES PLAYED FOR THE MALE WORLD CHAMPION TITLE 1886 - 2000

Match Information			Strategies Mix				Average Payoff Per Strategy				Statistics		
Year	White	Black	c4	d4	e4	Nf3	c4	d4	e4	Nf3	Pearson statistic	Degrees of freedom	p-value
1886	Steinitz, W	Zukertort, J				10				0.65			
1886	Zukertort, J	Steinitz, W		9	1			0.44	0.00		1.1111	2	0.5737
1889	Chigorin, M	Steinitz, W			21				0.60				
1889	Steinitz, W	Chigorin, M			8	11			0.56	0.91	4.2025	2	0.1223
1890	Gunsberg, I	Steinitz, W		1	8			0.00	0.50		1.4062	2	0.4950
1890	Steinitz, W	Gunsberg, I		6		4		0.50	0.00	0.62	0.7738	2	0.6791
1894	Lasker, E	Steinitz, W		3	15			1.00	0.73		1.3846	2	0.5004
1894	Steinitz, W	Lasker, E		12	6			0.42	0.42		1.9285	2	0.3812
1907	Lasker, E	Marshall, F			7				0.79				
1907	Marshall, F	Lasker, E		7	1			0.29	0.00		1.1428	2	0.2850
1908	Lasker, E	Tarrasch, S		2	6			0.75	0.75		1.0666	2	0.5866
1908	Tarrasch, S	Lasker, E			8				0.44				
1909	Janowski, D	Lasker, E		5	5			0.20	0.30		1.3333	2	0.5134
1909	Lasker, E	Janowski, D		4	7			0.88	0.93		0.1964	1	0.6576
1910	Lasker, E	Schlechter, C		1	4			1.00	0.50		5.0000	1	0.0253 **
1910	Schlechter, C	Lasker, E			5				0.60				
1921	Capablanca, J	Lasker, E		6	1			0.67	0.50		0.4666	1	0.4945
1921	Lasker, E	Capablanca, J		4	3			0.38	0.33		0.0583	1	0.8091
1927	Alekhine, A	Capablanca, J		17				0.59					
1927	Capablanca, J	Alekhine, A		16	1			0.53	0.00		4.9583	2	0.0838 *
1929	Alekhine, A	Bogoljubow, E		24	1	1		0.60	1.00	0.50	2.7771	4	0.5957
1929	Bogoljubow, E	Alekhine, A		19	4	2		0.37	0.50	0.50	3.9802	4	0.4086
1935	Alekhine, A	Euwe, M		17	7	6		0.65	0.79	0.67	1.5016	4	0.8263
1935	Euwe, M	Alekhine, A	1	28		1	0.50	0.61		1.00	2.8365	4	0.5885
1951	Botvinnik, M	Bronstein, D		11	1			0.45	0.50		1.5272	2	0.4659
1951	Bronstein, D	Botvinnik, M	1	9	1	1	0.50	0.50	0.00	0.50	5.9259	6	0.4315
1954	Botvinnik, M	Smyslov, V	17	14		3	0.65	0.54		0.33	4.7661	4	0.3121
1954	Smyslov, V	Botvinnik, M	1	4	28	2	0.50	0.75	0.57	0.50	4.6568	6	0.5885
1960	Botvinnik, M	Tal, M	7	14			0.71	0.61			1.0000	2	0.6065
1960	Tal, M	Botvinnik, M	2	1	17	1	0.75	0.50	0.59	1.00	3.2580	6	0.7758

Note: The null hypothesis is that the payoff is equal accros opening moves used by White (Proposition 1.)
Under the null hypothesis, the Pearson statistic is distributed Chi squared with the appropriate number of degrees of freedom.
* Indicates rejection at the 10-percent level of significance.
** Indicates rejection at the 5-percent level of significance.

TABLE 3 - Continued

Match Information			Strategies Mix				Average Payoff Per Strategy				Statistics		
Year	White	Black	c4	d4	e4	Nf3	c4	d4	e4	Nf3	Pearson statistic	Degrees of freedom	p-value
1963	Botvinnik, M	Petrosian, T	1	10			0.50	0.45			0.2444	2	0.8849
1963	Petrosian, T	Botvinnik, M	6	5			0.75	0.50			1.9250	2	0.3819
1966	Petrosian, T	Spassky, B	14	6		3	0.46	0.75		0.67	5.5446	4	0.2358
1966	Spassky, B	Petrosian, T	1	4	19		1.00	0.25	0.63		9.0526	4	0.0597 *
1972	Fischer, R	Spassky, B	4		5		0.75		0.60		0.9000	1	0.6376
1972	Spassky, B	Fischer, R		4	7			0.38	0.43		1.0607	2	0.5883
1978	Karpov, A	Kortschnoj, V	4		21		0.50		0.62		2.6785	2	0.2620
1978	Kortschnoj, V	Karpov, A	22	1		1	0.48	0.50		0.50	1.3090	4	0.8598
1984	Karpov, A	Kasparov, G	1	39	15	17	1.00	0.63	0.47	0.50	11.4347	6	0.0758 *
1984	Kasparov, G	Karpov, A	9	35	24	4	0.56	0.57	0.62	0.50	6.9500	6	0.3254
1993	Karpov, A	Timman, J		9		1		0.67		0.00	4.4444	2	0.1083
1993	Timman, J	Karpov, A	2	3	4	2	0.50	0.33	0.38	0.50	1.4768	3	0.6876
1993	Short, N	Kasparov, G			10				0.50				
1993	Kasparov, G	Short, N		4	6			0.75	0.75		0.0000	1	1.0000
1995	Anand, V	Kasparov, G			9				0.44				
1995	Kasparov, G	Anand, V		1	7	1		0.50	0.64	0.50	0.7346	2	0.6925
1996	Kamsky, G	Karpov, A		4	5			0.62	0.40		0.9000	2	0.6376
1996	Karpov, A	Kamsky, G		8		1		0.69		0.50	0.5625	1	0.4532
2000	Kasparov, G	Kramnik, V	2	1	5		0.50	0.50	0.50			0	
2000	Kramnik, V	Kasparov, G		6		1		0.67		0.50	0.4666	1	0.4945

Note: The null hypothesis is that the payoff is equal accros opening moves used by White (Proposition 1.)
Under the null hypothesis, the Pearson statistic is distributed Chi squared with the appropriate number of degrees of freedom.
* Indicates rejection at the 10-percent level of significance.
** Indicates rejection at the 5-percent level of significance.

TABLE 4: ROBUSTNESS CHECK OF TESTS OF PROPOSITION 2

IS WHITE MORE LIKELY TO CHANGE THE OPENING MOVE OR TO USE A LESS FREQUENTLY USED MOVE WHEN MEETING A WEAKER PLAYER?

	Dependent variables			
Independent Variables	White changes the opening move from the one used before (COM)	The past frequency of use of the current opening move (F)	White uses a move used less frequently in the past (LFM)	White changes the opening move to a less frequently used move (COM x LFM)
Black's ELO / 1000	-0.141***	0.027	-0.041	-0.077***
	(0.035)	(0.019)	(0.033)	(0.024)
Black Becomes A Professional Player	0.004	-0.001	0.013	0.003
	(0.007)	(0.004)	(0.008)	(0.006)
Black's Career Length	0.001	-0.000	0.002	0.002
	(0.001)	(0.000)	(0.001)	(0.001)
Black's Number of Games Recorded	0.000	-0.000	0.000	-0.000
	(0.0000)	(0.000)	(0.000)	(0.000)
Black's Experience	-0.002*	0.002*	-0.003**	-0.002**
	(0.001)	(0.001)	(0.001)	(0.001)
Black's Experience squared	0.000*	-0.000*	0.000**	0.000**
	(0.000)	(0.000)	(0.000)	(0.000)
White's Age	0.015***	-0.014**	0.027***	0.014***
	(0.005)	(0.006)	(0.009)	(0.003)
White's Age squared	-0.000***	0.000**	-0.000***	-0.000***
	(0.000)	(0.000)	(0.000)	(0.000)
White player fixed effects included?	Yes	Yes	Yes	Yes
R-squared:	0.200	0.530	0.244	0.1190
Number of observations	42,449	42,449	42,449	42,449

Notes: The dependent variable is a characteristic of the opening move in each model. The independent variables are measures of Black's productivity or variables correlated with Black's unobserved productivity. All models include White fixed effects. If White changes the opening move or uses a less frequently used move when meeting a less qualified Black, then the coefficients of the variables measuring Black's productivity or on the variables correlated with Black's productivity should be negative. If fewer Black players are less qualified as White becomes older, then the coefficient of White's age should be negative.

* - Indicates significance at 10-percent significance level

** - Indicates significance at 5-percent significance level

*** - Indicates significance at 1-percent significance level